LECTURES

ON THE

DIFFERENTIAL GEOMETRY

OF CURVES AND SURFACES

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DIFFERENTIAL GEOMETRY

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PREFACE.

THE material of the present volume consists of the substance of lectures delivered, from time to time, during my tenure of the Sadlerian professorship of pure mathematics in the University of Cambridge. The last occasion, when such lectures were given by me, was during the Michaelmas Term of 1909.

As the volume does not pretend to be a complete treatise on differential geometry, and as it is restricted to the contents of my lectures, readers will find that not a few sections of the vast range of the subject are discussed only shortly and that some are left undiscussed. In lectures, my aim was to expound those elements with which eager and enterprising students should become acquainted; they could thus, in my opinion, be best prepared for the penetrating consideration, which is suited for the private study rather than for the lecture-room or the examination-room. No lack of individual interest was implied in omitted branches of the subject; to give an instance of a purely personal kind, my lectures never even mentioned the application of Lie's theory of continuous groups to the construction of the differential invariants for space and for surfaces in space—a matter to which, elsewhere, I had devoted some attention. One of my ideals, in lecturing to students, was to provide them with some of the instruments for research; consequently this volume is mainly intended for students who, later, may devote themselves to original work.

The book can be regarded as composed of three main sections; its divisions are only partially indicated by the chapters, which are numbered consecutively. Throughout, it deals solely with configurations in ordinary Euclidean space.

In the first section, consisting of a single chapter, the properties of skew curves and of their associated lines and planes are expounded, without regard to any family or families of surfaces upon which the curves may happen to lie.

In the second section, consisting of chapters II—VI, the subject-matter is the properties of curves upon any general surface in space. Some classes of these curves (*e.g.* lines of curvature) are organically connected with the surface; they are completely determined by the elements of the surface to which they belong. Other curves, such as geodesics, have an equally organic relation with the surface; but they are not determined solely by the elements of the surface, for they can satisfy some arbitrarily assigned condition or conditions. Again, quite arbitrary curves and families of curves can be assumed upon a surface; not a little attention has been devoted to methods for constructing differential invariants which, being in value independent of parameters of reference, express the geometrical magnitudes of the curves, subject, of course, to the dominance of the intrinsic magnitudes of the surface containing the curve or curves.

In the third section, consisting of chapters VII—XI, the subject-matter is surfaces in general, rather than particular configurations on surfaces. The most ordinary methods of point-to-point correspondence and comparison of surfaces are explained. Surfaces, which are defined (wholly or partially) by intrinsic properties, are considered, special attention being paid to minimal surfaces. Families of surfaces are discussed, according to the respective definitions that ultimately establish the families; the most obvious instance relates to those surfaces which have plane or spherical sets of lines of curvature. Lastly, a brief sketch of

the simplest fundamental properties of triply orthogonal systems is given.

The book concludes with a single chapter that contains an introduction to the elementary theory of congruences of curves, specially of straight lines and of circles.

Scattered throughout the book, examples (over two hundred in number) will be found; many of them are extracted from memoirs by various authors. At the end, there is a set of miscellaneous examples collected from Cambridge examination papers in recent years; for the collection, I am indebted to Mr R. A. Herman.

To facilitate reference, I have constructed a customary table of contents at the beginning of the book and a customary subject-index at the end; and, because a more or less persistent significance is assigned to many of the symbols that are used, I have given (at the end of the table of contents) a list of these symbols with the passages where the significance is first stated.

From the frequent references throughout, as well as in the references in the brief half-historical introductions to most of the chapters, it will be seen that one of my special desires has been to direct students to the work of the mathematician who, I think, would be generally hailed as the greatest living master of the subject. The treatise by Darboux must remain, at least for this generation, the classical exposition of Differential Geometry.

In exposition, it may have been rash on my part to restrict myself throughout to a treatment, which is based mainly upon the analysis used by Gauss and by those who followed him in its use. Certainly I have made no attempt to give what could only have been a rather faint reproduction of Darboux's treatment, which centres round the tri-rectangular trihedron at any point of a curve or surface or system. My hope is that students may experience an added stimulus of interest when they find that different methods combine in the development of growing knowledge.

Of course, in so extensive a subject, indebtedness naturally is not confined to one great worker alone. The names quoted in the course of my pages (and all have been quoted, whose work has been used by me) will give some hint of the multitude of workers who, through the long sequence of years, have constructed the immense fabric of acquired knowledge. Great as many of those names are, I wish here to place on record my own sense of gratitude to Darboux and to his work. My tribute of homage is gladly rendered in this year, the jubilee of his doctorate at Paris.

For valuable help given to me in many ways during the revision of the proof-sheets, as well as for suggestions and criticisms that proved useful to me, I tender my most cordial thanks to my friend Mr R. A. Herman, Fellow and Lecturer of Trinity College, Cambridge, and University Lecturer in Mathematics.

Finally, in past years and on other occasions, it has been my good fortune to receive the unfailing assistance of the staff of the University Press at Cambridge. On this occasion, their assistance has been forthcoming in the same generous and unstinted measure as before. To them, as only is their due, my thanks once more are given.

A. R. F.

February, 1912.

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SYMBOLS USED, AND THEIR SIGNIFICANCE.

THE following list of symbols has been framed for convenience of reference. The meanings assigned are those which are most frequently used; they are given in the definitions on the respective pages indicated by the numbers. It should be understood, however, that other meanings are occasionally and temporarily assigned to them; and it will be found that some symbols, such as those which have a significance limited to a special investigation, are not included.

A	binary form connected with the curvature of a normal section, 190.
A	magnitude for a ruled surface, 380.
A, B, C	$=EM-FL,\ EN-GL,\ FN-GM$, 95.
A, B, C, F, G, H	quantities connected with triply orthogonal systems, 432.
a, a', a''	direction-cosines of the tangent to a skew curve, 20.
a, b, c	parameters of plane or spherical lines of curvature, 310.
a, b, c	direction-cosines of generator of a ruled surface, 380.
a, b, b', c	quantities connected with a rectilinear congruence, 475.
B	magnitude for a ruled surface, 380.
B, B'	$[B=d\phi/dn=\{\Delta(\phi)\}^{\frac{1}{2}}]$, covariants, 219, 230.
b, b', b''	direction-cosines of the principal normal to a skew curve, 20.
c, c', c''	direction-cosines of the binormal to a skew curve, 20.
D	multiplier connected with geodesic polar coordinates, 89.
D	magnitude for a ruled surface, 380.
D, D_1, D_2	quantities in the equations of geodesics, 190.
E, F, G	fundamental magnitudes of the first order for a surface, 33.
$\mathbf{E}, \mathbf{F}, \mathbf{G}$	fundamental magnitudes of first order for first sheet of centro-surface 110.
$\mathbf{E}', \mathbf{F}', \mathbf{G}'$	fundamental magnitudes of first order for second sheet of centro surface, 110.
$\mathfrak{E}$	excess-function in calculus of variations, 127.
e, f, g	fundamental quantities for a spherical image, 254.
e, f, g	quantities connected with a rectilinear congruence, 476.
f	a relative invariant, 210.
f_1	critical function for range of geodesics, 126.
(f, Δ)	Jacobian magnitude in partial differential equations, 175.
$f=0,\ g=0$	equations of congruences of curves, 467.
g	multiplier connected with geodesic polar coordinates, 89.
H	mean curvature of a surface at a point, 44.

$\mathbf{H}, \mathbf{H}'$	measures of mean curvature for centro-surface, 111.
H_1, H_2, H_3	magnitudes for triply orthogonal surfaces, 410.
h	binary form connected with two curves on a surface, 230.
h_1, h_2, h_3	magnitudes for triply orthogonal surfaces, 409.
I	zero or unity, in connection with a binary form, 190.
I	an invariant connected with a curve, 217.
$di - \delta i$	geodesic contingence of a curve on a surface, 149.
i, j	angles between geodesic and parametric curves, 148.
J	Jacobian in a congruence, 470.
$\overline{J}$	binary form connected with a curve on a surface, 230.
J, J', J''	three binary forms connected with a curve, 217, 218, 229.
$\mathbf{J}$	Jacobian for two sets of parametric variables, 204.
K	specific (or Gauss measure of) curvature of a surface at a point, 44.
$\mathbf{K}, \mathbf{K}'$	measures of specific curvature for centro-surface, 111.
l	parameter of plane or spherical lines of curvature, 310.
l_1, l_2	focal lengths along a ray, 480.
L, M, N	fundamental magnitudes of the second order for a surface, 38.
$\mathbf{L}, \mathbf{M}, \mathbf{N}$	fundamental magnitudes of second order for first sheet of centro-surface, 111.
$\mathbf{L}', \mathbf{M}', \mathbf{N}'$	fundamental magnitudes of second order for second sheet of centro-surface, 111.
l, m, n	direction-cosines of binormal of a skew curve, 5.
m, m', m''	derivatives of magnitudes of first order for a surface, 44.
$\frac{d}{dn}, \frac{d}{dn'}$	differentiation along geodesic normals to curves on a surface, 218, 230.
n, n', n''	derivatives of magnitudes of first order for a surface, 44.
P	(sometimes) an arbitrary function of p only.
p, q	quantities proportional to direction-cosines of the normal to a plane, 16, 60.
p, q	parameters of a current point on a surface, 32.
p, q	parameters of congruences of curves, 468.
p, q, r	coordinates of point on directrix curve of a ruled surface, 380.
P, Q, R, S	derived magnitudes of third order for a surface, 56.
Q	(sometimes) an arbitrary function of q only.
R	radius of spherical curvature of a skew curve, 7.
r	distance of a point on a surface of revolution from the axis, 82.
r, s, t	second derivatives of z with respect to x and y, 60.
dS	element of arc in a spherical image, 254.
s	arc along a curve, 2.
ds	element of arc along a curve, in space, 2, on a surface, 33.
$\frac{d}{ds}, \frac{d}{ds'}$	differentiation along curves on a surface, 218, 230.
T	$=(LN - M^2)^{\frac{1}{2}}$, a magnitude of the second order for a surface, 38.

T	tangential coordinate of a surface, 260.
t	parameter along a curve, 2.
$\frac{d}{dt}$	differentiation along a geodesic tangent to a curve on a surface, 223.
t, t_1, t_2	distances along a ray, 478, 479.
u	length along generator of ruled surface, 380.
u	parameter of plane or spherical lines of curvature, 310.
du	shortest distance between two consecutive rays in a congruence, 477.
u, u'	binary forms connected with a curve, 229.
u_{mn}	double-suffix notation for derivatives, 210.
$u_1, u_2, u_3, \ldots$	derivatives of u, 409.
u, v, w	parameters of triply orthogonal surfaces, 409.
u, v	Weierstrass parameters for minimal surface, 280, 291 foot-note.
$[u, v]$	connected with Lamé relations, 419.
$\{u, v\}$	connected with Lamé relations, 419.
u, v	parameters of nul lines (symmetric variables) on a surface, 76; or lines of curvature, 93.
V	$=(EG-F^2)^{\frac{1}{2}}$, a magnitude of the first order for a surface, 34.
v	a fundamental quantity for a spherical image, 257.
v, v'	binary forms connected with a curve on a surface, 230.
W	binary form connected with lines of curvature, 190.
w	binary form connected with two curves, 229.
w	a complex variable in a relation $F(w, z)=0$, 238.
w_2, w_2', w_2'', w_3	four binary forms connected with a curve, 217.
$x', x'', \ldots$	derivatives along a curve with regard to the arc, 2.
$x_1, x_2, x_{11}, \ldots$	derivatives with regard to parameters, 33.
$x_1, x_2, x_3, \ldots$	derivatives of x, 409.
X, Y, Z	direction-cosines of the normal to a surface, 36, 471, coordinates in spherical image, 254, tangential coordinates, 260.
X, Y, Z	(sometimes) functions of x alone, of y alone, of z alone.
X, Y, Z	direction-cosines of a ray in a congruence, 475, 484.
$\left.\begin{matrix} X, Y, Z \\ X', Y', Z' \\ Z'' \end{matrix}\right\}$	elements for infinitesimal deformation of a surface, 394, 396.
x, y, z	coordinates of a point on a curve or a surface.
x_0, y_0, z_0	point on an adjoint minimal surface, 298.
z, z_0	$(=x+iy, x-iy)$ conjugate complex variables in a plane, 236.
α	radius of curvature of surface along one line of curvature, $p=$constant, 64.
α, β, γ	direction-angles of the tangent to a skew curve, 17.
α, β, γ	parameters of plane or spherical lines of curvature, 310.
$\alpha, \beta, \gamma, \delta, \epsilon$	derived magnitudes of the fourth order for a surface, 57.
β	radius of curvature of surface along one line of curvature, $q=$constant, 64.

$\Gamma, \Gamma', \Gamma''$	quantities connected with magnitudes of first order for a surface, 45.
γ	radius of geodesic curvature of any curve, 149, 192.
$\gamma, \gamma', \gamma''$	quantities connected with fundamental quantities for a spherical image, 259.
γ', γ''	radii of geodesic curvature of parametric curves, 150.
$\Delta, \Delta', \Delta''$	quantities connected with magnitudes of first order for a surface, 45.
$\Delta\phi$	Beltrami's first differential parameter, 164.
$\Delta_2(\phi)$	Beltrami's second differential parameter, 207.
∇	binary form connected with two curves, 229.
$\Delta(\phi, \psi)$	a covariant intermediate to two curves, 206.
δ, δ'	two binary forms connected with a curve, 217.
$\delta, \delta', \delta''$	quantities connected with fundamental quantities for a spherical image, 259.
$d\epsilon$	angle of contingence of a skew curve, 4.
Θ	$=(E\phi_2{}^2-2F\phi_1\phi_2+G\phi_1{}^2)^{\frac{1}{2}}$, 153.
θ	angle between tangent to a curve on a surface and a line of curvature, 192.
θ	inclination of generator of ruled surface to directrix curve, 380.
$\Theta(t, t_0)$	critical function for range of geodesics, 126.
Λ	binary form connected with two curves on a surface, 230.
λ	quantity of first order when a surface is referred to its nul lines, 80.
λ	angle at which two curves intersect, 230.
λ	parameter of plane or spherical lines of curvature, 314.
$\Lambda, \Lambda', \Lambda'', \Lambda'''$	quantities connected with derived magnitudes of the third order for a surface, 59.
$\lambda, \lambda', \lambda'', \lambda'''$	quantities connected with derived magnitudes of the third order for a surface, 59.
λ, μ, ν	direction-angles of the binormal to a skew curve, 17.
μ, μ', μ''	derivatives of fundamental quantities for a spherical image, 259.
ν, ν', ν''	derivatives of fundamental quantities for a spherical image, 259.
ξ, η, ζ	direction-angles of the principal normal to a skew curve, 17.
ξ, η, ζ	centre of curvature on first sheet of centro-surface, 108.
ξ', η', ζ'	centre of curvature on second sheet of centro-surface, 108.
$\xi(p, q),\ \eta(p, q)$	quantities in infinitesimal transformation, 210.
ρ	radius of circular curvature of a skew curve, 4, of a curve on a surface, 192.
ρ	radius of curvature of a normal section of a surface, 41.
ρ'	radius of curvature of normal section of a surface, 151, 192.
ρ''	radius of curvature of a second normal section of a surface, 230.
$\rho, \rho', \rho'', \rho'''$	quantities connected with derived magnitudes of the third order for a surface, 59.
σ	radius of torsion of a skew curve, 5, of a curve on a surface, 192.
σ', σ''	radius of torsion of geodesic tangent, 192, 230.

$d\tau$	angle of torsion of a skew curve, 5.
$d\tau'$	angle of torsion of geodesic tangent to a curve, 154.
υ	parameter of plane or spherical lines of curvature, 314.
Φ	quantity connected with geodesics, 191.
ϕ	azimuth of point on a surface of revolution, 132.
ϕ	central function in Weingarten deformations, 401.
$\phi=b$	family of geodesic parallels, 165.
$\phi(p, q)=c$	equation of curve on surface, 34, 194, 210.
$\Phi(\phi, \psi)$	a covariant intermediate to two curves, 207.
$\psi=\dfrac{\partial\phi}{\partial a}=c$	family of geodesics, 166.
$d\chi$	angle of screw curvature of a skew curve, 12.
Ω	binary cubic connected with variation of curvature, 192.
ω	angle between parametric curves on a surface, 34.
ϖ	inclination of principal normal of curve on a surface to normal of the surface, 151, 192.
ω'	angle between parametric curves in a spherical image, 257.

CHAPTER I.

Curves in Space.

Among the books to be consulted on the matter of this chapter, one is the classical treatise by Monge, *Applications de l'analyse à la géométrie*; the most useful edition is that by Liouville (1850), which also contains the famous memoir by Gauss on the general theory of surfaces, as well as various Notes by Liouville, Serret, and others.

The portions of Darboux's great treatise*, *Théorie générale des surfaces*, that should be consulted, are the first four chapters of the first volume and Note IV appended to the fourth volume. Of Bianchi's treatise†, *Lezioni di geometria differenziale*, which also is excellent, the first chapter will repay reference in the present connection.

This chapter deals solely with real curves in space. Certain imaginary curves in space (such as minimal or nul lines, and some curves of constant torsion) have important relations with real surfaces. The consideration of such curves, other than nul lines, belongs to a discussion of differential geometry more extensive than is here possible; but nul lines will be considered later (§§ 55—59) in connection with surfaces.

1. Curves in space, when they are not plane, are called skew, or twisted, or curves of double curvature (of flexion or circular curvature, and of torsion); when an epithet is necessary, the word *skew* will be used.

Skew curves occur in various manners. The two simplest of these modes arise by analytical definition and by the expression of organic properties.

When a curve is defined analytically, the coordinates of a current point are usually expressed in terms of a variable parameter. Sometimes an equivalent (but more cumbrous) definition is adopted when the curve is the whole, or a part, of the intersection of two surfaces; it is then given by combining the equations of the surfaces.

When a curve is defined by an organic property, that property is often relative to some surface upon which the curve lies. Thus lines of curvature, asymptotic lines, geodesics, are families of curves, characterised by their respective relations to the surfaces on which they exist. Consequently it is necessary to deal with surfaces in general, before the adequate expressions

* It will usually be cited as *Théorie générale* or as *Darboux*.

† It will usually be cited as *Geometria differenziale* or as *Bianchi*; the references will be to the second (Italian) edition.

for curves defined by organic properties can be obtained; only the elements of the general theory are required for the purpose.

We shall be concerned with intrinsic properties of curves and of surfaces, almost without exception. The position in space, and the orientation, of curves and of surfaces retain in this theory nothing of the significance and the importance that usually belong to them in algebraic geometry. The properties and relations are obtained by means of the differential coefficients of the magnitudes connected with the curves and the surfaces; hence the subject is often called differential geometry.

Moreover, except in rare instances, we shall avoid singular points of all kinds on curves and surfaces, and also singular lines on surfaces, in spite of their importance in other branches of geometry and in the theory of algebraic functions. Our purpose is the formulation of the fundamental properties of the curves and surfaces within a range of the geometrical configuration that is devoid of singularities.

Principal Lines and Planes of a Curve.

2. Let the coordinates of a current point on a skew curve be expressed in terms of a parameter t in the form

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

As we are dealing with an ordinary range of the curve, the functions $x(t)$, $y(t)$, $z(t)$ are taken to be regular throughout the range of the parameter; and we assume the positive direction of currency along the curve to be that which is given by increasing values of t.

The arc measured along the curve from some fixed point is denoted by s; we have

$$\frac{ds}{dt} = \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}^{\frac{1}{2}},$$

where the positive sign is taken for the square root. Occasionally the arc s is the dependent variable in an investigation; then it is usually convenient to keep s a function of t. Otherwise, there is convenience in making the arc s the actual parameter; in all such cases, we denote the first derivatives of x, y, z by x', y', z'; and similarly for derivatives of higher orders. Clearly

$$x'^2 + y'^2 + z'^2 = 1.$$

If ξ, η, ζ are the coordinates of a point Q on the curve, whose arc-distance from P is u, then

$$\left.\begin{aligned} \xi &= x + ux' + \tfrac{1}{2}u^2x'' + \tfrac{1}{6}u^3x''' + \dots \\ \eta &= y + uy' + \tfrac{1}{2}u^2y'' + \tfrac{1}{6}u^3y''' + \dots \\ \zeta &= z + uz' + \tfrac{1}{2}u^2z'' + \tfrac{1}{6}u^3z''' + \dots \end{aligned}\right\},$$

where the coefficients of the powers of u are the values of the derivatives at P.

3. The *tangent* is the limiting position of a secant through P and a consecutive point; hence the equations of the tangent are

$$\frac{X-x}{x'}=\frac{Y-y}{y'}=\frac{Z-z}{z'},$$

where X, Y, Z are current coordinates along the line. The direction-cosines of the tangent at P are x', y', z'; the positive direction of the tangent is taken to be that in which s and t increase.

The plane through P perpendicular to the tangent at P is the *normal plane*; its equation is

$$(X-x)\,x'+(Y-y)\,y'+(Z-z)\,z'=0.$$

Every line passing through P in this plane is a normal to the curve.

Any number of planes pass through the tangent at P; their general equation is

$$(X-x)\,l+(Y-y)\,m+(Z-z)\,n=0,$$

with the condition

$$lx'+my'+nz'=0.$$

The *osculating plane* at P is defined as the one of these planes through the tangent at P which also contains the tangent at a consecutive point; as the direction-cosines of this consecutive tangent are proportional to

$$x'+ux''+\ldots,\quad y'+uy''+\ldots,\quad z'+uz''+\ldots,$$

we have, for the osculating plane,

$$l\,(x'+ux''+\ldots)+m\,(y'+uy''+\ldots)+n\,(z'+uz''+\ldots)=0,$$

that is, using $lx'+my'+nz'=0$, we have

$$lx''+my''+nz''=0$$

in the limit. Hence the equation of the osculating plane is

$$\begin{vmatrix} X-x, & Y-y, & Z-z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}=0.$$

As the tangent at P is the limiting position of a secant through P and a consecutive point P', and the tangent at P' is the limiting position of a secant through P' and another consecutive point P'', the osculating plane at P is the limiting position of a plane through P and two consecutive points. Three points usually suffice to determine a plane uniquely; and so the osculating plane at P is the plane which, of all the planes through P, has the closest contact with the curve. Moreover, through three points a unique circle can be drawn; hence, lying in the osculating plane, there is a circle which is the limiting position of the circle through P and two points on the curve consecutive to P. It is sometimes called the osculating circle; its radius is definite in position and magnitude, and is called the *radius of circular curvature* (sometimes the radius of flexion, sometimes the radius of

curvature simply), while the curvature of the circle is called the *circular curvature* of the curve (sometimes the flexion, sometimes the curvature simply).

It is easy to see that the intersection of two consecutive osculating planes is a tangent to the curve.

4. Among the normals at P to the curve, there is one which lies in the osculating plane; it is called the *principal normal.* The centre of circular curvature lies on this principal normal, and is the intersection of two consecutive normal planes and the osculating plane; hence it is given by

$$(\xi - x)\,x' + (\eta - y)\,y' + (\zeta - z)\,z' = 0,$$
$$(\xi - x)\,x'' + (\eta - y)\,y'' + (\zeta - z)\,z'' = x'^2 + y'^2 + z'^2 = 1,$$
$$(\xi - x)\,(y'z'' - z'y'') + (\eta - y)\,(z'x'' - x'z'') + (\zeta - z)\,(x'y'' - y'x'') = 0.$$

It follows that

$$\frac{\xi - x}{x''} = \frac{\eta - y}{y''} = \frac{\zeta - z}{z''} = \frac{1}{x''^2 + y''^2 + z''^2};$$

and therefore, denoting the radius of circular curvature by ρ, so that

$$\rho^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2,$$

we have

$$\frac{1}{\rho^2} = x''^2 + y''^2 + z''^2.$$

We select the positive sign for $(x''^2 + y''^2 + z''^2)^{-\frac{1}{2}}$ as giving the value of ρ. The positive direction of the principal normal is taken as towards the centre of curvature from the point on the curve; and therefore the direction-cosines of the principal normal are $\rho x''$, $\rho y''$, $\rho z''$.

Further, let $d\epsilon$ be the angle between two consecutive tangents at P and P', and let ds denote the arc PP'. Then

$$\frac{1}{\rho} = \frac{d\epsilon}{ds},$$

so that

$$\left(\frac{d\epsilon}{ds}\right)^2 = \frac{1}{\rho^2} = x''^2 + y''^2 + z''^2.$$

The angle $d\epsilon$, being the angle between consecutive tangents, is sometimes called the *angle of contingence*; and the circular curvature is sometimes called the curvature of contingence.

5. Among the normals at P to the curve, there is one which is perpendicular to the osculating plane; as it is perpendicular to two consecutive tangents, it is called the *binormal.* The equations of the binormal at P are

$$\frac{X - x}{y'z'' - z'y''} = \frac{Y - y}{z'x'' - x'z''} = \frac{Z - z}{x'y'' - y'x''};$$

and its direction-cosines are

$$\pm \rho (y'z'' - z'y''), \quad \pm \rho (z'x'' - x'z''), \quad \pm \rho (x'y'' - y'x'').$$

The direction-cosines of any line are customarily taken to be the direction-cosines of its positive direction. For the tangent and for the principal normal, these have been settled; the binormal is merely perpendicular to the osculating plane, and so the choice between the two possibilities for the positive direction is a matter of convention. We shall choose the positive direction of the binormal so that the positive direction of the tangent PT, the positive direction of the principal normal PN (the curve being concave to N), and the binormal PB, stand to one another in the same way as do the coordinate axes Ox, Oy, Oz in the usual rectangular configuration; and then the direction-cosines of the binormal are

$$\rho (y'z'' - z'y''), \quad \rho (z'x'' - x'z''), \quad \rho (x'y'' - y'x'').$$

The figure formed by the three lines and the three planes is called the *trihedron* of the curve at P (sometimes the principal trihedron, sometimes the moving trihedron); and the lines are sometimes called the *principal axes* or *lines* of the curve at the point.

6. The *angle of torsion* is the angle between consecutive osculating planes or between consecutive binormals. If this angle be denoted by $d\tau$, the quantity $d\tau/ds$ measures the rate per unit of arc at which the osculating plane turns round the tangent. It is usually denoted by $1/\sigma$, so that

$$\frac{1}{\sigma} = \frac{d\tau}{ds};$$

and σ is usually called the *radius of torsion*, while $1/\sigma$ is often called the curvature of torsion, or simply the torsion. But there is no circle of torsion associated with the curve in the same kind of way as the circle of curvature; the radius of torsion is devoid of direction, though the torsion itself has a sign that will be used (§ 9) with the foregoing convention. If l, m, n be the direction-cosines of the binormal at P, and $l + dl$, $m + dm$, $n + dn$ be those of the consecutive binormal, then

$$\sin^2 d\tau = \Sigma \{m (n + dn) - n (m + dm)\}^2,$$

that is,

$$\frac{1}{\sigma} = \frac{d\tau}{ds} = \{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2\}^{\frac{1}{2}}.$$

Now

$$m = \rho (z'x'' - x'z''), \quad m' = \rho (z'x''' - x'z''') + \rho' (z'x'' - x'z''),$$

$$n = \rho (x'y'' - y'x''), \quad n' = \rho (x'y''' - y'x''') + \rho' (x'y'' - y'x'');$$

hence

$$mn' - m'n = \rho^2 \{(z'x'' - x'z'')(x'y''' - y'x''') - (x'y'' - y'x'')(z'x''' - x'z''')\}$$
$$= \rho^2 x' \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}.$$

Similarly for the other two quantities in the expression for $1/\sigma$. Substituting, and taking the positive sign for the square root, we have

$$\frac{1}{\rho^2\sigma} = \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix},$$

thus leading to an expression for the torsion.

Also, as $ll' + mm' + nn' = 0$, we have

$$\frac{1}{\sigma} = (l'^2 + m'^2 + n'^2)^{\frac{1}{2}}.$$

Now

$$l' = \rho(y'z''' - z'y''') + \rho'(y'z'' - z'y''),$$

and so for the others; substituting, and evaluating, we have

$$\frac{1}{\sigma^2} = \rho^2(x'''^2 + y'''^2 + z'''^2) - \frac{1}{\rho^2} - \frac{\rho'^2}{\rho^2},$$

another expression for σ, which will be deduced otherwise in another connection.

7. These particular results as regards the expressions for $d\epsilon$ and $d\tau$, and other results specially relating to inclinations of lines organically related to any curve, can be obtained by the use of the *spherical indicatrix*. Through the centre of a sphere of radius unity, let a radius be drawn parallel to a line whose direction-cosines are α, β, γ; the extremity of the radius can be regarded as representing the line. Thus, corresponding to all the tangents of the curve, there will exist a continuous curve upon the sphere which consequently provides an image of the sheaf of tangents.

Let another radius be drawn parallel to a consecutive line whose direction-cosines are $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$. The angle between this line, and the line that has α, β, γ for its direction-cosines, is equal to the length of the arc between the representative points on the spherical indicatrix; hence it is equal to

$$\{(d\alpha)^2 + (d\beta)^2 + (d\gamma)^2\}^{\frac{1}{2}}.$$

Thus the angle of contingence is

$$= \{(dx')^2 + (dy')^2 + (dz')^2\}^{\frac{1}{2}}$$
$$= (x''^2 + y''^2 + z''^2)^{\frac{1}{2}}\, ds;$$

and the angle of torsion is

$$\begin{aligned}&= \{(dl)^2 + (dm)^2 + (dn)^2\}^{\frac{1}{2}} \\ &= (l'^2 + m'^2 + n'^2)^{\frac{1}{2}}\, ds \\ &= \{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2\}^{\frac{1}{2}}\, ds,\end{aligned}$$

as above.

8. Through the circle of curvature at P, any number of spheres can be drawn; their centres lie on a straight line, through the centre of curvature at P and perpendicular to the osculating plane; and each of the spheres contains the three consecutive points which determine the circle of curvature. A sphere is, in general, uniquely determined by four points; hence, when we choose that one of the spheres which passes through four consecutive points on the curve, we have the sphere which has the closest contact with the curve. It is called the *sphere of curvature*; its centre is called the centre of spherical curvature; and its radius is the radius of spherical curvature. Let X_0, Y_0, Z_0 be the centre of the sphere of curvature, and R its radius; then the equation

$$(X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2 = R^2$$

must be satisfied at P and at three points consecutive to P. Thus

$$\begin{aligned}(x - X_0)^2 \quad &+ (y - Y_0)^2 \quad + (z - Z_0)^2 \quad = R^2, \\ (x - X_0)\, x' \ &+ (y - Y_0)\, y' \ + (z - Z_0)\, z' \ = 0, \\ (x - X_0)\, x'' &+ (y - Y_0)\, y'' + (z - Z_0)\, z'' = - x'^2 - y'^2 - z'^2 = -1, \\ (x - X_0)\, x''' &+ (y - Y_0)\, y''' + (z - Z_0)\, z''' = - x'x'' - y'y'' - z'z'' = 0.\end{aligned}$$

From the last three equations, we have

$$(x - X_0) \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} = \begin{vmatrix} 0, & y', & z' \\ -1, & y'', & z'' \\ 0, & y''', & z''' \end{vmatrix},$$

that is,

$$x - X_0 = \rho^2\sigma\,(y'z''' - z'y''');$$

and similarly

$$\begin{aligned}y - Y_0 &= \rho^2\sigma\,(z'x''' - x'z'''), \\ z - Z_0 &= \rho^2\sigma\,(x'y''' - y'x''').\end{aligned}$$

The first of the equations then gives

$$\begin{aligned}R^2 &= \rho^4\sigma^2 \{(y'z''' - z'y''')^2 + (z'x''' - x'z''')^2 + (x'y''' - y'x''')^2\} \\ &= \rho^4\sigma^2 \{(x'^2 + y'^2 + z'^2)(x'''^2 + y'''^2 + z'''^2) - (x'x''' + y'y''' + z'z''')^2\} \\ &= \rho^4\sigma^2 (x'''^2 + y'''^2 + z'''^2) - \sigma^2,\end{aligned}$$

because

$$x'x'' + y'y'' + z'z'' = 0,$$
$$x'x''' + y'y''' + z'z''' = -x''^2 - y''^2 - z''^2 = -1/\rho^2.$$

Again,

$$\frac{1}{\rho^4\sigma^2} = \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}^2$$
$$= \begin{vmatrix} \Sigma x'^2, & \Sigma x'x'', & \Sigma x'x''' \\ \Sigma x'x'', & \Sigma x''^2, & \Sigma x''x''' \\ \Sigma x'x''', & \Sigma x''x''', & \Sigma x'''^2 \end{vmatrix}$$
$$= \begin{vmatrix} 1, & 0, & -1/\rho^2 \\ 0, & 1/\rho^2, & -\rho'/\rho^3 \\ -1/\rho^2, & -\rho'/\rho^3, & (R^2+\sigma^2)/\rho^4\sigma^2 \end{vmatrix}$$
$$= \frac{1}{\rho^6\sigma^2} R^2 - \frac{1}{\rho^6}\rho'^2,$$

so that

$$R^2 = \rho^2 + \sigma^2\rho'^2.$$

If C be the centre of circular curvature at P and S be the centre of spherical curvature,

$$CS = \sigma\rho',$$

numerically, since CS is the perpendicular through C to the osculating plane at P.

9. The perpendicular distance of a point Q on the curve from the plane through the tangent and the binormal (commonly called the rectifying plane), the arc-distance of Q from P being u, is

$$(\xi - x)\,\rho x'' + (\eta - y)\,\rho y'' + (\zeta - z)\,\rho z''$$
$$= \frac{u^2}{2\rho} + \text{higher powers of } u;$$

that is, the curve at P lies entirely on one side of the rectifying plane.

The perpendicular distance of the point Q on the curve from the osculating plane at P is

$$= (\xi - x)\,\rho\,(y'z'' - z'y'') + (\eta - y)\,\rho\,(z'x'' - x'z'') + (\zeta - z)\,\rho\,(x'y'' - y'x'')$$
$$= \tfrac{1}{6}u^3\rho \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} + \text{higher powers of } u$$
$$= \tfrac{1}{6}\frac{u^3}{\rho\sigma} + \text{higher powers of } u:$$

that is, the curve crosses its osculating plane at P.

The normal distance of the point Q on the curve from the sphere of curvature at P being n, we have

$$(R+n)^2 = (\xi - X_0)^2 + (\eta - Y_0)^2 + (\zeta - Z_0)^2.$$

Retaining only the lowest power of n, and the lowest power of u that is significant, we find

$$2Rn = \tfrac{1}{12} u^4 \mu,$$

where

$$\mu = \rho^2\sigma \begin{vmatrix} x', & y', & z' \\ x''', & y''', & z''' \\ x'''', & y'''', & z'''' \end{vmatrix} - 1/\rho^2.$$

Denoting this determinant by D, we have

$$\frac{D}{\rho^2\sigma} = \begin{vmatrix} x', & y', & z' \\ x''', & y''', & z''' \\ x'''', & y'''', & z'''' \end{vmatrix} \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma x'^2 \;, & \Sigma x'x'' \;, & \Sigma x'x''' \\ \Sigma x'x''', & \Sigma x''x''', & \Sigma x'''^2 \\ \Sigma x'x'''', & \Sigma x''x'''', & \Sigma x'''x'''' \end{vmatrix}.$$

Now

$$\Sigma x'x''' = -1/\rho^2, \quad \Sigma x''x''' = -\rho'/\rho^3, \quad \Sigma x'x'''' = 3\rho'/\rho^3;$$

$$\Sigma x'''^2 = \rho'^2/\rho^4 + 1/\rho^4 + 1/\rho^2\sigma^2, \quad = U, \text{ say};$$

$$\Sigma x''x'''' = -\rho''/\rho^3 + 3\rho'^2/\rho^4 - U, \quad \Sigma x'''x'''' = \tfrac{1}{2}U'.$$

Substituting these values, we have

$$\frac{D}{\rho^2\sigma} = -\tfrac{1}{2}\rho' U'/\rho^3 + (\rho''/\rho^3 - 3\rho'^2/\rho^4 + U)(U - 1/\rho^4) - 3\rho'^2/\rho^8$$

$$= \rho''/\rho^5\sigma^2 + \rho'\sigma'/\rho^5\sigma^3 + 1/\rho^4\sigma^4 + 1/\rho^6\sigma^2;$$

and therefore

$$\mu = \rho''/\rho + \rho'\sigma'/\rho\sigma + 1/\sigma^2$$

$$= \frac{1}{\rho\sigma}\frac{d}{ds}(\sigma\rho') + \frac{1}{\sigma^2}.$$

Hence the normal distance of Q from the surface of the sphere of curvature at P is

$$= \frac{u^4}{24R}\left\{\frac{1}{\sigma^2} + \frac{1}{\rho\sigma}\frac{d}{ds}(\sigma\rho')\right\} = \frac{1}{24}\frac{u^4}{\rho\sigma^2}\frac{dR}{d\rho};$$

that is, the curve at P lies entirely on one side of the surface of its sphere of curvature at the point.

When P is taken as an origin, and the three principal lines at P are

taken as axes of reference, the most important terms* in the expressions for the coordinates of Q are

$$\xi = u - \frac{1}{6}\frac{u^3}{\rho^2} + \frac{1}{8}\frac{\rho'}{\rho^3}u^4 + \ldots, \quad \eta = \frac{1}{2}\frac{u^2}{\rho} - \frac{1}{6}\frac{\rho'}{\rho^2}u^3 + \ldots, \quad \zeta = \frac{1}{6}\frac{u^3}{\rho\sigma} + \ldots.$$

When σ is positive, the current point of the curve passes at P from the negative to the positive side of the osculating plane; when σ is negative, the passage of the current point is from the positive to the negative side of that plane.

Routh's Diagram.

10. The association of the kinematics of a rigid system with geometry is of ancient occurrence; and it has been much used by writers on geometry, very specially by Darboux†. A simple and effective use of the notion in discussing the properties of skew curves has been made by Routh‡.

In the accompanying figure, drawn for the case of positive torsion, PT, PN, PB are the tangent, the principal normal, and the binormal, of a curve at a point P, so that BPN is the normal plane, TPN is the osculating plane, and TPB is the rectifying plane; C is the centre of circular curvature, and S is the centre of spherical curvature, so that CS is perpendicular to the osculating plane TPN. The principal normal at a consecutive point Q distant ds from P is QC', which does not meet PC because it lies in the consecutive osculating plane at Q; the centre of circular curvature at Q is C'; and PQC' is the osculating plane at Q. The centre of spherical curvature at Q is S'; so that $C'S'$, which is the intersection of two consecutive normal planes at Q (and therefore passes through S, the intersection of three consecutive normal planes at P), is perpendicular to the plane PQC'; thus S, C', C, P lie on a circle, for both the angles SCP and $SC'P$ are right. Then

$$PC = \rho, \quad QC' = \rho + d\rho = PC', \quad KC' = d\rho,$$

neglecting powers of small quantities higher than those retained. Also

$$\begin{aligned} d\epsilon &= \text{angle of contingence} \\ &= \text{inclination of the consecutive normal planes } SC'P,\ SC'Q \\ &= \text{angle } PC'Q, \end{aligned}$$

and

$$\begin{aligned} d\tau &= \text{angle of torsion} \\ &= \text{inclination of the consecutive osculating planes } CPQ,\ C'PQ \\ &= \text{angle } CPC' = \text{angle } CSC'; \end{aligned}$$

* For higher terms, see Mathews, *Quart. Journ. Math.*, vol. xxvi (1893), pp. 27—30.

† It is made fundamental in his treatment of the subject: see, passim, his treatise *Théorie générale*.

‡ *Quart. Journ. Math.*, vol. vii (1866), pp. 37—44.

and therefore

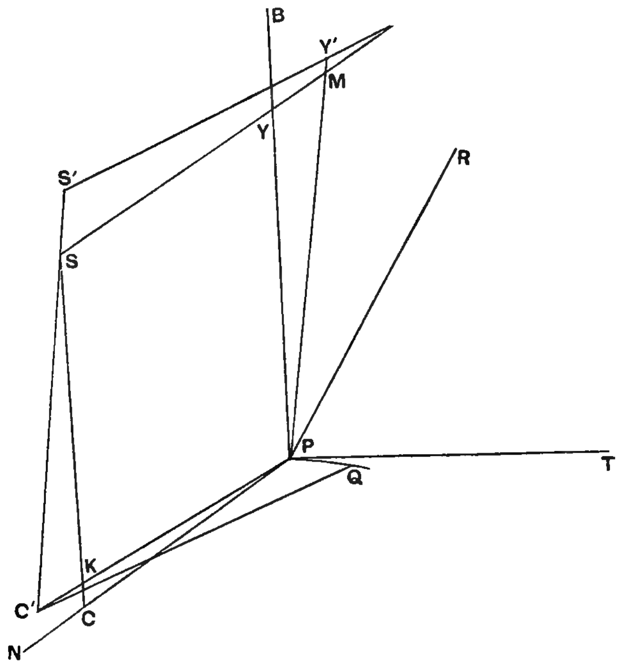
$$KC = \rho d\tau = \rho ds/\sigma,$$
$$CS = KC'/\text{angle } CSC' = d\rho/d\tau = \sigma\rho'.$$

Further, $PS = R$, so that

$$R^2 = PC^2 + CS^2 = \rho^2 + \sigma^2\rho'^2,$$

while

$$\tan CPS = \sigma\rho'/\rho.$$



Again, as regards the locus of S, we have

$dE =$ its angle of contingence

$= CSC' = d\tau,$

$dT =$ its angle of torsion

$=$ inclination of CSC' (a normal plane) to the consecutive normal plane

$= d\epsilon.$

Further, taking SY parallel to CP and PM parallel to $C'S'$, we have

$$Y'M = S'C' - SC = \text{variation of } SC = \frac{d^2\rho}{d\tau^2}\, d\tau,$$

$$SS' = S'C' - SC' = S'C' - SC + SC - SC' = \frac{d^2\rho}{d\tau^2}\, d\tau + CK$$
$$= \left(\frac{d^2\rho}{d\tau^2} + \rho\right) d\tau\,;$$

and therefore, for the locus of S,

$$\text{the radius of circular curvature } (\rho_1) = \rho + \frac{d^2\rho}{d\tau^2},$$

$$\text{the radius of torsion } (\sigma_1) = \frac{\rho}{\sigma}\left(\rho + \frac{d^2\rho}{d\tau^2}\right).$$

11. The use of the diagram can be developed. Thus PC and QC' do not intersect; so the principal normals of the curve have no envelope. Let dc be the arc-element of the locus of C; then

$$(dc)^2 = (CK)^2 + (C'K)^2 = (\rho\, d\tau)^2 + (d\rho)^2 = R^2 (d\tau)^2,$$

so that

$$\left(\frac{dc}{ds}\right)^2 = \frac{R^2}{\sigma^2} = \frac{\rho^2}{\sigma^2} + \rho'^2,$$

while, if ϕ denotes the inclination of the tangent CC' to the principal normal at P (being equal to the angle CSP), we have

$$\cot\phi = \sigma\rho'/\rho.$$

Next, denoting by $d\chi$ the angle between PC and QC', we have, from the spherical indicatrix,

$$d\chi = \{(d\epsilon)^2 + (d\tau)^2\}^{\frac{1}{2}},$$

and so

$$\frac{d\chi}{ds} = \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^{\frac{1}{2}},$$

a magnitude sometimes called the *screw curvature* of the curve at the point.

12. Two consecutive normal planes at P intersect in the line CS, which is called the *polar line.* The plane TPB, perpendicular to the principal normal PC, is called the *rectifying plane*; it contains the binormal PB, but two consecutive rectifying planes do not intersect in the binormal. Their intersection, a line PR through P, is called the *rectifying line*; it can be obtained as follows. The equations of QC', the radius of curvature at Q, are

$$\frac{X - ds}{-d\epsilon} = \frac{Y}{1} = \frac{Z}{d\tau};$$

and therefore the equation of the rectifying plane at Q, which is perpendicular to QC', is

$$-(X - ds)\, d\epsilon + Y + Z\, d\tau = 0.$$

Where this plane cuts $Y = 0$, the rectifying plane at P, we have

$$-(X - ds)\, d\epsilon + Z\, d\tau = 0,$$

or, ultimately,

$$-X\, d\epsilon + Z\, d\tau = 0;$$

hence the equations of the rectifying line PR are

$$Y = 0, \quad Z = X\, d\epsilon/d\tau = X\sigma/\rho.$$

Thus the inclination of PR to the binormal is $\tan^{-1}(\rho/\sigma)$.

Associated Developables.

13. The equation of any plane, organically connected with a skew curve, contains a single parameter; the envelope of the planes is therefore a developable surface. Among these, the most interesting are the envelopes of the principal planes of the curve.

On the surface, which is the envelope of the osculating planes, the original curve is the edge of regression (or cuspidal locus). To the consideration of this developable we shall return in § 16.

The envelope of the normal plane is called* the *polar developable.* Its equation is obtained by eliminating the parameter between

$$(X - x)\,x' + (Y - y)\,y' + (Z - z)\,z' = 0,$$
$$(X - x)\,x'' + (Y - y)\,y'' + (Z - z)\,z'' = x'^2 + y'^2 + z'^2 = 1.$$

When these equations are taken together, without elimination of the variable, they are the equations of the polar line; they can be changed into the form

$$\frac{X - (x + \rho^2 x'')}{y'z'' - z'y''} = \frac{Y - (y + \rho^2 y'')}{z'x'' - x'z''} = \frac{Z - (z + \rho^2 z'')}{x'y'' - y'x''},$$

verifying the property that it passes through the centre of circular curvature and is perpendicular to the osculating plane; and any point on it is a pole of the circle of curvature. Moreover, being the intersection of two consecutive planes which are tangent planes to the polar developable, the polar line is a generator of that surface.

The edge of regression of the polar developable is the locus of the centres of spherical curvature; and therefore (by § 8) its equations are

$$\frac{X - x}{y'z''' - z'y'''} = \frac{Y - y}{z'x''' - x'z'''} = \frac{Z - z}{x'y''' - y'x'''} = -\rho^2\sigma.$$

Also, the osculating plane of the edge of regression at X, Y, Z is the normal plane of the original curve at x, y, z; and the normal plane of the edge of regression at X, Y, Z is parallel to the osculating plane of the original curve at x, y, z.

14. The envelope of the rectifying plane TPB is usually called the *rectifying developable.*

The reason for using the epithet arises from an intrinsic property of the surface. The principal normal of the original curve is PN, perpendicular to the plane TPB, and therefore coinciding with the normal to the rectifying developable; hence the original curve is a geodesic (a line of shortest distance) upon the surface. When a surface is deformed without stretching

* The names of the various surfaces were assigned by Monge, *Applications de l'analyse à la géométrie* (1795), quoted on p. 1.

or tearing, there is no change in the length of any portion of any curve; when a developable surface is developed into a plane, every geodesic becomes a straight line. Thus, when the rectifying developable is developed into a plane, the original curve becomes a straight line; hence the name of the surface.

The equation of the surface can be obtained by eliminating the parameter between the equations

$$(X-x)x'' + (Y-y)y'' + (Z-z)z'' = 0,$$
$$(X-x)x''' + (Y-y)y''' + (Z-z)z''' = x'x'' + y'y'' + z'z'' = 0.$$

When these equations are taken together, without elimination of the variable, they are the equations of the rectifying line through P. They can be taken in the equivalent form

$$\frac{X-x}{y''z'''-z''y'''} = \frac{Y-y}{z''x'''-x''z'''} = \frac{Z-z}{x''y'''-y''x'''}.$$

Since

$$\begin{aligned}(y''z'''-z''y''')^2 &+ (z''x'''-x''z''')^2 + (x''y'''-y''x''')^2\\ &= (x''^2+y''^2+z''^2)(x'''^2+y'''^2+z'''^2) - (x''x'''+y''y'''+z''z''')^2\\ &= \frac{\rho^2+\sigma^2}{\rho^6\sigma^2},\end{aligned}$$

the cosine of the inclination of the rectifying line to the tangent is

$$\frac{\rho^3\sigma}{(\rho^2+\sigma^2)^{\frac{1}{2}}}\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix},$$

which is equal to $\rho(\rho^2+\sigma^2)^{-\frac{1}{2}}$, agreeing with a former result.

The edge of regression of the rectifying developable is given by the equations

$$(X-x)x'' + (Y-y)y'' + (Z-z)z'' = 0,$$
$$(X-x)x''' + (Y-y)y''' + (Z-z)z''' = x'x'' + y'y'' + z'z'' = 0,$$
$$(X-x)x'''' + (Y-y)y'''' + (Z-z)z'''' = x'x''' + y'y''' + z'z''' = -1/\rho^2;$$

and therefore the point corresponding to P is given by

$$\frac{X-x}{y''z'''-z''y'''} = \frac{Y-y}{z''x'''-x''z'''} = \frac{Z-z}{x''y'''-y''x'''} = -\frac{1}{\rho^2 E},$$

where E is the determinant

$$\begin{vmatrix} x'', & y'', & z'' \\ x''', & y''', & z''' \\ x'''', & y'''', & z'''' \end{vmatrix}.$$

The value of E can be found in the same way as the value of D in § 9. We have

$$\frac{1}{\rho^2\sigma}E = \begin{vmatrix} \Sigma x'x'', & \Sigma x''^2, & \Sigma x''x''' \\ \Sigma x'x''', & \Sigma x''x''', & \Sigma x'''^2 \\ \Sigma x'x'''', & \Sigma x''x'''', & \Sigma x'''x'''' \end{vmatrix}.$$

When the values of the constituents in this determinant are substituted, we find*

$$E = \frac{1}{\rho^5}\frac{d}{ds}\left(\frac{\rho}{\sigma}\right).$$

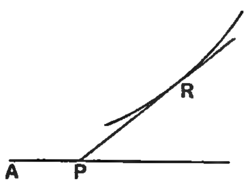
15. The rectifying developable can be used† to *determine curves the ratio of whose curvatures is a known variable‡ function of the arc.*

Take any such curve, and construct its rectifying developable. The curve is a geodesic upon this surface and cuts the rectifying line at an angle ψ, where

$$\rho = \sigma \cot \psi,$$

while the rectifying line is a generator of the developable.

Now suppose the surface developed into a plane. The assumed curve remains a geodesic and so becomes a straight line; take this straight line for the axis of x. The edge of regression becomes a curve in the plane; and the tangents to this curve are the developed tangents to the edge of regression, that is, are the developed rectifying lines. Let the initial point for measuring the arc along the assumed curve be taken as origin; let this be A, let P be the current point, and let (x, y) be the point R on the developed edge of regression where the rectifying line at P touches the curve. Then for the plane curve, we have



$$dy/dx = p = \tan \psi,$$

and for the distance s (which is AP) we have

$$s = x - \frac{y}{p}.$$

But along the curve, we are to have σ/ρ equal to some given variable function of s; let this be expressed by the relation

$$s = G\left(\frac{\sigma}{\rho}\right).$$

* The value also can be obtained from the Routh diagram (p. 11), by noting that the distance, from P along PR, of the point on the edge of regression is $-\dfrac{ds \cos i}{di}$, where i, $=\tan^{-1}(\rho/\sigma)$, is the inclination of the rectifying line to the binormal.

† Pirondini, *Crelle*, t. cix (1893), p. 238, *Ann. di Mat.*, 2ª Ser. t. xix (1892), p. 213.

‡ The case when the ratio is constant is treated in a different manner: see § 20, post.

Then the plane curve into which the edge of regression has been developed satisfies the equation

$$x - \frac{y}{p} = G(p).$$

The primitive of this Clairaut equation is

$$y = cx - cG(c),$$

giving the aggregate of tangents; and the singular solution, being their envelope, is given by the equations

$$\left.\begin{aligned} y &= cx - cG(c) \\ 0 &= x - G(c) - cG'(c) \end{aligned}\right\},$$

which thus is the equation of the developed edge of regression. Hence we have the result:—

To construct skew curves satisfying the relation $s = G(\sigma/\rho)$, form the plane curve

$$x = G(c) + cG'(c), \quad y = c^2 G'(c);$$

bend the plane about the tangents to this curve, according to any assigned law, so as to form a developable surface; the original axis of x in the plane becomes, on the developable surface, a skew curve having the required property.

16. The *osculating developable* is the envelope of the osculating plane of the curve. Its generators are the tangents to the curve; and its edge of regression is the curve itself.

This property suggests another method of analytical definition of a curve in which the initial element is not a point of the curve as in the preceding investigations, but is the variable osculating plane. This method was adopted by Serret*, who has deduced by its means a number of results. The equation of the osculating plane is taken in the form

$$z = px + qy - u,$$

where p and q are functions of the single parameter u. The envelope is, of course, a developable surface; its generators are given by the equations

$$\left.\begin{aligned} z &= px + qy - u \\ 0 &= p'x + q'y - 1 \end{aligned}\right\},$$

which thus are the equations of the tangent to the curve; and its edge of regression is given by the equations

$$\left.\begin{aligned} z &= px + qy - u \\ 0 &= p'x + q'y - 1 \\ 0 &= p''x + q''y \end{aligned}\right\}.$$

* *Liouville's Journal*, t. xiii (1848), p. 353.

Thus the current point on the curve is given by the equations

$$\frac{x}{q''} = \frac{y}{-p''} = \frac{z+u}{pq''-qp''} = \frac{1}{p'q''-q'p''}.$$

Let

$$\Delta = (q''p''' - p''q''')(p'q'' - q'p'')^{-2}, \quad T = \{p'^2 + q'^2 + (pq' - qp')^2\}^{\frac{1}{2}};$$

then

$$\frac{dx}{du} = q'\Delta, \quad \frac{dy}{du} = -p'\Delta, \quad \frac{dz}{du} = (pq' - qp')\Delta, \quad \frac{ds}{du} = T\Delta.$$

Hence the direction-cosines of the tangent are given by the equations

$$\cos\alpha = q'/T, \quad \cos\beta = -p'/T, \quad \cos\gamma = (pq' - qp')/T,$$

and the direction-cosines of the binormal by the equations

$$\frac{\cos\lambda}{-p} = \frac{\cos\mu}{-q} = \frac{\cos\nu}{1} = (1 + p^2 + q^2)^{-\frac{1}{2}}.$$

We at once find

$$\left(\frac{d\cos\alpha}{du}\right)^2 + \left(\frac{d\cos\beta}{du}\right)^2 + \left(\frac{d\cos\gamma}{du}\right)^2 = (1 + p^2 + q^2)(q'p'' - p'q'')^2 T^{-4},$$

$$\left(\frac{d\cos\lambda}{du}\right)^2 + \left(\frac{d\cos\mu}{du}\right)^2 + \left(\frac{d\cos\nu}{du}\right)^2 = (1 + p^2 + q^2)^{-2} T^2;$$

and therefore (§ 7) the radius of circular curvature is given by

$$\rho = T^3\Delta(1 + p^2 + q^2)^{-\frac{1}{2}}(p'q'' - q'p'')^{-1},$$

while the radius of torsion is given by

$$\sigma = (1 + p^2 + q^2)\Delta.$$

Serret-Frenet formulæ.

17. The preceding results are conclusions derived from the analytical definition of a curve by means of the coordinates of a current point. Another method is founded upon certain differential relations belonging to all curves; and these relations are made precise, generically for families of curves, individually for particular curves, by the assignment of some intrinsic property or properties.

These general relations exist between the derivatives of the direction-cosines of the edges of the principal trihedron at any point: sometimes they are called* after Serret, sometimes after Frenet. They can be obtained as follows.

The direction-cosines of the principal lines at any point of the curve possess many notations; we shall take

$\cos\alpha$, $\cos\beta$, $\cos\gamma$, (and a, a', a'') as the direction-cosines of the tangent,
$\cos\xi$, $\cos\eta$, $\cos\zeta$, (and b, b', b'') „ „ „ principal normal,
$\cos\lambda$, $\cos\mu$, $\cos\nu$, (and c, c', c'') „ „ „ binormal,

* They are given in a memoir by Serret, *Liouville's Journal*, t. xvi (1851), p. 193: also in a memoir (which had been a thesis) by Frenet, *ib.*, t. xvii (1852), p. 437.

with the convention already adopted (§ 5), whereby these lines could be displaced into coincidence with a set of coordinate axes without changing the sense of any line*. Then

$$\begin{vmatrix} \cos\alpha, & \cos\beta, & \cos\gamma \\ \cos\xi, & \cos\eta, & \cos\zeta \\ \cos\lambda, & \cos\mu, & \cos\nu \end{vmatrix} = 1;$$

and each constituent of the determinant is equal to its minor. Also

$\cos\alpha$, $\cos\xi$, $\cos\lambda$ are the direction-cosines of the axis of x,

$\cos\beta$, $\cos\eta$, $\cos\mu$ „ „ „ „ y,

$\cos\gamma$, $\cos\zeta$, $\cos\nu$ „ „ „ „ z,

when the principal lines of the curve are taken as the axes of reference. Now

$$\cos\alpha = x', \quad \cos\xi = \rho x'';$$

hence

$$\frac{d\cos\alpha}{ds} = \frac{\cos\xi}{\rho},$$

together with two similar relations for the derivatives of the other two direction-cosines of the tangent. Again, we have

$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu = 0,$$

so that, because

$$\cos\xi\cos\lambda + \cos\eta\cos\mu + \cos\zeta\cos\nu = 0,$$

it follows that

$$\cos\alpha\frac{d\cos\lambda}{ds} + \cos\beta\frac{d\cos\mu}{ds} + \cos\gamma\frac{d\cos\nu}{ds} = 0.$$

Also

$$\cos\lambda\frac{d\cos\lambda}{ds} + \cos\mu\frac{d\cos\mu}{ds} + \cos\nu\frac{d\cos\nu}{ds} = 0;$$

hence

$$\frac{d\cos\lambda}{ds}\frac{1}{\cos\mu\cos\gamma - \cos\nu\cos\beta} = \ldots = \ldots = \theta,$$

that is,

$$\frac{1}{\cos\xi}\frac{d\cos\lambda}{ds} = \frac{1}{\cos\eta}\frac{d\cos\mu}{ds} = \frac{1}{\cos\zeta}\frac{d\cos\nu}{ds} = \theta.$$

But

$$\cos\lambda = \rho(y'z'' - z'y'');$$

hence

$$\rho(y'z''' - z'y''') + \rho'(y'z'' - z'y'') = \theta\rho x'',$$
$$\rho(z'x''' - x'z''') + \rho'(z'x'' - x'z'') = \theta\rho y'',$$
$$\rho(x'y''' - y'x''') + \rho'(x'y'' - y'x'') = \theta\rho z''.$$

Multiplying by x'', y'', z'' respectively, and adding, we find

$$-\rho\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} = \frac{\theta}{\rho},$$

* The alternative convention leads to a change in the sign of σ throughout.

so that

$$\theta = -\frac{1}{\sigma};$$

hence

$$\frac{d\cos\lambda}{ds} = -\frac{\cos\xi}{\sigma},$$

together with two similar relations for the derivatives of the other two direction-cosines of the binormal. Further,

$$\cos\xi = \cos\mu\cos\gamma - \cos\nu\cos\beta,$$

and therefore

$$\begin{aligned}\frac{d\cos\xi}{ds} &= \cos\mu\frac{d\cos\gamma}{ds} - \cos\nu\frac{d\cos\beta}{ds} + \cos\gamma\frac{d\cos\mu}{ds} - \cos\beta\frac{d\cos\nu}{ds}\\ &= \frac{1}{\rho}(\cos\mu\cos\zeta - \cos\nu\cos\eta) - \frac{1}{\sigma}(\cos\gamma\cos\eta - \cos\beta\cos\zeta)\\ &= -\frac{\cos\alpha}{\rho} + \frac{\cos\lambda}{\sigma},\end{aligned}$$

together with two similar relations for the derivatives of the other two direction-cosines of the principal normal.

These are the *Serret-Frenet formulæ* satisfied by the derivatives of the direction-cosines of the principal lines. They are taken by Darboux in the form*

$$\frac{da}{ds} = \frac{b}{\rho}, \quad \frac{db}{ds} = \frac{c}{\sigma} - \frac{a}{\rho}, \quad \frac{dc}{ds} = -\frac{b}{\sigma}.$$

Particular sets of simultaneous solutions of these equations are

$$a,\ b,\ c = \cos\alpha,\ \cos\xi,\ \cos\lambda,$$
$$a,\ b,\ c = \cos\beta,\ \cos\eta,\ \cos\mu,$$
$$a,\ b,\ c = \cos\gamma,\ \cos\zeta,\ \cos\nu.$$

The complete resolution of the equations can be made to depend upon that of a single equation. Let†

$$l = \frac{a+ib}{1-c},$$

where i denotes $\sqrt{-1}$; then

$$\begin{aligned}\frac{dl}{ds} &= \frac{1}{1-c}\left(\frac{da}{ds} + i\frac{db}{ds}\right) + \frac{a+ib}{(1-c)^2}\frac{dc}{ds}\\ &= \frac{i}{2\sigma}l^2 - \frac{i}{\rho}l - \frac{i}{2\sigma},\end{aligned}$$

* As compared with Darboux's earlier form, there is a change of the sign of σ (or of c), due to the convention concerning the axes. But Darboux's later preference, *Théorie générale*, t. iv, p. 428, is for the form here adopted.

† This combination of direction-cosines is of frequent use in differential geometry; its effective introduction appears to be due to Weierstrass, though it occurs earlier in the work of Lagrange and of Gauss on the conformal representation of a spherical surface upon a plane.

which is an equation for l of the Riccati form*, when ρ and σ are regarded as known functions of s. When l is known, however it has been obtained, the complex quantity conjugate to l is known; hence, writing

$$l = \frac{a+ib}{1-c}, \quad -\frac{1}{m} = \frac{a-ib}{1-c},$$

we have

$$a = \frac{1-lm}{l-m}, \quad b = i\frac{1+lm}{l-m}, \quad c = \frac{l+m}{l-m}.$$

18. These Frenet-Serret formulæ can be obtained by another process†, which is based directly upon their significance in relation to the curvature and the tortuosity.

We take the direction-cosines of the principal lines at a point according to the tableau

$$\begin{pmatrix} a, & a', & a'' \\ b, & b', & b'' \\ c, & c', & c'' \end{pmatrix}.$$

The direction-cosines of the principal lines at a consecutive point, referred to the principal lines at the original point, are given by

$$\begin{array}{llll} 1\ , & d\epsilon\ , & 0\ , & \text{for the tangent,} \\ -d\epsilon, & 1\ , & d\tau, & \text{,, ,, principal normal,} \\ 0\ , & -d\tau, & 1\ , & \text{,, ,, binormal.} \end{array}$$

Hence

$$a + da = a + bd\epsilon,$$

$$b + db = -ad\epsilon + b + cd\tau,$$

$$c + dc = -bd\tau + c;$$

and therefore

$$\frac{da}{ds} = \frac{b}{\rho}, \quad \frac{db}{ds} = \frac{c}{\sigma} - \frac{a}{\rho}, \quad \frac{dc}{ds} = -\frac{b}{\sigma},$$

with similar relations between a', b', c'; and a'', b'', c''.

These relations are of fundamental importance in the theory of skew curves. The present process of establishing them is independent of the earlier analysis; and so they can be used to obtain, easily, many of the results already given. Thus

$$a = \frac{dx}{ds}, \quad a' = \frac{dy}{ds}, \quad a'' = \frac{dz}{ds};$$

and therefore

$$b = \rho\frac{d^2x}{ds^2} = \rho x'', \quad b' = \rho\frac{d^2y}{ds^2} = \rho y'', \quad b'' = \rho\frac{d^2z}{ds^2} = \rho z'',$$

* For some of the properties of this equation, see my *Treatise on Differential Equations*, (3rd ed.), § 110.

† I am indebted to Mr R. A. Herman for this process.

so that

$$\rho^2(x''^2+y''^2+z''^2)=1.$$

Similarly

$$x'''=\frac{d^2a}{ds^2}=\frac{d}{ds}\left(\frac{b}{\rho}\right)=\frac{1}{\rho}\left(\frac{c}{\sigma}-\frac{a}{\rho}\right)-\frac{b}{\rho^2}\rho',$$

and so for y''' and z'''; thus

$$\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} = \begin{vmatrix} a, & a', & a'' \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix} \frac{1}{\rho^2\sigma}=\frac{1}{\rho^2\sigma}.$$

Manifestly $\dfrac{d^nx}{ds^n}$ can be expressed in a form $au_n+bv_n+cw_n$, where u_n, v_n, w_n are determined by the equations

$$u_{n+1}=\frac{du_n}{ds}-\frac{v_n}{\rho},\quad v_{n+1}=\frac{dv_n}{ds}-\frac{w_n}{\sigma}+\frac{u_n}{\rho},\quad w_{n+1}=\frac{dw_n}{ds}+\frac{v_n}{\sigma};$$

and so for the derivatives of y and of z.

19. We proceed to make some applications of the Serret-Frenet formulæ.

A curve is uniquely defined, except as to position and orientation in space, when its two curvatures are given as functions of its arc.

Let there be two such curves, different if possible; denote the radii for one of the curves by ρ and σ, and for the other curve by ρ' and σ', so that we have

$$\rho'=\rho,\quad \sigma'=\sigma.$$

At the current point on the one curve determined by the arc s, we have

$$\frac{d\cos\alpha}{ds}=\frac{\cos\xi}{\rho},\quad \frac{d\cos\xi}{ds}=-\frac{\cos\alpha}{\rho}+\frac{\cos\lambda}{\sigma},\quad \frac{d\cos\lambda}{ds}=-\frac{\cos\xi}{\sigma};$$

and at the current point on the other curve determined by the same arc s, we have

$$\frac{d\cos\alpha'}{ds}=\frac{\cos\xi'}{\rho},\quad \frac{d\cos\xi'}{ds}=-\frac{\cos\alpha'}{\rho}+\frac{\cos\lambda'}{\sigma},\quad \frac{d\cos\lambda'}{ds}=-\frac{\cos\xi'}{\sigma}.$$

Hence

$$\frac{d}{ds}(\cos\alpha\cos\alpha'+\cos\xi\cos\xi'+\cos\lambda\cos\lambda')=0,$$

and therefore

$$\cos\alpha\cos\alpha'+\cos\xi\cos\xi'+\cos\lambda\cos\lambda'=\text{constant}.$$

Now suppose the two curves so placed in space that the two respective initial points from which the arcs are measured coincide; and suppose the two curves to be so orientated at that point that their principal lines coincide

there in direction. Then at the point we have $\alpha_0 = \alpha_0'$, $\xi_0 = \xi_0'$, $\lambda_0 = \lambda_0'$, and so the constant is equal to unity at the point; that is,

$$\cos\alpha\cos\alpha' + \cos\xi\cos\xi' + \cos\lambda\cos\lambda' = 1.$$

Also

$$\cos^2\alpha + \cos^2\xi + \cos^2\lambda = 1,$$

$$\cos^2\alpha' + \cos^2\xi' + \cos^2\lambda' = 1;$$

hence

$$\cos\alpha = \cos\alpha', \quad \cos\xi = \cos\xi', \quad \cos\lambda = \cos\lambda',$$

the first of which is

$$\frac{dx}{ds} = \frac{dx'}{ds}.$$

Similarly, we have

$$\frac{dy}{ds} = \frac{dy'}{ds}, \quad \frac{dz}{ds} = \frac{dz'}{ds};$$

and therefore

$$x - x' = \text{constant}, \quad y - y' = \text{constant}, \quad z - z' = \text{constant}.$$

The initial point has the same coordinates for the two curves, so that each of these constants is zero; hence

$$x - x' = 0, \quad y - y' = 0, \quad z - z' = 0,$$

and therefore the two curves everywhere coincide. But the only changes made in the second curve were in its position and its orientation in space; thus the two curves were originally the same, save for position and orientation in space. Hence the proposition.

We can at once infer one result. It is known that both the curvatures of a helix on a circular cylinder are constant; hence every curve, which has both its curvatures constant, is a helix on a circular cylinder.

More generally, it follows that all magnitudes, intrinsically belonging to the curve, can be expressed in terms of ρ and σ and of their derivatives.

The main inference from the general proposition is that, for the intrinsic and unique specification of a curve, we need to have the values of ρ and σ given as functions of s; any complete equivalent of such data would also be needed, if they were not provided; and the data are sufficient. If then only a single equation is given, of any form, between ρ, σ, s, we must expect some arbitrary element to exist in the equations of the most general curve which satisfies the condition implied by the single equation. In other words, we shall have a family of curves: and a curve will be selected from the family by the assignment of some special form to the arbitrary element.

Thus it has been seen that curves can be constructed satisfying an equation $s = G(\sigma/\rho)$. For the purpose, it is sufficient to have a family of developable surfaces bound by the property that, when the surfaces are developed, the edges of regression become one and the same curve in the

plane; and the curves, satisfying the equation, are given by taking one curve upon each member of the family of surfaces.

We shall now take a number of other examples* of this general result.

Curves having their Curvatures in a Constant Ratio†.

20. Let $\rho/\sigma = k = \tan A$, suppose, where A is a constant. The Serret-Frenet equations now are

$$\frac{da}{ds} = \frac{b}{\rho}; \quad \frac{db}{ds} = -\frac{a}{\rho} + \frac{c}{\sigma} = \frac{-(a - ck)}{\rho}, \quad \frac{dc}{ds} = -\frac{b}{\sigma} = -\frac{bk}{\rho},$$

which are linear in a, b, c; hence

$$\frac{d}{ds}(a \cos A - c \sin A + ib) = -\frac{i}{\rho \cos A}(a \cos A - c \sin A + ib).$$

Let

$$u = \sec A \int \frac{ds}{\rho},$$

so that u is a real quantity, being a function of s; then

$$a \cos A - c \sin A + ib = Re^{-iu},$$

where R is an arbitrary constant. We must suppose that R is complex; let

$$R = Pe^{-i\delta},$$

where P and δ are real; then

$$a \cos A - c \sin A + ib = Pe^{-i(u+\delta)}.$$

Consequently

$$a \cos A - c \sin A - ib = Pe^{i(u+\delta)};$$

and

$$a^2 + b^2 + c^2 = 1.$$

Solving these three equations, we find

$$\left.\begin{aligned} a \cos A - c \sin A &= P \cos(u + \delta) = \sin p \cos(u + \delta) \\ -b &= P \sin(u + \delta) = \sin p \sin(u + \delta) \\ a \sin A + c \cos A &= (1 - P^2)^{\frac{1}{2}} \quad = \cos p \end{aligned}\right\},$$

giving the values of a, b, c, the cosines of the inclinations of the three principal lines to the axis of x.

Similarly, let p' and δ' be the constants of integration for a', b', c', and p'' and δ'' be the constants of integration for a'', b'', c'', the respective cosines of the inclinations of the three principal lines to the axis of y and the axis of z.

* The reader would do well to consider Darboux's treatment of these examples, and of others, in his *Théorie générale*, t. i, §§ 6—12, 32—39.

† This is the one case not covered by the example in § 15. It appears to have been discussed first by Puiseux, *Liouville's Journal*, t. vii (1842), pp. 65—69. The analysis, which follows, is more detailed than the treatment in Darboux and in Bianchi; it is given so as to secure the most explicit form of the analytical definition of the curves.

Then

$$\left.\begin{aligned} a'\cos A - c'\sin A &= \sin p'\cos(u+\delta') \\ -b' &= \sin p'\sin(u+\delta') \\ a'\sin A + c'\cos A &= \cos p' \end{aligned}\right\}, \quad \left.\begin{aligned} a''\cos A - c''\sin A &= \sin p''\cos(u+\delta'') \\ -b'' &= \sin p''\sin(u+\delta'') \\ a''\sin A + c''\cos A &= \cos p'' \end{aligned}\right\}.$$

The primitive of all the three sets of equations, in this form, apparently involves six constants; but they reduce to three. The three lines having a, b, c; a', b', c'; a'', b'', c''; for their direction-cosines are perpendicular to one another; the necessary conditions are satisfied by the relations

$$\frac{\cot p \cot p'}{\cos(\delta-\delta')} = \frac{\cot p' \cot p''}{\cos(\delta'-\delta'')} = \frac{\cot p'' \cot p}{\cos(\delta''-\delta)} = -1.$$

To obtain the analytical definition of the curve, we note that

$$a = \cos A \sin p \cos(u+\delta) + \cos p \sin A,$$

so that

$$x - x_0 = s \cos p \sin A + \cos A \sin p \int \cos(u+\delta)\, ds,$$

and similarly

$$\left.\begin{aligned} y - y_0 &= s \cos p' \sin A + \cos A \sin p' \int \cos(u+\delta')\, ds \\ z - z_0 &= s \cos p'' \sin A + \cos A \sin p'' \int \cos(u+\delta'')\, ds \end{aligned}\right\}.$$

where

$$u = \sec A \int \frac{ds}{\rho},$$

and x_0, y_0, z_0 are arbitrary constants. The new arbitrary constants x_0, y_0, z_0 affect the position of the curve in space: the surviving constants δ, δ', δ'' affect its orientation.

There is nothing in the problem to limit the value of ρ. Hence it may be taken to be an arbitrary function of s; and so, for the range of variation of this arbitrary function, we have a family of curves intrinsically distinct from one another. But all the curves of the family have two properties in common. We have

$$a\sin A + c\cos A = \cos p,\quad a'\sin A + c'\cos A = \cos p',\quad a''\sin A + c''\cos A = \cos p'';$$

hence

$$\sin A = a\cos p + a'\cos p' + a''\cos p'',$$
$$\cos A = c\cos p + c'\cos p' + c''\cos p''.$$

The first of these two relations shews that the tangent to the curve is at a constant inclination $\frac{1}{2}\pi - A$ to the line whose direction-cosines are $\cos p$, $\cos p'$, $\cos p''$ (for $\Sigma \cos^2 p = 1$), that is, to a fixed line; and the second shews that the binormal is at a constant angle A to the same line. Moreover

$$0 = b\cos p + b'\cos p' + b''\cos p'',$$

that is, the principal normal is perpendicular to the same line. It therefore

follows that this line is the rectifying line of the curve: that is, *along any curve the rectifying line has a constant direction, and the rectifying developable is a cylinder. The generators are the rectifying lines: and the curve is a geodesic on the surface.*

A curve on a surface which makes a constant angle with a fixed direction is called a *helix*. It therefore follows from the preceding investigation that a curve, having the ratio of its curvatures constant, is a helix. The establishment of the converse proposition—that a helix has its curvatures in a constant ratio—is left as an exercise.

Curves having assigned Torsion, variable or constant.

21. Let the torsion be given as a function of the arc. With a, a', a''; b, b', b''; c, c', c''; as the direction-cosines of the principal lines, we have

$$a = b'c'' - b''c', \quad \frac{dc'}{ds} = -\frac{b'}{\sigma}, \quad \frac{dc''}{ds} = -\frac{b''}{\sigma}.$$

Therefore

$$a = -\sigma \left(c'' \frac{dc'}{ds} - c' \frac{dc''}{ds}\right),$$

with two similar equations; so that

$$\left.\begin{aligned} dx &= a\,ds = -\sigma(c''dc' - c'\,dc'') \\ dy &= a'ds = -\sigma(c\ dc'' - c''dc\ ) \\ dz &= a''ds = -\sigma(c'\,dc - c\ dc') \end{aligned}\right\}.$$

Also we have

$$c^2 + c'^2 + c''^2 = 1;$$

and from the value of the torsion in general, we have

$$\left(\frac{dc}{ds}\right)^2 + \left(\frac{dc'}{ds}\right)^2 + \left(\frac{dc''}{ds}\right)^2 = \frac{1}{\sigma^2}.$$

Now, when the torsion is given, σ is a known function of s; and therefore the quantities c, c', c'' are three functions of s, subject to these two equations, that is, one arbitrary element survives among the three functions.

The first of the equations is satisfied by taking

$$c = \sin\theta\cos\phi, \quad c' = \sin\theta\sin\phi, \quad c'' = \cos\theta,$$

for any values of θ and ϕ: and then the second of the equations is satisfied, provided

$$(d\theta)^2 + \sin^2\theta\,(d\phi)^2 = \left(\frac{ds}{\sigma}\right)^2.$$

With these values, we have

$$\left.\begin{aligned} dx &= -\sigma(\cos\theta\cos\phi\sin\theta\,d\phi + \sin\phi\,d\theta) \\ dy &= -\sigma(\cos\theta\sin\phi\sin\theta\,d\phi - \cos\phi\,d\theta) \\ dz &= \quad \sigma\sin^2\theta\,d\phi \end{aligned}\right\}.$$

All the magnitudes involved are functions of one parameter, which can be chosen at will; we choose z to be the parameter. As already indicated, an arbitrary element will remain in the equations; accordingly, we assume

$$\tan\phi = f(z),$$

where f is an arbitrary function of z. Then

$$d\phi = \frac{f'}{1+f^2}\,dz,$$

$$\sin^2\theta = \frac{1+f^2}{\sigma f'}, \quad \cos^2\theta = \frac{\sigma f' - 1 - f^2}{\sigma f'};$$

and therefore

$$d\theta = \left(\frac{ff'}{1+f^2} - \frac{1}{2}\frac{f''}{f'} - \frac{1}{2}\frac{d\sigma}{ds}\frac{ds}{dz}\right)\left(\frac{1+f^2}{\sigma f' - 1 - f^2}\right)^{\frac{1}{2}} dz.$$

Consequently

$$\frac{f'}{\sigma(1+f^2)} + \frac{1+f^2}{\sigma f' - 1 - f^2}\left(\frac{ff'}{1+f^2} - \frac{1}{2}\frac{f''}{f'} - \frac{1}{2}\frac{d\sigma}{ds}\frac{ds}{dz}\right)^2 = \left(\frac{1}{\sigma}\frac{ds}{dz}\right)^2,$$

a relation between z and s, involving the arbitrary function f; in particular, it expresses ds/dz in terms of z and s. Also

$$\left.\begin{aligned} dx + i\,dy &= -\,e^{i\phi}\ \cot\theta\,(dz - i\sigma\tan\theta\,d\theta) \\ dx - i\,dy &= -\,e^{-i\phi}\cot\theta\,(dz + i\sigma\tan\theta\,d\theta) \end{aligned}\right\},$$

which are the (integrable) analytical equations of the curve when substitution is made for ϕ, $\tan\theta$, $d\theta$; and they involve an arbitrary function f, while z is the parameter of the equations.

As is to be expected, the simplest case arises when the *torsion is constant.* It is not necessary, for the construction of the analytical equations of the curve, that the equation giving ds/dz should be retained. We have

$$\left.\begin{aligned} dx + i\,dy &= -\,e^{i\phi}\ \left(\frac{\sigma f' - 1 - f^2}{1+f^2}\right)^{\frac{1}{2}}\left\{1 + \tfrac{1}{2}i\sigma\frac{f''(1+f^2) - 2ff'^2}{(\sigma f' - 1 - f^2)f'}\right\}dz \\ dx - i\,dy &= -\,e^{-i\phi}\left(\frac{\sigma f' - 1 - f^2}{1+f^2}\right)^{\frac{1}{2}}\left\{1 - \tfrac{1}{2}i\sigma\frac{f''(1+f^2) - 2ff'^2}{(\sigma f' - 1 - f^2)f'}\right\}dz \end{aligned}\right\},$$

as the equations of the curve; or, substituting for ϕ, we find

$$\left.\begin{aligned} dx &= -\,\frac{\sigma f'^2 - f' - \tfrac{1}{2}\sigma ff''}{f'(\sigma f' - 1 - f^2)^{\frac{1}{2}}}\,dz \\ dy &= -\,\frac{\tfrac{1}{2}\sigma f'' - ff'}{f'(\sigma f' - 1 - f^2)^{\frac{1}{2}}}\,dz \end{aligned}\right\}$$

as the analytical equations of curves of constant torsion $1/\sigma$, where f, $= f(z)$, is arbitrary in the equations*.

* This is Serret's form of the equations of curves of constant torsion: see Liouville's edition of Monge (quoted p. 1), p. 566.

Curves of constant torsion have formed a subject of many investigations in comparatively recent years, especially those which are algebraic curves. Thus taking the formulæ

$$\left.\begin{aligned} dx &= -\sigma(c''dc' - c'\,dc'') \\ dy &= -\sigma(c\,dc'' - c''dc\,) \\ dz &= -\sigma(c'\,dc - c\,dc') \end{aligned}\right\},$$

where σ now is supposed constant, Fabry assumes that c, c', c'' are integral functions of sines and cosines of integer multiples of a parameter t, such that each of the quantities on the right-hand side is devoid of a term not involving sines or cosines when expressed as a sum of terms each involving only one sine or cosine. Again, Fouché takes the Weierstrass expressions (§ 17)

$$c = \frac{1-\alpha\beta}{\alpha-\beta}, \quad c' = i\frac{1+\alpha\beta}{\alpha-\beta}, \quad c'' = \frac{\alpha+\beta}{\alpha-\beta},$$

with α and β as algebraic functions of a parameter: and imposing the conditions that dx, dy, dz must be the exact differentials of some algebraical functions, he obtains a critical equation that admits many evident solutions*. And from the relation

$$R^2 = \rho^2 + \sigma^2\left(\frac{d\rho}{ds}\right)^2,$$

it can be proved that no algebraic curve of constant non-zero torsion exists on a sphere.

Curves having assigned Circular Curvature.

22. Let the radius of circular curvature be given as a function of the arc. Then the quantities a, a', a'' satisfy the relations

$$a^2 + a'^2 + a''^2 = 1,$$

$$\left(\frac{da}{ds}\right)^2 + \left(\frac{da'}{ds}\right)^2 + \left(\frac{da''}{ds}\right)^2 = \frac{1}{\rho^2},$$

where ρ is a known function of s; and therefore a, a', a'' are three functions of s, subject to these two equations, that is, one arbitrary element survives among the three functions. Also

$$dx = a\,ds = a\rho\,dS, \quad dy = a'\,ds = a'\rho\,dS, \quad dz = a''\,ds = a''\rho\,dS,$$

where S is a new parameter related to s by the equation

$$\frac{ds}{\rho} = dS;$$

* References to the memoirs by Fabry and by Fouché as well as to other papers on the subject are given by Darboux, *Théorie générale*, t. iv, p. 429, in the course of a Note on the torsion of skew curves, which is specially commended to the reader's attention.

and then

$$x - x_0 = \int a\rho dS, \quad y - y_0 = \int a'\rho dS, \quad z - z_0 = \int a''\rho dS,$$

where a, a', a'' are functions of S such that

$$a^2 + a'^2 + a''^2 = 1,$$

$$\left(\frac{da}{dS}\right)^2 + \left(\frac{da'}{dS}\right)^2 + \left(\frac{da''}{dS}\right)^2 = 1.$$

The case, when ρ is constant and equal (say) to k, is not analytically simpler than the case when ρ is variable; the parameter S is merely s/k.

EXAMPLES.

1. When the circular curvature of a curve is zero at all points, the curve is a straight line; and when the curvature of torsion is zero at all points, the curve is plane.

2. Shew that the determinant

$$\begin{vmatrix} x'', & y'', & z'' \\ x''', & y''', & z''' \\ x'''', & y'''', & z'''' \end{vmatrix}$$

vanishes for a helix; and conversely.

3. Prove that the radius of circular curvature of the locus of the centre of spherical curvature of a curve is $R\frac{dR}{d\rho}$; and indicate analogies between the formulæ of plane curves, connecting the magnitudes usually denoted by r, p, ϕ, ψ, ρ, with the formulæ of skew curves connecting the magnitudes denoted in the text by R, ρ, ϕ, τ, ρ_1.

Prove that, for any curve drawn upon a sphere, the relation

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left(\sigma\frac{d\rho}{ds}\right) = 0, \quad \text{or} \quad \rho + \frac{d^2\rho}{d\tau^2} = 0,$$

is satisfied.

4. A helix is drawn on a circular cylinder of radius a and cuts the generators at a constant angle a; shew that both the circular curvature and the torsion are constant, that the rectifying line at any point is the generator of the cylinder, and that the locus of the centre of circular curvature is another helix upon a coaxial cylinder.

Hence shew how to construct the circular cylinder which contains the helix having at a point the closest (three-point) contact with a curve.

5. Prove that, if a curve be drawn so that its tangent has a constant inclination to a fixed direction in space, the ratio of its curvatures is constant.

6. Shew that, for a spherical helix,

$$\rho = a \sec a\, (\cos^2 a - \cos^2\theta)^{\frac{1}{2}}, \quad \sigma = a \operatorname{cosec} a\, (\cos^2 a - \cos^2\theta)^{\frac{1}{2}},$$

and that the cross-section of its rectifying developable is an epicycloid.

7. A loxodrome is drawn on a sphere of radius a, cutting a set of meridians at a constant angle a. Shew that, at an angular distance θ from the pole, its radius of circular curvature is $a(1 - \cos^2\theta\cos^2 a)^{-\frac{1}{2}}\sin\theta$, and its radius of torsion is

$$a(1 - \cos^2\theta\cos^2 a)\sec a \operatorname{cosec} a.$$

8. Prove that the radius of circular curvature of the edge of regression of the rectifying developable at the point corresponding to P is $\sec i \dfrac{d}{di}\left(\cos^2 i \dfrac{ds}{di}\right)$, where i is the inclination of the rectifying line to the binormal at P; and that the radius of torsion is

$$\rho \frac{d}{ds}\left(\cos^2 i \frac{ds}{di}\right).$$

9. Prove that the circular curvature of the locus of the centre of circular curvature of a skew curve is

$$\left[\left\{\frac{\rho^2\sigma}{R^3}\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)-\frac{1}{R}\right\}^2+\frac{\rho'^2\sigma^4}{\rho^2R^4}\right]^{\frac{1}{2}}.$$

10. Obtain the direction-cosines of the rectifying line in the form

$$\{\rho x' + \rho\sigma(y'z'' - z'y'')\}(\rho^2+\sigma^2)^{-\frac{1}{2}},$$

with two similar expressions.

11. Denoting $\dfrac{d^lx}{ds^l}$ by x_l, and similarly for derivatives of y and z, prove that

$$x_lx_m + y_ly_m + z_lz_m$$

and

$$\begin{vmatrix} x_l, & x_m, & x_n \\ y_l, & y_m, & y_n \\ z_l, & z_m, & z_n \end{vmatrix}$$

are independent of the axes of reference.

12. Shew that the torsion of the curve

$$\left.\begin{aligned} x &= \{bc(b-c)\}^{\frac{1}{2}}\int\{(b-t)(c-t)\}^{-\frac{1}{2}}dt \\ y &= \{ca(c-a)\}^{\frac{1}{2}}\int\{(c-t)(a-t)\}^{-\frac{1}{2}}dt \\ z &= \{ab(a-b)\}^{\frac{1}{2}}\int\{(a-t)(b-t)\}^{-\frac{1}{2}}dt \end{aligned}\right\}$$

is constant. Indicate the character of the spherical indicatrix of its tangents and of its binormals.

13. Prove that the radii of curvature and torsion of an involute of a curve are

$$\frac{\sigma}{(\rho^2+\sigma^2)^{\frac{1}{2}}}s, \quad \frac{\rho^2+\sigma^2}{\rho(\sigma\rho'-\rho\sigma')}s.$$

14. In the Serret-Frenet formulæ, let

$$\frac{a+ib}{1-c} = e^{(v-\epsilon)i},$$

where ϵ denotes the integral $\int \rho^{-1}ds$; shew that v satisfies the equation

$$\frac{dv}{ds} = \frac{i}{\sigma}\sin(v-\epsilon).$$

15. In a particular curve, the direction-cosines of the binormal are given by

$$c = \lambda\cos\theta + \mu\cos 3\theta, \quad c' = \lambda'\sin\theta + \mu\sin 3\theta, \quad c'' = \kappa + 2d\cos 2\theta,$$

where θ is a current parameter, and λ, λ', μ, κ, d are constants such that

$$\lambda = -\frac{d^2}{\mu} + \frac{1}{\mu}(d^4 - 3\mu^4)^{\frac{1}{2}}, \quad \lambda' = \frac{d^2}{\mu} + \frac{1}{\mu}(d^4 - 3\mu^4)^{\frac{1}{2}}, \quad \kappa = \frac{d^2-\mu^2}{2d\mu^2}(d^4 - 3\mu^4)^{\frac{1}{2}},$$

$$d^8 + 6d^6\mu^2 + 6d^4\mu^4 - 2d^2\mu^6 - 3\mu^8 = 4d^2\mu^4.$$

Prove that the curve, if of constant torsion, is algebraic.

16. Curves (often called Bertrand curves*) are such that the relation

$$\frac{m}{\rho}+\frac{n}{\sigma}=1$$

is satisfied, where m and n are constants: shew that the curve is analytically defined by the relations

$$\left.\begin{aligned} dx &= mA\,dS-n\,(A''dA'-A'dA'') \\ dy &= mA'dS-n\,(A\,dA''-A''dA\ ) \\ dz &= mA''dS-n\,(A'dA\ -A\,dA'\ ) \end{aligned}\right\},$$

where A, A', A'' are three functions of a parameter subject to the conditions

$$A^2+A'^2+A''^2=1,$$

$$(dS)^2=(dA)^2+(dA')^2+(dA'')^2.$$

17. Prove that, if two curves have the same principal normals, their osculating planes cut at a constant angle a; and shew that they are Bertrand curves.

Also prove that, if c denote the common distance of corresponding points,

$$\sigma\sigma'=c^2\operatorname{cosec}^2 a.$$

18. Shew that a curve, intrinsically defined by the equations

$$\rho=ks, \quad \sigma=ls,$$

where k and l are constants, lies on a circular cone, and is a helix on that cone.

19. Prove that, for any skew curve,

$$x'''=-\frac{1}{\rho^2}x'-\frac{\rho'}{\rho}x''+\frac{1}{\sigma}(y'z''-z'y''),$$

with corresponding expressions for y''', z'''; also

$$x''''=3\frac{\rho'}{\rho^3}x'-\left(\frac{1}{\rho^2}+\frac{\rho''}{\rho}-2\frac{\rho'^2}{\rho^2}+\frac{1}{\sigma^2}\right)x''-\left(2\frac{\rho'}{\rho\sigma}+\frac{\sigma'}{\sigma^2}\right)(y'z''-z'y''),$$

with corresponding expressions for y'''', z''''; and indicate a method of obtaining the general form of the expressions for the nth derivatives of x, y, z with regard to s.

Shew that the value of $x''''^2+y''''^2+z''''^2$ is

$$\frac{1}{\rho^2}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+5\frac{\rho'^2}{\rho^2}+2\frac{\rho''}{\rho}\right)+\left(\frac{\rho''}{\rho^2}-2\frac{\rho'^2}{\rho^3}\right)^2-6\frac{\rho'}{\rho^4\sigma}\frac{d}{ds}\left(\frac{\rho}{\sigma}\right)+\frac{1}{\rho^4}\left\{\frac{d}{ds}\left(\frac{\rho}{\sigma}\right)\right\}^2.$$

20. Denoting four consecutive points, at equal small intervals ds on a curve, by 1, 2, 3, 4, by $\overline{12}$ the chord joining the points 1 and 2, and so for the other chords, prove that

$$\overline{12}\,.\,\overline{34}+\overline{23}\,.\,\overline{14}-\overline{13}\,.\,\overline{24}=\frac{1}{6}\frac{R^2}{\rho^4\sigma^2}(ds)^6.$$

21. A helix is drawn on a surface making a constant angle a with the axis of z. Shew that its curvature is given by the equation

$$\sin a=n_1\left\{\frac{(R_1\cos\chi-\rho)\,R_2}{\rho\,(R_1-R_2)}\right\}^{\frac{1}{2}}+n_2\left\{\frac{(\rho-R_2\cos\chi)\,R_1}{\rho\,(R_1-R_2)}\right\}^{\frac{1}{2}},$$

where R_1 and R_2 are the radii of curvature of the principal sections, n_1 and n_2 are the direction-cosines with regard to the axis of z of the tangents to the sections, and χ is given by the equation

$$\cos a\sin\chi=n_3,$$

where $\cos^{-1}n_3$ is the angle between the axis of z and the normal to the surface.

* After Bertrand's memoir, *Liouville's Journal*, t. xv (1850), p. 332.

22. The six coordinates of the principal lines of a curve with regard to fixed rectangular axes are $a, a', a'', \alpha, \alpha', \alpha''$; $b, b', b'', \beta, \beta', \beta''$; $c, c', c'', \gamma, \gamma', \gamma''$; the first three, in each case, being the direction-cosines. Prove that

$$\frac{d\alpha}{ds}=\frac{\beta}{\rho}, \qquad \frac{d\beta}{ds}=\frac{\gamma}{\sigma}-\frac{\alpha}{\rho}+c, \qquad \frac{d\gamma}{ds}=-\frac{\beta}{\sigma}-b.$$

23. A curve is given as the intersection of two surfaces

$$\phi(x, y, z)=0, \quad \psi(x, y, z)=0;$$

the quantities D, E, F denote the determinants

$$D, E, F = \left\| \begin{matrix} \phi_x, & \phi_y, & \phi_z \\ \psi_x, & \psi_y, & \psi_z \end{matrix} \right\|;$$

k denotes $(D^2+E^2+F^2)^{-\frac{1}{2}}$, and a derivative of any quantity U is denoted by U', where

$$U'=\left(D\frac{\partial}{\partial x}+E\frac{\partial}{\partial y}+F\frac{\partial}{\partial z}\right)U.$$

Obtain the following results* for the curve at any point :—

(i) The equation of the osculating plane is

$$(X-x)(EF'-E'F)+(Y-y)(FD'-F'D)+(Z-z)(DE'-D'E)=0;$$

(ii) The radius of circular curvature and the radius of torsion are given by the relations

$$\frac{1}{\rho^2}=k^6\{(EF'-E'F)^2+(FD'-F'D)^2+(DE'-D'E)^2\},$$

$$\frac{1}{\rho^2\sigma}=k^6 \begin{vmatrix} D, & E, & F \\ D', & E', & F' \\ D'', & E'', & F'' \end{vmatrix}.$$

* Other results are given by Frost, *Solid Geometry*, (3rd ed., 1886), §§ 628 et seq.

CHAPTER II.

General Theory of Surfaces.

Much of the present chapter is founded upon the memoir by Gauss, *Disquisitiones generales circa superficies curvas*, (Ges. Werke, t. iv, pp. 217 et seq.); and some account of the memoir is given in Salmon's *Analytical Geometry of Three Dimensions.*

Frequent reference should be made to portions of the first volume and the third volume of Darboux's treatise. Much of chapter III and chapter IV of Bianchi's treatise will be found useful, as also will the first section of Knoblauch's *Einleitung in die allgemeine Theorie der krummen Flächen.*

It is unnecessary to give copious references in detail; the subject-matter is bound to be treated in any book on differential geometry.

Fundamental Magnitudes of the First Order.

23. In the discussion of the intrinsic properties of a surface, the position of the surface relative to coordinate axes is not of importance; and therefore there is convenience in substituting, for the equation of the surface in Cartesian form, other equivalent equations that shall have more direct reference to variation upon the surface itself. This usually is effected by expressing the coordinates of any point on the surface in terms of two independent parameters p and q, through relations

$$x = x(p, q), \quad y = y(p, q), \quad z = z(p, q);$$

the elimination of p and q between these relations leads to the equation of the surface, if it should be required. We shall assume, unless there is explicit statement to the contrary, that we have to deal with surfaces or portions of surfaces, which are regular in character, and within the range of which no singularities (whether of point or line) occur. The parameters p and q are not necessarily real; often it will be expedient to take conjugate or other complex variables as the parameters of reference. Within the range considered, the functions $x(p, q)$, $y(p, q)$, $z(p, q)$ are finite and continuous, usually uniform; if they are multiform, we shall usually restrict the variations to regions which admit no interchange of branches of the functions. Also, a representation in terms of two parameters is not unique; for if we make p and q two independent functions of two new

parameters p' and q', we shall have x, y, z given as functions of p' and q', of the same type as before.

A curve drawn upon the surface can be represented by some relation between p and q, say

$$\phi(p, q) = 0,$$

whether the relation be integral or differential. Sometimes the curve can be obtained by making p and q functions of a single parameter; for instance, geodesics are discussed by this method of representation among others.

A notation for derivatives with respect to p and q will be required; we write

$$\frac{\partial x}{\partial p} = x_1, \quad \frac{\partial x}{\partial q} = x_2,$$

$$\frac{\partial^2 x}{\partial p^2} = x_{11}, \quad \frac{\partial^2 x}{\partial p \partial q} = x_{12}, \quad \frac{\partial^2 x}{\partial q^2} = x_{22},$$

and so on, with corresponding symbols for derivatives of y and of z. The notation will occasionally be used for derivatives of other magnitudes as they arise.

24. Take any point on the surface, determined by p and q; and consider a neighbouring point, also on the surface, determined by the values $p + dp$ and $q + dq$. When we retain only the first powers of small quantities, the distance between the two points measures the infinitesimal arc on the surface; denoting it by ds, we have*

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= E dp^2 + 2F dp dq + G dq^2, \end{aligned}$$

where

$$\left.\begin{aligned} E &= x_1^2 + y_1^2 + z_1^2 = \Sigma x_1^2 \\ F &= x_1 x_2 + y_1 y_2 + z_1 z_2 = \Sigma x_1 x_2 \\ G &= x_2^2 + y_2^2 + z_2^2 = \Sigma x_2^2 \end{aligned}\right\} .$$

These quantities E, F, G are independent of the particular selection of perpendicular coordinate axes; for when we effect an orthogonal transformation

$$\begin{aligned} x' &= a + \lambda x + \mu y + \nu z, \\ y' &= b + \lambda' x + \mu' y + \nu' z, \\ z' &= c + \lambda'' x + \mu'' y + \nu'' z, \end{aligned}$$

we have

$$\begin{aligned} E' &= \Sigma x_1'^2 \\ &= x_1^2 \Sigma \lambda^2 + 2x_1 y_1 \Sigma \lambda\mu + 2x_1 z_1 \Sigma \lambda\nu + y_1^2 \Sigma \mu^2 + 2y_1 z_1 \Sigma \mu\nu + z_1^2 \Sigma \nu^2 \\ &= x_1^2 + y_1^2 + z_1^2 = E, \end{aligned}$$

* We shall always write dx^2 instead of $(dx)^2$, and similarly for other powers and for other quantities.

and similarly for F and G. Hence E, F, G are often called the *fundamental magnitudes of the first order*, sometimes the primary quantities. It is convenient to have a symbol for $EG - F^2$; accordingly, we write

$$V^2 = EG - F^2,$$

so that E, G, V^2 are greater than zero, while we take V to be positive, on a real surface when p and q are real. And, unless there is a specific statement to the contrary, we shall assume that p and q are real.

25. Any curve upon the surface can be represented by an equation $\phi(p, q) = 0$. The simplest of such equations are

$$p = \text{constant}, \quad q = \text{constant};$$

the curves, thus represented, are called the *parametric curves*. We take the positive direction along the curve $p = a$ at any point to be that in which q increases, and the positive direction along the curve $q = b$ at any point to be that in which p increases.

The element of arc along $p = a$ is $G^{\frac{1}{2}} dq$, and its direction-cosines are $x_2 G^{-\frac{1}{2}}$, $y_2 G^{-\frac{1}{2}}$, $z_2 G^{-\frac{1}{2}}$; the sign of $G^{\frac{1}{2}}$ being taken positive.

The element of arc along $q = b$ is $E^{\frac{1}{2}} dp$, and its direction-cosines are $x_1 E^{-\frac{1}{2}}$, $y_1 E^{-\frac{1}{2}}$, $z_1 E^{-\frac{1}{2}}$; the sign of $E^{\frac{1}{2}}$ being taken positive.

The angle at which the parametric curves cut is usually denoted by ω; then

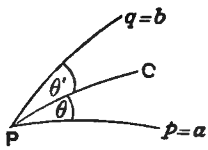
$$\cos\omega = \Sigma\, x_2 G^{-\frac{1}{2}} . x_1 E^{-\frac{1}{2}} = F(EG)^{-\frac{1}{2}},$$
$$\sin\omega = V(EG)^{-\frac{1}{2}}, \quad \tan\omega = V/F.$$

Let dS be the element of area of the surface bounded by the parametric curves p, q, $p + dp$, $q + dq$, each constant; then

$$dS = E^{\frac{1}{2}} dp \,.\, G^{\frac{1}{2}} dq \,.\, \sin\omega = V\,dp\,dq.$$

26. Let PC be the curve defined by

$$\phi(p, q) = c,$$



and let ds be the element of arc along PC; then

$$\phi_1\, dp + \phi_2\, dq = 0,$$
$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$

so that

$$\frac{1}{\phi_2}\frac{dp}{ds} = \frac{1}{-\phi_1}\frac{dq}{ds} = (E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2)^{-\frac{1}{2}}.$$

The direction-cosines of the tangent at P to PC are

$$x_1\frac{dp}{ds} + x_2\frac{dq}{ds}, \ \ldots, \ \ldots;$$

and so, if θ is the angle (taken as in the figure) at P between PC and $p = a$, we have

$$\cos\theta = G^{-\frac{1}{2}} \Sigma x_2 \left(x_1 \frac{dp}{ds} + x_2 \frac{dq}{ds} \right)$$

$$= G^{-\frac{1}{2}} \left(F \frac{dp}{ds} + G \frac{dq}{ds} \right)$$

$$= (F\phi_2 - G\phi_1) \{G (E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2)\}^{-\frac{1}{2}},$$

$$\sin\theta = G^{-\frac{1}{2}} V \frac{dp}{ds}$$

$$= V\phi_2 \{G (E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2)\}^{-\frac{1}{2}}.$$

Similarly, if θ' be the angle (taken as in the figure) at P between PC and $q = b$, so that

$$\theta + \theta' = \omega,$$

we have

$$\cos\theta' = E^{-\frac{1}{2}} \left(E \frac{dp}{ds} + F \frac{dq}{ds} \right)$$

$$= (E\phi_2 - F\phi_1) \{E (E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2)\}^{-\frac{1}{2}},$$

$$\sin\theta' = E^{-\frac{1}{2}} V \frac{dq}{ds}$$

$$= - V\phi_1 \{E (E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2)\}^{-\frac{1}{2}}.$$

Next, let another curve PC' be given by

$$\psi(p, q) = c',$$

and let δs, δp, δq represent small variations along the curve at the point P. Let χ denote the angle at P between PC and PC'; then

$$ds\, \delta s \cos\chi = \Sigma (x_1\, dp + x_2\, dq)(x_1\, \delta p + x_2\, \delta q)$$

$$= E dp\, \delta p + F (dp\, \delta q + dq\, \delta p) + G dq\, \delta q,$$

$$ds\, \delta s \sin\chi = V (dp\, \delta q - \delta p\, dq),$$

so that

$$\cos\chi = \frac{E\phi_2\psi_2 - F(\phi_2\psi_1 + \psi_2\phi_1) + G\phi_1\psi_1}{(E\phi_2^2 - 2F\phi_2\phi_1 + G\phi_1^2)^{\frac{1}{2}} (E\psi_2^2 - 2F\psi_2\psi_1 + G\psi_1^2)^{\frac{1}{2}}},$$

$$\sin\chi = \frac{V(\phi_2\psi_1 - \psi_2\phi_1)}{(E\phi_2^2 - 2F\phi_2\phi_1 + G\phi_1^2)^{\frac{1}{2}} (E\psi_2^2 - 2F\psi_2\psi_1 + G\psi_1^2)^{\frac{1}{2}}}.$$

It follows that *two directions, given by* $dp : dq$, *and* $\delta p : \delta q$, *are perpendicular, if*

$$E\, dp\, \delta p + F (dp\, \delta q + dq\, \delta p) + G\, dq\, \delta q = 0;$$

and that, if two directions are given by *the quadratic equation*

$$\Theta dp^2 + 2\Phi\, dp\, dq + \Psi dq^2 = 0,$$

their inclination χ is given by

$$\frac{\sin\chi}{2V(\Phi^2-\Theta\Psi)^{\frac{1}{2}}}=\frac{\cos\chi}{E\Psi-2F\Phi+G\Theta},$$

so that they are perpendicular if

$$E\Psi-2F\Phi+G\Theta=0.$$

Thus the curves orthogonal to a family $f(p,q)=\alpha$, for varying values of α, are given by the differential equation

$$\left(E\frac{\partial f}{\partial q}-F\frac{\partial f}{\partial p}\right)dp+\left(F\frac{\partial f}{\partial q}-G\frac{\partial f}{\partial p}\right)dq=0.$$

27. Let X, Y, Z be the direction-cosines of the normal to the surface at P. It is perpendicular to every tangent line to the surface and therefore, in particular, to the tangents at P to the parametric curves; hence

$$Xx_1+Yy_1+Zz_1=0,\quad Xx_2+Yy_2+Zz_2=0.$$

Also

$$X^2+Y^2+Z^2=1;$$

consequently*

$$X=(y_1z_2-y_2z_1)V^{-1},\quad Y=(z_1x_2-z_2x_1)V^{-1},\quad Z=(x_1y_2-x_2y_1)V^{-1},$$

or, with a customary notation,

$$X,\ Y,\ Z=\frac{1}{V}\left\|\begin{matrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2\end{matrix}\right\|.$$

The following relations, capable of easy verification, may be noted for future use:

$$V=\begin{vmatrix} X, & Y, & Z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2\end{vmatrix};$$

$$\left.\begin{aligned} x_1Y-y_1X&=(z_1F-z_2E)V^{-1}\\ y_1Z-z_1Y&=(x_1F-x_2E)V^{-1}\\ z_1X-x_1Z&=(y_1F-y_2E)V^{-1}\end{aligned}\right\},\qquad \left.\begin{aligned} x_2Y-y_2X&=(z_1G-z_2F)V^{-1}\\ y_2Z-z_2Y&=(x_1G-x_2F)V^{-1}\\ z_2X-x_2Z&=(y_1G-y_2F)V^{-1}\end{aligned}\right\};$$

$$\left.\begin{aligned} \frac{\partial V}{\partial x_1}&=y_2Z-z_2Y\\ \frac{\partial V}{\partial y_1}&=z_2X-x_2Z\\ \frac{\partial V}{\partial z_1}&=x_2Y-y_2X\end{aligned}\right\},\qquad \left.\begin{aligned} \frac{\partial V}{\partial x_2}&=-y_1Z+z_1Y\\ \frac{\partial V}{\partial y_2}&=-z_1X+x_1Z\\ \frac{\partial V}{\partial z_2}&=-x_1Y+y_1X\end{aligned}\right\};$$

* As V is taken positive, the signs given to X, Y, Z are effectively a definition of the positive direction of the normal.

$$\left.\begin{aligned} V\frac{\partial^2 V}{\partial x_1^2} &= GX^2 \\ V\frac{\partial^2 V}{\partial y_1^2} &= GY^2 \\ V\frac{\partial^2 V}{\partial z_1^2} &= GZ^2 \end{aligned}\right\}, \quad \left.\begin{aligned} V\frac{\partial^2 V}{\partial x_1\,\partial x_2} &= -FX^2 \\ V\frac{\partial^2 V}{\partial y_1\,\partial y_2} &= -FY^2 \\ V\frac{\partial^2 V}{\partial z_1\,\partial z_2} &= -FZ^2 \end{aligned}\right\}, \quad \left.\begin{aligned} V\frac{\partial^2 V}{\partial x_2^2} &= EX^2 \\ V\frac{\partial^2 V}{\partial y_2^2} &= EY^2 \\ V\frac{\partial^2 V}{\partial z_2^2} &= EZ^2 \end{aligned}\right\};$$

$$\left.\begin{aligned} V\frac{\partial^2 V}{\partial x_1\,\partial y_1} &= GXY \\ V\frac{\partial^2 V}{\partial y_1\,\partial z_1} &= GYZ \\ V\frac{\partial^2 V}{\partial z_1\,\partial x_1} &= GZX \end{aligned}\right\}, \quad \left.\begin{aligned} V\frac{\partial^2 V}{\partial x_2\,\partial y_2} &= EXY \\ V\frac{\partial^2 V}{\partial y_2\,\partial z_2} &= EYZ \\ V\frac{\partial^2 V}{\partial z_2\,\partial x_2} &= EZX \end{aligned}\right\};$$

$$\left.\begin{aligned} V\frac{\partial^2 V}{\partial x_1\,\partial y_2} &= VZ - FXY \\ V\frac{\partial^2 V}{\partial y_1\,\partial z_2} &= VX - FYZ \\ V\frac{\partial^2 V}{\partial z_1\,\partial x_2} &= VY - FZX \end{aligned}\right\}, \quad \left.\begin{aligned} V\frac{\partial^2 V}{\partial x_2\,\partial y_1} &= -VZ - FXY \\ V\frac{\partial^2 V}{\partial y_2\,\partial z_1} &= -VX - FYZ \\ V\frac{\partial^2 V}{\partial z_2\,\partial x_1} &= -VY - FZX \end{aligned}\right\}.$$

An equation of the surface, in differential form, can be obtained at once. Let any direction at P in the tangent plane to the surface be denoted by dx, dy, dz; then, as it is perpendicular to the normal, we have

$$Xdx + Ydy + Zdz = 0,$$

which is the differential equation indicated.

When x, y, z are given (and therefore X, Y, Z are deduced) as functions of p and q, the equation is satisfied identically—a result to be expected because the integral equation is implicitly contained in the expressions for x, y, z.

When X, Y, Z are given as appropriate functions of x, y, z, the "condition of integrability" must be satisfied*. A verification that it is satisfied will be given in § 30; assuming this, we have (on integration) an integral of the surface in a form

$$U = \text{constant}.$$

Manifestly,

$$X : Y : Z = \frac{\partial U}{\partial x} : \frac{\partial U}{\partial y} : \frac{\partial U}{\partial z}.$$

Fundamental Magnitudes of the Second Order.

28. The primary quantities are constructed from the first derivatives of x, y, z with respect to p and q. We now proceed to construct quantities that involve their second derivatives. As before, denote small independent

* See my *Treatise on Differential Equations*, (3rd ed.), § 152, for the condition, and for the method of integration of the equation when the condition is satisfied.

variations of p and q by dp and dq; then the value of x belonging to $p+dp$, $q+dq$ has two forms, viz.

$$x + dx + \tfrac{1}{2}d^2x + \tfrac{1}{6}d^3x + \dots,$$

and

$$x + (x_1 dp + x_2 dq) + \tfrac{1}{2}(x_{11}dp^2 + 2x_{12}dpdq + x_{22}dq^2) + \dots;$$

and so, taking the small quantities of the second order, we have

$$d^2x = x_{11}dp^2 + 2x_{12}dpdq + x_{22}dq^2.$$

Similarly

$$d^2y = y_{11}dp^2 + 2y_{12}dpdq + y_{22}dq^2,$$
$$d^2z = z_{11}dp^2 + 2z_{12}dpdq + z_{22}dq^2.$$

The *fundamental magnitudes of the second order* (sometimes called the secondary quantities) are defined by the expressions

$$\left.\begin{aligned} L &= Xx_{11} + Yy_{11} + Zz_{11} \\ M &= Xx_{12} + Yy_{12} + Zz_{12} \\ N &= Xx_{22} + Yy_{22} + Zz_{22} \end{aligned}\right\}.$$

It is convenient to have a symbol* for $LN - M^2$; we write

$$LN - M^2 = T^2.$$

Though L, M, N are real on a real surface when p and q are real, it is not the fact that T^2 is necessarily positive.

Manifestly we have

$$VL = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_{11}, & y_{11}, & z_{11} \end{vmatrix}, \quad VM = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_{12}, & y_{12}, & z_{12} \end{vmatrix}, \quad VN = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_{22}, & y_{22}, & z_{22} \end{vmatrix}.$$

These secondary quantities, like the primary quantities, are independent of the particular selection of perpendicular coordinate axes; for when we effect the same orthogonal transformation as before (§ 24), we have

$$\begin{aligned} V'L' &= \begin{vmatrix} x_1', & y_1', & z_1' \\ x_2', & y_2', & z_2' \\ x_{11}', & y_{11}', & z_{11}' \end{vmatrix} \\ &= \begin{vmatrix} \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \\ \lambda'', & \mu'', & \nu'' \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_{11}, & y_{11}, & z_{11} \end{vmatrix} \\ &= 1 . VL, \end{aligned}$$

when the sets of axes are of the same type. Therefore

$$L' = L;$$

and similarly for M and N.

* The reader should be warned that, for the various quantities, there is no notation in general use by writers on the subject.

29. The quantities X, Y, Z are functions of p and q; their first derivatives with respect to p and q can be expressed by means of L, M, N. For

$$Xx_1 + Yy_1 + Zz_1 = 0,$$

and therefore

$$X_1x_1 + Y_1y_1 + Z_1z_1 = -(Xx_{11} + Yy_{11} + Zz_{11}) = -L.$$

Similarly, from $Xx_2 + Yy_2 + Zz_2 = 0$, we have

$$X_1x_2 + Y_1y_2 + Z_1z_2 = -(Xx_{12} + Yy_{12} + Zz_{12}) = -M;$$

and, as $X^2 + Y^2 + Z^2 = 1$, we have

$$X_1X + Y_1Y + Z_1Z = 0.$$

Solving these three equations for X_1, Y_1, Z_1, we have

$$X_1V = -L(y_2Z - z_2Y) - M(z_1Y - y_1Z),$$

and therefore (§ 27)

$$X_1V^2 = -L(x_1G - x_2F) - M(-x_1F + x_2E)$$

and similarly

$$\left.\begin{aligned} &= x_1(FM - GL) + x_2(-EM + FL) \\ Y_1V^2 &= y_1(FM - GL) + y_2(-EM + FL) \\ Z_1V^2 &= z_1(FM - GL) + z_2(-EM + FL) \end{aligned}\right\}.$$

In the same way, we find

$$\left.\begin{aligned} X_2V^2 &= x_1(FN - GM) + x_2(-EN + FM) \\ Y_2V^2 &= y_1(FN - GM) + y_2(-EN + FM) \\ Z_2V^2 &= z_1(FN - GM) + z_2(-EN + FM) \end{aligned}\right\}.$$

From these, we have

$$x_1T^2 = X_1(-EN + FM) + X_2(EM - FL),$$
$$x_2T^2 = X_1(-FN + GM) + X_2(FM - GL);$$

and similarly for y_1 and y_2 in terms of Y_1 and Y_2, and for z_1 and z_2 in terms of Z_1 and Z_2.

Also we have

$$V^4(Y_1Z_2 - Y_2Z_1) = \begin{vmatrix} y_1, & y_2 \\ z_1, & z_2 \end{vmatrix} \begin{vmatrix} FM - GL, & -EM + FL \\ FN - GM, & -EN + FM \end{vmatrix}$$
$$= VX \,.\, V^2T^2,$$

and so for the others; thus

$$\left.\begin{aligned} V(Y_1Z_2 - Y_2Z_1) &= T^2X \\ V(Z_1X_2 - Z_2X_1) &= T^2Y \\ V(X_1Y_2 - X_2Y_1) &= T^2Z \end{aligned}\right\}.$$

Again, writing

$$\left.\begin{aligned} V^2e &= EM^2 - 2FLM + GL^2 \\ V^2f &= EMN - F(LN + M^2) + GLM \\ V^2g &= EN^2 - 2FMN + GM^2 \end{aligned}\right\},$$

we similarly prove

$$\left.\begin{aligned} T^2(Y_1Z - YZ_1) &= V(fX_1 - eX_2) \\ T^2(Z_1X - ZX_1) &= V(fY_1 - eY_2) \\ T^2(X_1Y - XY_1) &= V(fZ_1 - eZ_2) \end{aligned}\right\}, \qquad \left.\begin{aligned} T^2(Y_2Z - YZ_2) &= V(gX_1 - fX_2) \\ T^2(Z_2X - ZX_2) &= V(gY_1 - fY_2) \\ T^2(X_2Y - XY_2) &= V(gZ_1 - fZ_2) \end{aligned}\right\},$$

results which will be useful when we come to deal with tangential coordinates. Lastly,

$$\begin{vmatrix} X, & Y, & Z \\ X_1, & Y_1, & Z_1 \\ X_2, & Y_2, & Z_2 \end{vmatrix} = T^2/V,$$

by using the above expressions for $Y_1Z_2 - Y_2Z_1$, $Z_1X_2 - Z_2X_1$, $X_1Y_2 - X_2Y_1$.

30. We now formally prove that, if X, Y, Z are given as functions of x, y, z and not as functions of p and q, the condition of integrability of the equation $Xdx + Ydy + Zdz = 0$ is satisfied. For then we have

$$X_1 = x_1\frac{\partial X}{\partial x} + y_1\frac{\partial X}{\partial y} + z_1\frac{\partial X}{\partial z},$$

and

$$X_2 = x_2\frac{\partial X}{\partial x} + y_2\frac{\partial X}{\partial y} + z_2\frac{\partial X}{\partial z};$$

hence

$$x_2X_1 - x_1X_2 = V\left(-Z\frac{\partial X}{\partial y} + Y\frac{\partial X}{\partial z}\right),$$

that is,

$$Y\frac{\partial X}{\partial z} - Z\frac{\partial X}{\partial y} = V^{-1}(x_2X_1 - x_1X_2).$$

Similarly,

$$Z\frac{\partial Y}{\partial x} - X\frac{\partial Y}{\partial z} = V^{-1}(y_2Y_1 - y_1Y_2),$$

$$X\frac{\partial Z}{\partial y} - Y\frac{\partial Z}{\partial x} = V^{-1}(z_2Z_1 - z_1Z_2);$$

and therefore

$$\begin{aligned} X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) &+ Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) \\ &= -V^{-1}(x_2X_1 + y_2Y_1 + z_2Z_1) + V^{-1}(x_1X_2 + y_1Y_2 + z_1Z_2) \\ &= V^{-1}M - V^{-1}M \\ &= 0, \end{aligned}$$

which is the required condition of integrability.

Curvature: the Gauss Measure and Characteristic Equation.

31. The primary quantities involve only the first derivatives of x, y, z; hence they can only be concerned with arc-lengths upon the surface, and with tangential properties. The secondary quantities involve the second derivatives of x, y, z; hence it is to be expected that they will be concerned with curvature properties, among others. Their simplest occurrence is in connection with the curvature of the normal section of the surface.

Let a normal section of the surface be drawn through any tangential direction at a point. It is a plane curve; and so its radius of curvature lies in that plane and is perpendicular to the tangent, that is, it lies along the normal to the surface. Instead of taking the radius of curvature to be always positive (as in § 4), let us assume it to be positive, when the normal section is concave to the side of the surface which is taken as positive, and assume it to be negative, when the normal section is convex to that side of the surface. Then, denoting the radius of curvature by ρ, we have always

$$\rho x'' = X, \quad \rho y'' = Y, \quad \rho z'' = Z,$$

and therefore

$$\frac{1}{\rho} = Xx'' + Yy'' + Zz''.$$

Now

$$x'' = x_{11} p'^2 + 2x_{12} p'q' + x_{22} q'^2 + x_1 p'' + x_2 q'',$$

and similarly for y'', z''; consequently

$$\begin{aligned} \frac{1}{\rho} &= Xx'' + Yy'' + Zz'' \\ &= Lp'^2 + 2Mp'q' + Nq'^2 \\ &= \frac{Ldp^2 + 2Mdpdq + Ndq^2}{Edp^2 + 2Fdpdq + Gdq^2}, \end{aligned}$$

thus giving the curvature of the normal section of the surface through the direction $dp : dq$.

32. It is known from the elementary properties of surfaces that the normals at contiguous points do not necessarily intersect; and that, at an ordinary point, there are two directions on the surface such that normals at contiguous points in either of those directions meet the normal at the point. Proceeding thus from point to point in a continuous direction at each point, we obtain a locus upon the surface; this locus is called a *line of curvature.* Through an ordinary point there pass two lines of curvature; and on the surface there are two systems of lines of curvature. But while the normals to the surface at successive points along a line of curvature intersect, they are not necessarily (nor even generally) the principal normals of the curve; in other words, the osculating plane of a line of curvature does not, in general, give a normal section of a surface.

The intersection of consecutive normals along a line of curvature is called a *centre of curvature* of the surface; as there are two lines of curvature at each point, there are two centres of curvature and both of them lie upon the normal. As we pass over the surface, we have two such points associated with each point on the surface; the locus of these points is called the *surface of centres*. The distance between the point and a centre of curvature, with its proper sign, is called a *radius of curvature* of the surface; thus at any point there are two radii of curvature, sometimes called the *principal radii*. They are the radii of curvature (as defined in § 31) of normal sections through the respective directions.

At a point x, y, z, let r be a radius of curvature, and let ξ, η, ζ denote the corresponding centre of curvature; then

$$\xi = x + rX, \quad \eta = y + rY, \quad \zeta = z + rZ.$$

For a normal at a consecutive point on a line of curvature, the quantities ξ, η, ζ, r are unaltered; hence first variations along the line of curvature are such that

$$0 = dx + r dX, \quad 0 = dy + r dY, \quad 0 = dz + r dZ,$$

and therefore, for the line of curvature, we have

$$\begin{aligned} 0 &= (x_1 + rX_1)\,dp + (x_2 + rX_2)\,dq, \\ 0 &= (y_1 + rY_1)\,dp + (y_2 + rY_2)\,dq, \\ 0 &= (z_1 + rZ_1)\,dp + (z_2 + rZ_2)\,dq. \end{aligned}$$

These are apparently three equations; in reality, they are equivalent to only two equations because, multiplying them by X, Y, Z respectively and adding, we have a nul result.

Multiplying the equations by x_1, y_1, z_1 respectively and adding, we have

$$\begin{aligned} 0 &= (E + r\Sigma x_1 X_1)\,dp + (F + r\Sigma x_1 X_2)\,dq \\ &= (E - rL)\,dp + (F - rM)\,dq\,; \end{aligned}$$

and multiplying them by x_2, y_2, z_2 respectively and adding, we have

$$\begin{aligned} 0 &= (F + r\Sigma x_2 X_1)\,dp + (G + r\Sigma x_2 X_2)\,dq \\ &= (F - rM)\,dp + (G - rN)\,dq, \end{aligned}$$

so that along any line of curvature we have

$$\left.\begin{aligned} 0 &= (E - rL)\,dp + (F - rM)\,dq \\ 0 &= (F - rM)\,dp + (G - rN)\,dq \end{aligned}\right\}.$$

These two equations, combined, determine the directions of the lines of curvature and the radii of curvature at any point.

For the directions, we have

$$\begin{aligned} 0 &= E dp + F dq - r(L dp + M dq), \\ 0 &= F dp + G dq - r(M dp + N dq)\,; \end{aligned}$$

they are given by

$$\begin{vmatrix} Edp+Fdq, & Fdp+Gdq \\ Ldp+Mdq, & Mdp+Ndq \end{vmatrix}=0,$$

that is, by

$$(EM-FL)\,dp^2+(EN-GL)\,dpdq+(FN-GM)\,dq^2=0.$$

Unless the equation is evanescent, it is quadratic in the ratio dp/dq; and therefore at any point of a surface there are generally two lines of curvature. Moreover, as

$$E(FN-GM)-F(EN-GL)+G(EM-FL)=0,$$

the two lines are perpendicular to one another (§ 26) at the point.

Exception to the conclusion, that there are two lines of curvature at a point, occurs when the equation giving those directions is evanescent. We then have

$$\frac{L}{E}=\frac{M}{F}=\frac{N}{G},=\frac{1}{\kappa},$$

say. The radius of curvature of a normal section of the surface through any direction, being

$$\frac{Edp^2+2Fdpdq+Gdq^2}{Ldp^2+2Mdpdq+Ndq^2},$$

is equal to κ, independent of the direction and therefore the same for all directions through the point. Such a point is an *umbilicus* on the surface; the character of the surface in the vicinity of such a point will be considered later.

To determine the magnitude of the radii, we eliminate dp/dq between the equations. Then

$$\begin{vmatrix} E-rL, & F-rM \\ F-rM, & G-rN \end{vmatrix}=0,$$

that is,

$$T^2r^2-(GL-2FM+EN)\,r+V^2=0,$$

so that there are two values, respectively corresponding to the two directions. These must be associated correctly. Should a value of r be given, the value of dp/dq (which determines the direction) is equal to either of the fractions

$$-\frac{F-rM}{E-rL},\quad -\frac{G-rN}{F-rM}.$$

Should a direction be given, the radius of curvature of the surface (which, in general, is not equal to the radius of curvature of the curve) is equal to either of the fractions

$$\frac{Edp+Fdq}{Ldp+Mdq};\quad \frac{Fdp+Gdq}{Mdp+Ndq}.$$

33. A pair of symmetric combinations of the radii are of importance. These are the *mean curvature* H, where

$$H = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{V^2}(GL - 2FM + EN);$$

and the *total curvature* (or the *specific curvature* or *Gauss measure*) K, where

$$K = \frac{1}{\alpha\beta} = \frac{T^2}{V^2};$$

the quantities α and β denoting the two radii. It will be proved that T^2 is expressible in terms of derivatives of E, F, G, so that the total curvature depends only upon the fundamental magnitudes of the first order. The same property does not belong to the mean curvature.

Later, it will be seen that, for a minimal surface (that is, the surface of least area with any assigned boundary), the mean curvature H is zero, so that the equation

$$GL - 2FM + EN = 0$$

is characteristic of a minimal surface. But this equation may be satisfied along a line or lines, on any surface.

The Gauss measure of curvature is positive for a synclastic surface or for the synclastic portions of a surface, that is, at places where all the surface near the point lies on the same side of the tangent plane; familiar instances of synclastic surfaces are provided by the inside of a bowl, a closed soap-bubble, and the palm of a hand. The Gauss measure of curvature is negative for an anticlastic surface or for the anticlastic portions of a surface, that is, at places where different adjacent parts of the surface lie on different sides of the tangent plane; familiar instances are provided by a saddle-back, the top of a mountain pass, and a ridge between two fingers of a hand. The Gauss measure of curvature is zero for a developable surface; familiar instances are provided by the rolling shutter of a desk, and a crumpled piece of paper.

34. To establish the result just stated as regards the total curvature, as well as to establish the intrinsic significance of the six fundamental magnitudes which have been introduced, it is necessary to obtain further relations; and then it will appear that the six magnitudes are not functionally independent of one another. Let quantities m, m', m'', n, n', n'' be defined by the equations

$$\left.\begin{aligned}
m &= x_1x_{11} + y_1y_{11} + z_1z_{11} = \tfrac{1}{2}E_1\\
m' &= x_1x_{12} + y_1y_{12} + z_1z_{12} = \tfrac{1}{2}E_2\\
m'' &= x_1x_{22} + y_1y_{22} + z_1z_{22} = F_2 - \tfrac{1}{2}G_1\\
n &= x_2x_{11} + y_2y_{11} + z_2z_{11} = F_1 - \tfrac{1}{2}E_2\\
n' &= x_2x_{12} + y_2y_{12} + z_2z_{12} = \tfrac{1}{2}G_1\\
n'' &= x_2x_{22} + y_2y_{22} + z_2z_{22} = \tfrac{1}{2}G_2
\end{aligned}\right\};$$

other quantities Γ, Γ', Γ''; Δ, Δ', Δ''; will be required, as defined by the equations

$$\left.\begin{aligned} V^2\Gamma &= mG - nF \\ V^2\Gamma' &= m'G - n'F \\ V^2\Gamma'' &= m''G - n''F \end{aligned}\right\}, \qquad \left.\begin{aligned} V^2\Delta &= nE - mF \\ V^2\Delta' &= n'E - m'F \\ V^2\Delta'' &= n''E - m''F \end{aligned}\right\},$$

which also give

$$\left.\begin{aligned} m &= E\Gamma + F\Delta \\ n &= F\Gamma + G\Delta \end{aligned}\right\}, \quad \left.\begin{aligned} m' &= E\Gamma' + F\Delta' \\ n' &= F\Gamma' + G\Delta' \end{aligned}\right\}, \quad \left.\begin{aligned} m'' &= E\Gamma'' + F\Delta'' \\ n'' &= F\Gamma'' + G\Delta'' \end{aligned}\right\}.$$

Solving the equations

$$Xx_{11} + Yy_{11} + Zz_{11} = L,$$

$$x_1x_{11} + y_1y_{11} + z_1z_{11} = m,$$

$$x_2x_{11} + y_2y_{11} + z_2z_{11} = n,$$

for x_{11}, y_{11}, z_{11}, we have

$$Vx_{11} = L(y_1z_2 - z_1y_2) + m(y_2Z - z_2Y) + n(z_1Y - y_1Z)$$

$$= LVX + \frac{m}{V}(x_1G - x_2F) + \frac{n}{V}(-x_1F + x_2E),$$

that is,

$$x_{11} = LX + x_1\Gamma + x_2\Delta.$$

Similarly for y_{11} and z_{11}; the values are

$$\left.\begin{aligned} x_{11} &= LX + x_1\Gamma + x_2\Delta \\ y_{11} &= LY + y_1\Gamma + y_2\Delta \\ z_{11} &= LZ + z_1\Gamma + z_2\Delta \end{aligned}\right\}.$$

We proceed in the same way to obtain the other second derivatives of x, y, z; their values are

$$\left.\begin{aligned} x_{12} &= MX + x_1\Gamma' + x_2\Delta' \\ y_{12} &= MY + y_1\Gamma' + y_2\Delta' \\ z_{12} &= MZ + z_1\Gamma' + z_2\Delta' \end{aligned}\right\},$$

$$\left.\begin{aligned} x_{22} &= NX + x_1\Gamma'' + x_2\Delta'' \\ y_{22} &= NY + y_1\Gamma'' + y_2\Delta'' \\ z_{22} &= NZ + z_1\Gamma'' + z_2\Delta'' \end{aligned}\right\}.$$

For the moment, let

$$l = x_{11}x_{22} + y_{11}y_{22} + z_{11}z_{22},$$

$$l' = x_{12}^2 + y_{12}^2 + z_{12}^2;$$

then

$$m_2' - m_1'' = l' - l, \quad n_1' - n_2 = l' - l,$$

and the common value gives

$$l' - l = \tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}).$$

Now

$$\begin{aligned} x_{12}^2 + y_{12}^2 + z_{12}^2 &= M(Xx_{12} + Yy_{12} + Zz_{12}) + \Gamma'(x_1x_{12} + y_1y_{12} + z_1z_{12}) \\ &\qquad + \Delta'(x_2x_{12} + y_2y_{12} + z_2z_{12}) \\ &= M^2 + m'\Gamma' + n'\Delta' \\ &= M^2 + (En'^2 - 2Fn'm' + Gm'^2)V^{-2} \\ &= M^2 + (E\Gamma'^2 + 2F\Gamma'\Delta' + G\Delta'^2), \end{aligned}$$

and

$$\begin{aligned} x_{11}x_{22} + y_{11}y_{22} + z_{11}z_{22} &= L(Xx_{22} + Yy_{22} + Zz_{22}) + \Gamma(x_1x_{22} + y_1y_{22} + z_1z_{22}) \\ &\qquad + \Delta(x_2x_{22} + y_2y_{22} + z_2z_{22}) \\ &= LN + m''\Gamma + n''\Delta \\ &= LN + \{Enn'' - F(nm'' + n''m) + Gmm''\}V^{-2} \\ &= LN + \{E\Gamma\Gamma'' + F(\Gamma\Delta'' + \Gamma''\Delta) + G\Delta\Delta''\}; \end{aligned}$$

hence

$$\begin{aligned} LN - M^2 + \{E(nn'' - n'^2) - F(nm'' - 2n'm' + n''m) + G(mm'' - m'^2)\}V^{-2} \\ = l - l' \\ = -\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}), \end{aligned}$$

and therefore

$$\begin{aligned} LN - M^2 = V^2K &= -\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}) \\ &\quad - \{E(nn'' - n'^2) - F(nm'' - 2n'm' + n''m) + G(mm'' - m'^2)\}V^{-2} \\ &= -\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}) + (E, F, G\between\Gamma', \Delta')^2 - (E, F, G\between\Gamma, \Delta\between\Gamma'', \Delta''). \end{aligned}$$

This is sometimes called* the *Gauss characteristic equation.* Its chief significance lies in the fact that $LN - M^2$ is expressible in terms of E, F, G and their derivatives of the first and the second order; hence it follows that *the total curvature is expressible in terms of the fundamental magnitudes of the first order and their derivatives,* a result that is important in connection with the deformation of surfaces.

Mainardi-Codazzi relations.

35. There are also two relations of a differential type. We have

$$\frac{\partial}{\partial q}x_{11} = \frac{\partial}{\partial p}x_{12},$$

and so, substituting for x_{11} and x_{12} their values that are linear in X, x_1, x_2 (§ 34), we find

$$\begin{aligned} LX_2 + x_1\Gamma_2 + x_2\Delta_2 + XL_2 + x_{12}\Gamma + x_{22}\Delta \\ = MX_1 + x_1\Gamma_1' + x_2\Delta_1' + XM_1 + x_{11}\Gamma' + x_{12}\Delta'. \end{aligned}$$

On substitution for X_1, X_2, x_{11}, x_{12}, x_{22} in terms of X, x_1, x_2 (§§ 29, 34), this equation becomes

$$X\Theta + x_1\Theta' + x_2\Theta'' = 0,$$

* It was obtained first by Gauss in his celebrated memoir of 1827 already (p. 32) quoted.

where

$$\left.\begin{aligned}\Theta &= L_2 - M_1 + M\Gamma + N\Delta - L\Gamma' - M\Delta' \\ \Theta' &= FK + \Gamma_2 - \Gamma_1' + \Gamma''\Delta - \Gamma'\Delta' \\ \Theta'' &= -EK + \Delta_2 - \Delta_1' + \Gamma\Delta' - \Gamma'\Delta + \Delta\Delta'' - \Delta'^2\end{aligned}\right\}.$$

Proceeding similarly from y_{11} and y_{12}, and from z_{11} and z_{12}, we have

$$Y\Theta + y_1\Theta' + y_2\Theta'' = 0,$$

$$Z\Theta + z_1\Theta' + z_2\Theta'' = 0.$$

It follows, from these three relations linear in Θ, Θ', Θ'', that

$$\Theta = 0, \quad \Theta' = 0, \quad \Theta'' = 0,$$

as their determinant is not zero.

When the same process is applied to x_{12} and x_{22}; to y_{12} and y_{22}; and to z_{12} and z_{22}; three relations are obtained which similarly lead to

$$\Phi = 0, \quad \Phi' = 0, \quad \Phi'' = 0,$$

where

$$\left.\begin{aligned}\Phi &= N_1 - M_2 + M\Delta'' + L\Gamma'' - M\Gamma' - N\Delta' \\ \Phi' &= -GK + \Gamma_1'' - \Gamma_2' + \Gamma\Gamma'' - \Gamma'^2 + \Gamma'\Delta'' - \Gamma''\Delta' \\ \Phi'' &= FK + \Delta_1'' - \Delta_2' + \Gamma''\Delta - \Gamma'\Delta'\end{aligned}\right\}.$$

Apparently, there are six relations; we proceed to shew that all of them are satisfied in virtue of

(i) the Gauss equation,

(ii) the relations $\Theta = 0$, $\Phi = 0$,

(iii) necessary identities.

The last are connected with the derivatives of various quantities that occur, and they are as follows.

Because

$$\tfrac{1}{2}E_1 = m = E\Gamma + F\Delta, \quad \tfrac{1}{2}E_2 = m' = E\Gamma' + F\Delta',$$

it follows that

$$\frac{\partial}{\partial q}(E\Gamma + F\Delta) = \frac{\partial}{\partial p}(E\Gamma' + F\Delta').$$

When this is expanded, and we substitute for $\Gamma_2 - \Gamma_1'$ and $\Delta_2 - \Delta_1'$ in terms of Θ' and Θ'', we find

$$E\Theta' + F\Theta'' = 0.$$

Similarly, proceeding from

$$\tfrac{1}{2}G_1 = n' = F\Gamma' + G\Delta', \quad \tfrac{1}{2}G_2 = n'' = F\Gamma'' + G\Delta'',$$

we find

$$F\Phi' + G\Phi'' = 0.$$

Again, because

$$n_2 - n_1' = F_{12} - \tfrac{1}{2}E_{22} - \tfrac{1}{2}G_{11},$$

when we expand the left-hand side and use the Gauss equation, we find

$$F\Theta' + G\Theta'' = 0.$$

Lastly, because

$$m_1'' - m_2' = F_{12} - \tfrac{1}{2}E_{22} - \tfrac{1}{2}G_{11},$$

when we expand the left-hand side and use the Gauss equation, we find

$$E\Phi' + F\Phi'' = 0.$$

Hence, retaining the Gauss equation, we have the relations

$$\left.\begin{array}{l} E\Theta' + F\Theta'' = 0 \\ F\Theta' + G\Theta'' = 0 \end{array}\right\}, \quad \left.\begin{array}{l} E\Phi' + F\Phi'' = 0 \\ F\Phi' + G\Phi'' = 0 \end{array}\right\},$$

which are satisfied in virtue of our necessary identities; that is, we have

$$\Theta' = 0, \quad \Theta'' = 0, \quad \Phi' = 0, \quad \Phi'' = 0,$$

because $V^2, = EG - F^2$, is not zero.

It therefore follows that the relations to be retained are the Gauss characteristic equation, together with the relations $\Theta = 0$, $\Phi = 0$. These two new equations are

$$\left.\begin{array}{l} L_2 + \Gamma M + \Delta N = M_1 + \Gamma' L + \Delta' M \\ M_2 + \Gamma' M + \Delta' N = N_1 + \Gamma'' L + \Delta'' M \end{array}\right\};$$

they frequently are called the *Mainardi-Codazzi relations** In all, therefore, there are three differential equations which the six fundamental magnitudes of the first and the second order must satisfy. It remains to be seen what element, if any, of generality is possessed by a surface if and when it is determined by six magnitudes, which are given initially and which satisfy the three differential equations.

The Mainardi-Codazzi relations can be expressed in a different (but equivalent) form. We have

$$\begin{aligned} \frac{\partial}{\partial q}\left(\frac{L}{V}\right) &= \frac{1}{V} L_2 - \frac{L}{V^2} V_2 \\ &= \frac{1}{V} L_2 - \frac{L}{V}(\Gamma' + \Delta''), \\ \frac{\partial}{\partial p}\left(\frac{M}{V}\right) &= \frac{1}{V} M_1 - \frac{M}{V^2} V_1 \\ &= \frac{1}{V} M_1 - \frac{M}{V}(\Gamma + \Delta'); \end{aligned}$$

* They were given, though not in the adopted form, by Mainardi, *Giorn. Ist. Lomb.*, t. ix (1856), p. 395; and, in the adopted form, by Codazzi, *Ann. di Mat.*, t. ii (1868), p. 273.

hence

$$\frac{\partial}{\partial q}\left(\frac{L}{V}\right)-\frac{\partial}{\partial p}\left(\frac{M}{V}\right)=\frac{1}{V}(L_2-M_1)-\frac{L}{V}(\Gamma'+\Delta'')+\frac{M}{V}(\Gamma+\Delta')$$

$$=\frac{1}{V}(-L\Delta''+2M\Delta'-N\Delta)$$

and similarly

$$\frac{\partial}{\partial p}\left(\frac{N}{V}\right)-\frac{\partial}{\partial q}\left(\frac{M}{V}\right)=\frac{1}{V}(-L\Gamma''+2M\Gamma'-N\Gamma),$$

which are the Mainardi-Codazzi relations in another form.

36. The necessary identities used in establishing the relations can be used also to obtain two other results, connected with the angle ω between the parametric lines. We have

$$\frac{\partial}{\partial q}\left(\frac{V\Delta}{E}\right)-\frac{\partial}{\partial p}\left(\frac{V\Delta'}{E}\right)$$

$$=\frac{V\Delta}{E}(\Gamma'+\Delta'')+\frac{V}{E}(\Delta_2-\Delta_1')-\frac{V\Delta'}{E}(\Gamma+\Delta')-\frac{V\Delta}{E^2}E_2+\frac{V\Delta'}{E^2}E_1$$

$$=\frac{V}{E}(\Delta_2-\Delta_1'+\Delta\Delta''-\Delta'^2+\Gamma\Delta'-\Gamma'\Delta)$$

$$=VK, \text{ in virtue of } \Theta''=0.$$

Similarly, in virtue of $\Phi''=0$, we have

$$\frac{\partial}{\partial p}\left(\frac{V\Gamma''}{G}\right)-\frac{\partial}{\partial q}\left(\frac{V\Gamma'}{G}\right)=VK.$$

Now the angle ω between the parametric lines is given by $\tan\omega=V/F$; hence

$$d\omega=\frac{FdV-VdF}{EG}$$

$$=\frac{1}{2EGV}(FGdE-2EGdF+EFdG).$$

Consequently

$$2EGV\omega_1=FGE_1-2EGF_1+EFG_1$$

$$=FG\,.\,2m-2EG(n+m')+EF\,.\,2n'$$

$$=-(2E\Gamma'+2G\Delta)V^2,$$

and therefore

$$\omega_1=-\left(\frac{V\Delta}{E}+\frac{V\Gamma'}{G}\right);$$

and similarly

$$\omega_2=-\left(\frac{V\Delta'}{E}+\frac{V\Gamma''}{G}\right),$$

so that

$$d\omega=\omega_1dp+\omega_2dq=-\left(\frac{V\Delta}{E}+\frac{V\Gamma'}{G}\right)dp-\left(\frac{V\Delta'}{E}+\frac{V\Gamma''}{G}\right)dq.$$

Moreover,

$$\begin{aligned} -\omega_{12} &= \frac{\partial}{\partial q}\left(\frac{V\Delta}{E} + \frac{V\Gamma'}{G}\right) \\ &= \frac{\partial}{\partial p}\left(\frac{V\Delta'}{E}\right) + \frac{\partial}{\partial q}\left(\frac{V\Gamma'}{G}\right) + VK \end{aligned}$$

and

$$\left.\begin{aligned} \omega_{12} &= -\frac{\partial}{\partial p}\left(\frac{V\Delta'}{E} + \frac{V\Gamma''}{G}\right) \\ &= -\frac{\partial}{\partial p}\left(\frac{V\Gamma''}{G}\right) - \frac{\partial}{\partial q}\left(\frac{V\Delta}{E}\right) + VK \end{aligned}\right\},$$

which are the two results in question. The second of them gives Liouville's form for the total curvature.

Bonnet's Theorem.

37. We now come to the theorem which is the essential justification for considering the differential geometry of surfaces in connection with the six fundamental magnitudes. It was proved* first by Bonnet, and may be enunciated as follows:—

When six fundamental magnitudes are given, and when they satisfy the Gauss characteristic equation and the two Mainardi-Codazzi relations, they determine a surface uniquely save as to its position and orientation in space.

The equations, satisfied by X, x_1, x_2 when they are regarded as three dependent variables, are

$$\left.\begin{array}{lll} X_1 & -\frac{1}{V^2}(FM - GL)x_1 - \frac{1}{V^2}(-EM + FL)x_2 = 0 \\ x_{11} - LX & \quad - \Gamma x_1 & \quad - \Delta x_2 = 0 \\ x_{12} - MX & \quad - \Gamma' x_1 & \quad - \Delta' x_2 = 0 \end{array}\right\} \text{......(i)},$$

and

$$\left.\begin{array}{lll} P = X_2 & -\frac{1}{V^2}(FN - GM)x_1 - \frac{1}{V^2}(-EN + FM)x_2 = 0 \\ Q = x_{12} - MX & \quad - \Gamma' x_1 & \quad - \Delta' x_2 = 0 \\ R = x_{22} - NX & \quad - \Gamma'' x_1 & \quad - \Delta'' x_2 = 0 \end{array}\right\} \text{...(ii)}.$$

Both sets are linear in the dependent variables; derivatives with regard to p occur only in set (i) and with regard to q only in set (ii).

The primitive† of the linear set (i) is of the form

$$\left.\begin{aligned} X &= \xi A + \eta B + \zeta C \\ x_1 &= \xi a_1 + \eta b_1 + \zeta c_1 \\ x_2 &= \xi a_2 + \eta b_2 + \zeta c_2 \end{aligned}\right\} \text{..........................(iii)},$$

* *Journ. Éc. Polytech.*, cah. xlii (1867), p. 31.

† See my *Treatise on Differential Equations*, (3rd ed.), § 173.

where ξ, η, ζ are arbitrary constants so far as concerns derivation with respect to p, that is, are arbitrary functions of q so far as concerns set (i). Also

$$X,\ x_1,\ x_2 = A,\ a_1,\ a_2;$$
$$X,\ x_1,\ x_2 = B,\ b_1,\ b_2;$$
$$X,\ x_1,\ x_2 = C,\ c_1,\ c_2;$$

are particular sets of integrals of the equations, linearly independent of one another; and any linear combination of them with coefficients, that are independent of p and that combine them as in (iii), is also a set of integrals of the equations (i).

But the equations (ii) are to be satisfied simultaneously with (i). Consequently they must be satisfied by the quantities (iii); and the variables at our disposal for this purpose are ξ, η, ζ, which are functions of q alone. Accordingly, for the equations (ii), ξ, η, ζ become the dependent variables while q is the independent variable. Let

P, Q, R take values* P_1, Q_1, R_1 when $X, x_1, x_2 = A, a_1, a_2$;
.... P_2, Q_2, R_2 .... $= B, b_1, b_2$;
.... P_3, Q_3, R_3 .... $= C, c_1, c_2$;

then when the expressions (iii), which must satisfy the equations (ii), are substituted in those equations, we have

$$\left.\begin{aligned}
A\frac{d\xi}{dq}+B\frac{d\eta}{dq}+C\frac{d\zeta}{dq} &= -(\xi P_1+\eta P_2+\zeta P_3)\\
a_1\frac{d\xi}{dq}+b_1\frac{d\eta}{dq}+c_1\frac{d\zeta}{dq} &= -(\xi Q_1+\eta Q_2+\zeta Q_3)\\
a_2\frac{d\xi}{dq}+b_2\frac{d\eta}{dq}+c_2\frac{d\zeta}{dq} &= -(\xi R_1+\eta R_2+\zeta R_3)
\end{aligned}\right\} \quad \text{...........(iv).}$$

These equations (iv), regarded as determining ξ, η, ζ, must provide a primitive involving those quantities and expressing them as functions of q alone, even though the coefficients in these equations involve the quantity p, which now is parametric. The requirement will be met if the values of ξ, η, ζ, as given by (iv), also satisfy the relations

$$\left.\begin{aligned}
\frac{\partial A}{\partial p}\frac{d\xi}{dq}+\frac{\partial B}{\partial p}\frac{d\eta}{dq}+\frac{\partial C}{\partial p}\frac{d\zeta}{dq} &= -\left(\xi\frac{\partial P_1}{\partial p}+\eta\frac{\partial P_2}{\partial p}+\zeta\frac{\partial P_3}{\partial p}\right)\\
\frac{\partial a_1}{\partial p}\frac{d\xi}{dq}+\frac{\partial b_1}{\partial p}\frac{d\eta}{dq}+\frac{\partial c_1}{\partial p}\frac{d\zeta}{dq} &= -\left(\xi\frac{\partial Q_1}{\partial p}+\eta\frac{\partial Q_2}{\partial p}+\zeta\frac{\partial Q_3}{\partial p}\right)\\
\frac{\partial a_2}{\partial p}\frac{d\xi}{dq}+\frac{\partial b_2}{\partial p}\frac{d\eta}{dq}+\frac{\partial c_2}{\partial p}\frac{d\zeta}{dq} &= -\left(\xi\frac{\partial R_1}{\partial p}+\eta\frac{\partial R_2}{\partial p}+\zeta\frac{\partial R_3}{\partial p}\right)
\end{aligned}\right\} \quad \text{......(v);}$$

and these relations will be satisfied if it can be shewn that they are consequences of equations (iv) and of the earlier equations, regard being paid to the three differential relations satisfied by the fundamental magnitudes.

* The values Q_1, Q_2, Q_3 actually are zero because the third equation in (i) is the same as $Q=0$; the symbols are retained for symmetry.

The three equations in (v) can be taken separately and the process is mainly the same for each of them; we therefore set out the details in connection with the first.

Because of the first of equations (i), we have

$$V^2\frac{\partial A}{\partial p} = (FM - GL)\,a_1 + (-EM + FL)\,a_2$$
$$= sa_1 + ta_2,$$

say; and

$$V^2\frac{\partial B}{\partial p} = sb_1 + tb_2,$$

$$V^2\frac{\partial C}{\partial p} = sc_1 + tc_2.$$

Multiply the second of the equations in (iv) by s, the third by t, add, and use the immediately preceding results; we have

$$V^2\left(\frac{\partial A}{\partial p}\frac{d\xi}{dq} + \frac{\partial B}{\partial p}\frac{d\eta}{dq} + \frac{\partial C}{\partial p}\frac{d\zeta}{dq}\right) = -\xi\,(sQ_1 + tR_1) - \eta\,(sQ_2 + tR_2) - \zeta\,(sQ_3 + tR_3).$$

This will be the same as the first of the equations (v)—which accordingly will be a consequence of earlier equations—if

$$V^2\frac{\partial P_1}{\partial p} = sQ_1 + tR_1,$$

$$V^2\frac{\partial P_2}{\partial p} = sQ_2 + tR_2,$$

$$V^2\frac{\partial P_3}{\partial p} = sQ_3 + tR_3.$$

We proceed to shew that these relations are satisfied. We have

$$P_1 = \frac{\partial A}{\partial q} - \frac{1}{V^2}(FN - GM)\,a_1 - \frac{1}{V^2}(-EN + FM)\,a_2$$
$$= \frac{\partial A}{\partial q} - \frac{1}{V^2}(s'a_1 + t'a_2),$$

say; and, similarly,

$$Q_1 = \frac{\partial a_1}{\partial q} - MA - \Gamma' a_1 - \Delta' a_2,$$

$$R_1 = \frac{\partial a_2}{\partial q} - NA - \Gamma'' a_1 - \Delta'' a_2.$$

Hence

$$\frac{\partial P_1}{\partial p} = \frac{\partial^2 A}{\partial p\,\partial q} - a_1\frac{\partial}{\partial p}\left(\frac{s'}{V^2}\right) - a_2\frac{\partial}{\partial p}\left(\frac{t'}{V^2}\right) - \frac{s'}{V^2}\frac{\partial a_1}{\partial p} - \frac{t'}{V^2}\frac{\partial a_2}{\partial p}.$$

But

$$\frac{\partial A}{\partial p} = \frac{s}{V^2}a_1 + \frac{t}{V^2}a_2,$$

so that

$$\frac{\partial^2 A}{\partial p\,\partial q} = \frac{s}{V^2}\frac{\partial a_1}{\partial q} + \frac{t}{V^2}\frac{\partial a_2}{\partial q} + a_1\frac{\partial}{\partial q}\left(\frac{s}{V^2}\right) + a_2\frac{\partial}{\partial q}\left(\frac{t}{V^2}\right),$$

and therefore

$$\frac{\partial P_1}{\partial p} = \frac{s}{V^2}\frac{\partial a_1}{\partial q} + \frac{t}{V^2}\frac{\partial a_2}{\partial q}$$

$$+ a_1\left\{\frac{\partial}{\partial q}\left(\frac{s}{V^2}\right) - \frac{\partial}{\partial p}\left(\frac{s'}{V^2}\right)\right\} + a_2\left\{\frac{\partial}{\partial q}\left(\frac{t}{V^2}\right) - \frac{\partial}{\partial p}\left(\frac{t'}{V^2}\right)\right\}$$

$$- \frac{s'}{V^2}(LA + \Gamma a_1 + \Delta a_2) - \frac{t'}{V^2}(MA + \Gamma' a_1 + \Delta' a_2)$$

$$= \frac{s}{V^2}Q_1 + \frac{t}{V^2}R_1$$

$$+ a_1\left\{\frac{\partial}{\partial q}\left(\frac{s}{V^2}\right) - \frac{\partial}{\partial p}\left(\frac{s'}{V^2}\right) + (\Gamma' s - \Gamma s')/V^2 + (\Gamma'' t - \Gamma' t')/V^2\right\}$$

$$+ a_2\left\{\frac{\partial}{\partial q}\left(\frac{t}{V^2}\right) - \frac{\partial}{\partial p}\left(\frac{t'}{V^2}\right) + (\Delta' s - \Delta s')/V^2 + (\Delta'' t - \Delta' t')/V^2\right\}$$

$$+ \frac{A}{V^2}(sM + tN - s'L - t'M).$$

When the coefficient of a_1 is evaluated, it is found to vanish in virtue of the Mainardi-Codazzi relations. The coefficient of a_2 vanishes in the same way. And the coefficient of A vanishes identically. Hence

$$V^2\frac{\partial P_1}{\partial p} = sQ_1 + tR_1.$$

Similarly

$$V^2\frac{\partial P_2}{\partial p} = sQ_2 + tR_2,$$

$$V^2\frac{\partial P_3}{\partial p} = sQ_3 + tR_3.$$

Consequently, the first of the equations in (v) is satisfied in virtue of earlier equations and of the differential relations among the fundamental magnitudes.

Next, to obtain the second of the equations in (v), we multiply the three equations in (iv) by L, Γ, Δ, and add. After corresponding calculations similar to those just given, and by using the relations in § 35, satisfied in virtue of the Gauss equation and the Mainardi-Codazzi relations, we find that the result reduces to the required second equation in (v).

And to obtain the third of the equations in (v), we multiply the three equations in (iv) by M, Γ', Δ', and add. Calculations similar to those for the second equation are required; the result reduces to the required third equation in (v).

Thus the equations (v) are satisfied, in virtue of earlier equations that are retained, and in virtue of the differential relations among the fundamental magnitudes. Consequently the equations (iv) possess a primitive which expresses ξ, η, ζ as functions of q alone even though the coefficients in the

equations, in the form in which they actually occur in the general investigation, may involve p parametrically. This primitive is of the type

$$\left.\begin{aligned} \xi &= \lambda\xi_1 + \mu\xi_2 + \nu\xi_3 \\ \eta &= \lambda\eta_1 + \mu\eta_2 + \nu\eta_3 \\ \zeta &= \lambda\zeta_1 + \mu\zeta_2 + \nu\zeta_3 \end{aligned}\right\},$$

where λ, μ, ν are arbitrary constants; ξ_1, η_1, ζ_1, being functions of q alone, are a special set of integrals; and likewise for ξ_2, η_2, ζ_2; ξ_3, η_3, ζ_3; the three special sets being linearly independent. When these values of ξ, η, ζ are substituted in the expressions (iii), we have

$$\left.\begin{aligned} X &= \lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C} \\ x_1 &= \lambda\mathbf{a}_1 + \mu\mathbf{b}_1 + \nu\mathbf{c}_1 \\ x_2 &= \lambda\mathbf{a}_2 + \mu\mathbf{b}_2 + \nu\mathbf{c}_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{vi})_1,$$

where

$$\begin{aligned} \mathbf{A} &= \xi_1 A + \eta_1 B + \zeta_1 C, \\ \mathbf{a}_1 &= \xi_1 a_1 + \eta_1 b_1 + \zeta_1 c_1, \\ \mathbf{a}_2 &= \xi_1 a_2 + \eta_1 b_2 + \zeta_1 c_2, \end{aligned}$$

and so for $\mathbf{B}$, $\mathbf{b}_1$, $\mathbf{b}_2$; and $\mathbf{C}$, $\mathbf{c}_1$, $\mathbf{c}_2$. Thus $\mathbf{A}$, $\mathbf{a}_1$, $\mathbf{a}_2$; $\mathbf{B}$, $\mathbf{b}_1$, $\mathbf{b}_2$; $\mathbf{C}$, $\mathbf{c}_1$, $\mathbf{c}_2$; are three particular sets of solutions of the original six differential equations to be satisfied; and the values of X, x_1, x_2 in $(\text{vi})_1$ constitute the primitive of the six equations.

The equations, determining Y, y_1, y_2, are precisely the same in form as the six which determine X, x_1, x_2; and likewise those for Z, z_1, z_2. Hence the primitive of the equations for Y, y_1, y_2 is

$$\left.\begin{aligned} Y &= \lambda'\mathbf{A} + \mu'\mathbf{B} + \nu'\mathbf{C} \\ y_1 &= \lambda'\mathbf{a}_1 + \mu'\mathbf{b}_1 + \nu'\mathbf{c}_1 \\ y_2 &= \lambda'\mathbf{a}_2 + \mu'\mathbf{b}_2 + \nu'\mathbf{c}_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{vi})_2;$$

and the primitive of the equations for Z, z_1, z_2 is

$$\left.\begin{aligned} Z &= \lambda''\mathbf{A} + \mu''\mathbf{B} + \nu''\mathbf{C} \\ z_1 &= \lambda''\mathbf{a}_1 + \mu''\mathbf{b}_1 + \nu''\mathbf{c}_1 \\ z_2 &= \lambda''\mathbf{a}_2 + \mu''\mathbf{b}_2 + \nu''\mathbf{c}_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{vi})_3;$$

where, in $(\text{vi})_2$, λ', μ', ν'; and, in $(\text{vi})_3$, λ'', μ'', ν''; are arbitrary constants.

38. Thus the complete primitive appears to contain nine arbitrary constants which are produced in sets of three by the integration of the equations. But these equations are themselves merely inferences from earlier fundamental equations, among which are

$$\begin{aligned} X^2 + Y^2 + Z^2 &= 1, & x_1^2 + y_1^2 + z_1^2 &= E, \\ Xx_1 + Yy_1 + Zz_1 &= 0, & x_1x_2 + y_1y_2 + z_1z_2 &= F, \\ Xx_2 + Yy_2 + Zz_2 &= 0, & x_2^2 + y_2^2 + z_2^2 &= G; \end{aligned}$$

and therefore these equations must be satisfied.

Now, substituting in $X^2+Y^2+Z^2=1$ the values given by the primitives, we have

$$\mathbf{A}^2\Sigma\lambda^2+2\mathbf{AB}\Sigma\lambda\mu+\mathbf{B}^2\Sigma\mu^2+2\mathbf{AC}\Sigma\lambda\nu+2\mathbf{BC}\Sigma\mu\nu+\mathbf{C}^2\Sigma\nu^2=1.$$

But $\mathbf{A}=X$, $\mathbf{B}=Y$, $\mathbf{C}=Z$ are particular solutions, so that

$$\mathbf{A}^2+\mathbf{B}^2+\mathbf{C}^2=1.$$

Hence writing

$$k_1,\ k_2,\ k_3,\ k_4,\ k_5,\ k_6=\Sigma\lambda^2-1,\ \Sigma\lambda\mu,\ \Sigma\mu^2-1,\ \Sigma\lambda\nu,\ \Sigma\mu\nu,\ \Sigma\nu^2-1,$$

we have

$$\mathbf{A}^2k_1+2\mathbf{AB}k_2+\mathbf{B}^2k_3+2\mathbf{AC}k_4+2\mathbf{BC}k_5+\mathbf{C}^2k_6=0.$$

Similarly from $Xx_1+Yy_1+Zz_1=0$, we have

$$\mathbf{Aa}_1k_1+(\mathbf{Ab}_1+\mathbf{a}_1\mathbf{B})\,k_2+\mathbf{Bb}_1k_3+(\mathbf{Ac}_1+\mathbf{a}_1\mathbf{C})\,k_4+(\mathbf{Bc}_1+\mathbf{b}_1\mathbf{C})\,k_5+\mathbf{Cc}_1k_6=0\,;$$

and from the other four relations in turn, we have

$$\mathbf{Aa}_2k_1+(\mathbf{Ab}_2+\mathbf{a}_2\mathbf{B})\,k_2+\mathbf{Bb}_2k_3+(\mathbf{Ac}_2+\mathbf{a}_2\mathbf{C})\,k_4+(\mathbf{Bc}_2+\mathbf{b}_2\mathbf{C})\,k_5+\mathbf{Cc}_2k_6=0,$$

$$\mathbf{a}_1{}^2k_1+2\mathbf{a}_1\mathbf{b}_1k_2+\mathbf{b}_1{}^2k_3+2\mathbf{a}_1\mathbf{c}_1k_4+2\mathbf{b}_1\mathbf{c}_1k_5+\mathbf{c}_1{}^2k_6=0,$$

$$\mathbf{a}_1\mathbf{a}_2k_1+(\mathbf{a}_1\mathbf{b}_2+\mathbf{a}_2\mathbf{b}_1)\,k_2+\mathbf{b}_1\mathbf{b}_2k_3+(\mathbf{a}_1\mathbf{c}_2+\mathbf{a}_2\mathbf{c}_1)\,k_4+(\mathbf{b}_1\mathbf{c}_2+\mathbf{b}_2\mathbf{c}_1)\,k_5+\mathbf{c}_1\mathbf{c}_2k_6=0,$$

$$\mathbf{a}_2{}^2k_1+2\mathbf{a}_2\mathbf{b}_2k_2+\mathbf{b}_2{}^2k_3+2\mathbf{a}_2\mathbf{c}_2k_4+2\mathbf{b}_2\mathbf{c}_2k_5+\mathbf{c}_2{}^2k_6=0.$$

Thus there are six equations linear and homogeneous in the six constants k. The determinant of the coefficients on the left-hand sides is equal to

$$\begin{vmatrix} \mathbf{A}, & \mathbf{B}, & \mathbf{C} \\ \mathbf{a}_1, & \mathbf{b}_1, & \mathbf{c}_1 \\ \mathbf{a}_2, & \mathbf{b}_2, & \mathbf{c}_2 \end{vmatrix}^4,$$

that is, to V^4, and so it does not vanish. Hence all the constants k are zero, that is,

$$\begin{aligned} \Sigma\lambda^2-1&=0, & \Sigma\mu\nu&=0\,, \\ \Sigma\mu^2-1&=0, & \Sigma\nu\lambda&=0\,, \\ \Sigma\nu^2-1&=0, & \Sigma\lambda\mu&=0\,; \end{aligned}$$

and therefore the nine constants λ, μ, ν are the direction-cosines of three perpendicular lines.

Finally, we have, for the surface itself,

$$dx=x_1dp+x_2dq,$$

that is,

$$\begin{aligned} x-l&=\lambda\int(\mathbf{a}_1dp+\mathbf{a}_2dq)+\mu\int(\mathbf{b}_1dp+\mathbf{b}_2dq)+\nu\int(\mathbf{c}_1dp+\mathbf{c}_2dq), \\ &=\lambda u+\mu v+\nu w\,; \end{aligned}$$

and similarly

$$\begin{aligned} y-l'&=\lambda'u+\mu'v+\nu'w\,, \\ z-l''&=\lambda''u+\mu''v+\nu''w, \end{aligned}$$

where l, l', l'' are additive arbitrary constants of integration.

The relations between the constants λ leave the orientation of the surface undetermined; the existence of the constants l, l', l'' leaves the position of the surface undetermined.

And so we have Bonnet's theorem as enunciated.

39. It follows from this theorem that the six fundamental magnitudes E, F, G, L, M, N, together with their derivatives, are sufficient for the expression and the determination of all magnitudes and all properties that are intrinsically possessed by the surface. Later we shall see that, as is to be expected, some properties are common to all those surfaces which (roughly at the moment) may be described as having E, F, G in common without any regard to L, M, N other than the Gauss characteristic equation. At present, the important result is that the six magnitudes give uniquely the intrinsic determination of a surface and that therefore they suffice for the expression of all properties of the surface which are independent of its position and its orientation.

Derived Magnitudes.

40. It has just been stated, as an inference from Bonnet's theorem, that properties and magnitudes intrinsically possessed by a surface are expressible in terms of the three fundamental magnitudes E, F, G of the first order, of the three fundamental magnitudes L, M, N of the second order, and of their derivatives. Now it happens, as might be expected, that certain combinations involving first derivatives of L, M, N have relations with derivatives of x, y, z of the third order similar to those possessed by L, M, N with derivatives of x, y, z of the second order. Similarly certain combinations involving second derivatives of L, M, N have corresponding relations with derivatives of x, y, z of the fourth order; and so with the respective orders in succession. The combinations, which thus arise, are sometimes called fundamental magnitudes; having regard to the essential significance of the fundamental magnitudes of the first order and the second order, the new combinations may be called *derived magnitudes* of the various orders.

The derived magnitudes are perhaps most simply defined in connection with the variation of the curvature of the normal section of the surface along a curve* In particular, those of the *third order* are defined by the relation

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = Pp'^3 + 3Qp'^2q' + 3Rp'q'^2 + Sq'^3.$$

The values of P, Q, R, S are obtained as follows. As before (§ 31), we have

$$\frac{1}{\rho}X = x'' = x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2 + x_1p'' + x_2q'',$$

* This will be seen, at a later stage, to imply the consideration of geodesic tangents to the curve.

with two similar equations. Multiplying them by x_1, y_1, z_1, and adding, we have

$$Ep'' + Fq'' + mp'^2 + 2m'p'q' + m''q'^2 = 0;$$

multiplying them by x_2, y_2, z_2, and adding, we have

$$Fp'' + Gq'' + np'^2 + 2n'p'q' + n''q'^2 = 0.$$

Hence

$$\left.\begin{aligned} -p'' &= \Gamma p'^2 + 2\Gamma' p'q' + \Gamma'' q'^2 \\ -q'' &= \Delta p'^2 + 2\Delta' p'q' + \Delta'' q'^2 \end{aligned}\right\}.$$

Now

$$\frac{1}{\rho} = Lp'^2 + 2Mp'q' + Nq'^2,$$

and therefore

$$\begin{aligned} \frac{d}{ds}\left(\frac{1}{\rho}\right) &= L_1 p'^3 + (L_2 + 2M_1) p'^2 q' + (2M_2 + N_1) p'q'^2 + N_2 q'^3 \\ &\qquad + 2(Lp' + Mq') p'' + 2(Mp' + Nq') q'' \\ &= (L_1 - 2L\Gamma - 2M\Delta) p'^3 \\ &\qquad + (L_2 + 2M_1 - 4L\Gamma' - 2M\Gamma - 4M\Delta' - 2N\Delta) p'^2 q' \\ &\qquad + (2M_2 + N_1 - 2L\Gamma'' - 4M\Gamma' - 2M\Delta'' - 4N\Delta') p'q'^2 \\ &\qquad + (N_2 - 2M\Gamma'' - 2N\Delta'') q'^3, \end{aligned}$$

on substitution for p'' and q''. Having regard to the Mainardi-Codazzi relations, and reverting to the definitions of P, Q, R, S, we have values of the *derived magnitudes of the third order* in the form

$$\left.\begin{aligned} P &= L_1 - 2(L\Gamma + M\Delta) \\ Q &= L_2 - 2(L\Gamma' + M\Delta') \\ &= M_1 - (L\Gamma' + M\Delta') - (M\Gamma + N\Delta) \\ R &= M_2 - (M\Gamma' + N\Delta') - (L\Gamma'' + M\Delta'') \\ &= N_1 - 2(M\Gamma' + N\Delta') \\ S &= N_2 - 2(M\Gamma'' + N\Delta'') \end{aligned}\right\}.$$

It is not difficult to verify that these derived magnitudes of the third order satisfy the differential relations

$$\begin{aligned} P_2 - Q_1 &= 2(P\Gamma' + Q\Delta') - 2(Q\Gamma + R\Delta) + 2K(FL - EM), \\ Q_2 - R_1 &= P\Gamma'' + Q\Delta'' - (R\Gamma + S\Delta) + K(GL - EN), \\ R_2 - S_1 &= 2(Q\Gamma'' + R\Delta'') - 2(R\Gamma' + S\Delta') + 2K(GM - FN). \end{aligned}$$

41. The *derived magnitudes of the fourth order* are defined* in connection with the second derivative of the curvature along the normal section from point to point of any curve; they are such as to give

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = (\alpha, \beta, \gamma, \delta, \epsilon \between p', q')^4.$$

* See a paper by the author, *Messenger of Math.*, vol. xxxii (1903), pp. 68—80.

By a process similar to that used for the investigation of the values of P, Q, R, S, we find

$$\left.\begin{aligned}
\alpha &= P_1 - 3(P\Gamma + Q\Delta) \\
\beta &= P_2 - 3(P\Gamma' + Q\Delta') \qquad\qquad\qquad - \tfrac{3}{2}K(FL - EM) \\
&= Q_1 - 2(Q\Gamma + R\Delta) - (P\Gamma' + Q\Delta') + \tfrac{1}{2}K(FL - EM) \\
\gamma &= Q_2 - 2(Q\Gamma' + R\Delta') - (P\Gamma'' + Q\Delta'') - \tfrac{1}{2}K(GL - EN) \\
&= R_1 - 2(Q\Gamma' + R\Delta') - (R\Gamma + S\Delta) + \tfrac{1}{2}K(GL - EN) \\
\delta &= R_2 - 2(Q\Gamma'' + R\Delta'') - (R\Gamma' + S\Delta') - \tfrac{1}{2}K(GM - FN) \\
&= S_1 - 3(R\Gamma' + S\Delta') \qquad\qquad\qquad + \tfrac{3}{2}K(GM - FN) \\
\epsilon &= S_2 - 3(R\Gamma'' + S\Delta'')
\end{aligned}\right\}.$$

The derived maguitudes of any order m, thus defined, involve derivatives of x, y, z of order m.

42. The first derivatives of the Gauss measure of total curvature, and of the measure of mean curvature, can be expressed in terms of the derived magnitudes of the third order. We have

$$V^2K = LN - M^2;$$

and therefore

$$V^2K_1 = L_1N + LN_1 - 2MM_1 - 2\frac{V_1}{V}(LN - M^2).$$

When we substitute for L_1, M_1, N_1 in terms of P, Q, R and other magnitudes, also for V_1, and reduce, we find

and, similarly,

$$\left.\begin{aligned} V^2K_1 &= NP - 2MQ + LR \\ V^2K_2 &= NQ - 2MR + LS \end{aligned}\right\}.$$

In the same way for H, the measure of mean curvature, we find

$$\left.\begin{aligned} V^2H_1 &= GP - 2FQ + ER \\ V^2H_2 &= GQ - 2FR + ES \end{aligned}\right\}.$$

Cor. When a surface has the property that there is a functional relation between its principal radii of curvature, the relation can be expressed in a form

$$f(H, K) = 0,$$

or in the Jacobian form

$$H_1K_2 - H_2K_1 = 0;$$

thus its fundamental magnitudes must satisfy the equation

$$\frac{GP - 2FQ + ER}{NP - 2MQ + LR} = \frac{GQ - 2FR + ES}{NQ - 2MR + LS}.$$

Such a surface is usually called a *Weingarten surface*, and referred to as a *surface W*: some of the properties will be discussed hereafter (§§ 203—208).

43. The derivatives of x, y, z of the third order can be expressed as linear functions of X, x_1, x_2; Y, y_1, y_2; Z, z_1, z_2, the coefficients involving the derived quantities of the third order and derivatives of E, F, G. Taking the equation

$$x_{11} = LX + x_1\Gamma + x_2\Delta,$$

differentiating with respect to p, and substituting for X_1, x_{11}, x_{12} in terms of X, x_1, x_2, we have

$$x_{111} = \Lambda X + x_1\lambda + x_2\rho;$$

and so for the other derivatives. The results, after reduction, are as follows:

$$\left.\begin{aligned} x_{111} &= \Lambda X + x_1\lambda + x_2\rho \\ y_{111} &= \Lambda Y + y_1\lambda + y_2\rho \\ z_{111} &= \Lambda Z + z_1\lambda + z_2\rho \end{aligned}\right\}, \qquad \left.\begin{aligned} x_{112} &= \Lambda' X + x_1\lambda' + x_2\rho' \\ y_{112} &= \Lambda' Y + y_1\lambda' + y_2\rho' \\ z_{112} &= \Lambda' Z + z_1\lambda' + z_2\rho' \end{aligned}\right\},$$

$$\left.\begin{aligned} x_{122} &= \Lambda'' X + x_1\lambda'' + x_2\rho'' \\ y_{122} &= \Lambda'' Y + y_1\lambda'' + y_2\rho'' \\ z_{122} &= \Lambda'' Z + z_1\lambda'' + z_2\rho'' \end{aligned}\right\}, \qquad \left.\begin{aligned} x_{222} &= \Lambda''' X + x_1\lambda''' + x_2\rho''' \\ y_{222} &= \Lambda''' Y + y_1\lambda''' + y_2\rho''' \\ z_{222} &= \Lambda''' Z + z_1\lambda''' + z_2\rho''' \end{aligned}\right\},$$

where the coefficients are given by the equations

$$\left.\begin{aligned} \Lambda \;\; &= P + 3(L\Gamma + M\Delta) \\ \Lambda' \; &= Q + M\Gamma + N\Delta + 2(L\Gamma' + M\Delta') \\ \Lambda'' &= R + 2(M\Gamma' + N\Delta') + L\Gamma'' + M\Delta'' \\ \Lambda''' &= S + 3(M\Gamma'' + N\Delta'') \end{aligned}\right\},$$

$$\left.\begin{aligned} E\lambda + F\rho &= m_1 - L^2 - \frac{1}{V^2}\{En^2 - 2Fnm + Gm^2\} \\ F\lambda + G\rho &= n_1 - LM - \frac{1}{V^2}\{Enn' - F(nm' + n'm) + Gmm'\} \end{aligned}\right\},$$

$$\left.\begin{aligned} E\lambda' + F\rho' &= m_2 - LM - \frac{1}{V^2}\{Enn' - F(nm' + n'm) + Gmm'\} \\ F\lambda' + G\rho' &= n_2 - LN - \frac{1}{V^2}\{Enn'' - F(nm'' + n''m) + Gmm''\} \\ &= n_1 - M^2 - \frac{1}{V^2}\{En'^2 - 2Fn'm' + Gm'^2\} \end{aligned}\right\},$$

$$\left.\begin{aligned} E\lambda'' + F\rho'' &= m_1'' - LN - \frac{1}{V^2}\{Enn'' - F(nm'' + n''m) + Gmm''\} \\ &= m_2' - M^2 - \frac{1}{V^2}\{En'^2 - 2Fn'm' + Gm'^2\} \\ F\lambda'' + G\rho'' &= n_2' - MN - \frac{1}{V^2}\{En'n'' - F(n'm'' + n''m') + Gm'm''\} \end{aligned}\right\},$$

$$\left.\begin{aligned} E\lambda''' + F\rho''' &= m_2'' - MN - \frac{1}{V^2}\{En'n'' - F(n'm'' + n''m') + Gm'm''\} \\ F\lambda''' + G\rho''' &= n_2'' - N^2 - \frac{1}{V^2}\{En''^2 - 2Fn''m'' + Gm''^2\} \end{aligned}\right\}.$$

It is easy to see that the derivatives of x, y, z, of any order greater than unity, can be expressed similarly as linear combinations of X, x_1, x_2; the coefficients in the combinations involve the derived magnitudes and derivatives of the fundamental magnitudes E, F, G.

EXAMPLES.

1. Two directions at a point P on a surface are given by

$$\Theta\, dp^2 + 2\Phi\, dp\, dq + \Psi\, dq^2 = 0,$$

and a third direction is given by $\delta p, \delta q, \delta s$, making angles α and β with the former directions. Shew that, if

$$J = (E\delta p + F\delta q)^2\Psi - 2(E\delta p + F\delta q)(F\delta p + G\delta q)\Phi + (F\delta p + G\delta q)^2\Theta,$$

$$I = \{(E\Psi - 2F\Phi + G\Theta)^2 - 4(EG - F^2)(\Theta\Psi - \Phi^2)\}^{\frac{1}{2}},$$

then

$$\delta s^2 \cos\alpha\cos\beta = J/I,$$

$$\delta s^2 \sin\alpha\sin\beta = -\frac{V^2}{I}(\Theta\,\delta p^2 + 2\Phi\,\delta p\,\delta q + \Psi\,\delta q^2),$$

and

$$\delta s^2 \sin(\alpha - \beta) = \frac{2V}{I}\begin{vmatrix} E\delta p + F\delta q, & F\delta p + G\delta q \\ \Theta\delta p + \Phi\delta q, & \Phi\delta p + \Psi\delta q \end{vmatrix}.$$

2. Shew analytically that, if L, M, N vanish everywhere on a surface, the surface is plane.

3. A surface is given by a Cartesian equation in the form

$$z = f(x, y);$$

the partial derivatives of z are denoted, as usual, by $p, q; r, s, t$. Shew that

$$E = 1 + p^2, \quad F = pq, \quad G = 1 + q^2, \quad V^2 = 1 + p^2 + q^2;$$

$$\frac{X}{-p} = \frac{Y}{-q} = \frac{Z}{1} = \frac{1}{(1 + p^2 + q^2)^{\frac{1}{2}}};$$

$$\frac{L}{r} = \frac{M}{s} = \frac{N}{t} = \frac{1}{(1 + p^2 + q^2)^{\frac{1}{2}}}, \quad T^2 = \frac{rt - s^2}{1 + p^2 + q^2};$$

$$H = \frac{(1 + q^2)r - 2pqs + (1 + p^2)t}{(1 + p^2 + q^2)^{\frac{3}{2}}}, \quad K = \frac{rt - s^2}{(1 + p^2 + q^2)^2};$$

and obtain expressions for the derived magnitudes of the third order.

4. A surface is given by the equation $F(x, y, z) = 0$, so that the direction-cosines of the normal are given by

$$\frac{1}{X}\frac{\partial F}{\partial x} = \frac{1}{Y}\frac{\partial F}{\partial y} = \frac{1}{Z}\frac{\partial F}{\partial z} = \left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right\}^{\frac{1}{2}}.$$

Shew that the mean curvature H is

$$-\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) + \frac{X}{Z}\frac{\partial X}{\partial z} + \frac{Y}{Z}\frac{\partial Y}{\partial z},$$

with two similar expressions, and the total curvature K is

$$\frac{1}{X}\left\{X\frac{\partial(Y, Z)}{\partial(y, z)}+Y\frac{\partial(Z, X)}{\partial(z, x)}+Z\frac{\partial(X, Y)}{\partial(x, y)}\right\},$$

with two similar expressions.

5. The parametric curves are orthogonal, and a curve is drawn on the surface making a constant angle a with the curve p=constant; shew that the differential equation of the curve is

$$\frac{dp}{dq}=\left(\frac{G}{E}\right)^{\frac{1}{2}}\tan a.$$

6. Obtain the equation of the lines of curvature in the form

$$\begin{vmatrix} dX, & dY, & dZ \\ dx, & dy, & dz \\ X, & Y, & Z \end{vmatrix}=0;$$

and shew that, if

$$u=yZ-zY, \quad v=zX-xZ, \quad w=xY-yX,$$

(so that X, Y, Z, u, v, w are the six coordinates of the normal), the differential equation of the lines of curvature is

$$du\,dX+dv\,dY+dw\,dZ=0.$$

7. Shew that, at any point of a surface,

$$\left.\begin{aligned}\Sigma x_{111}{}^2&=\Lambda^2 +E\lambda^2 +2F\lambda\rho +G\rho^2\\ \Sigma x_{112}{}^2&=\Lambda'^2 +E\lambda'^2 +2F\lambda'\rho' +G\rho'^2\\ \Sigma x_{122}{}^2&=\Lambda''^2 +E\lambda''^2 +2F\lambda''\rho'' +G\rho''^2\\ \Sigma x_{222}{}^2&=\Lambda'''^2+E\lambda'''^2+2F\lambda'''\rho'''+G\rho'''^2\end{aligned}\right\},$$

where the symbols on the right-hand side have the same significance as in § 43.

8. A surface is given by the equations

$$\frac{x}{a}=\frac{1+uv}{u+v}, \quad \frac{y}{b}=\frac{u-v}{u+v}, \quad \frac{z}{c}=\frac{1-uv}{u+v};$$

shew that the equations of the lines of curvature are

$$(1-2au^2+u^4)^{-\frac{1}{2}}\,du \pm (1-2av^2+v^4)^{-\frac{1}{2}}\,dv=0,$$

where $a=(a^2-2b^2+c^2)/(a^2+c^2)$; and obtain an expression for the total curvature.

9. A skew surface is generated by the binormals of a curve. Prove that, at a point on a generator distant d from the curve, the total curvature is $-\sigma^2(\sigma^2+d^2)^{-2}$; and that, at the curve itself, the principal radii of curvature of the surface are given by the equation

$$\frac{r^2}{\sigma^2}+\frac{r}{\rho}-1=0.$$

10. A skew surface is generated by the radii of spherical curvature of a curve. Shew that, at the centre of spherical curvature of the curve, the total curvature of the surface is $-\sigma_1{}^{-2}$, where σ_1 is the radius of torsion of the locus of the centre of spherical curvature. Shew also that, at the curve itself, the total curvature of the surface is $-\sigma_1{}^2R^{-4}$, where R is the radius of spherical curvature.

11. A surface is generated by straight lines which meet the lines $y = \pm x \tan a$, $z = \pm\frac{1}{2}c$; and the intercepts between the axis of z and the points where the generator meets the lines are u and v, functions of a parameter. Shew that, at a point on the surface in the plane of xy, the total curvature is

$$\frac{-16c^2 u'^2 v'^2 \sin^2 2a}{\{c^2(u'^2 + 2u'v' \cos 2a + v'^2) + (uv' + u'v)^2 \sin^2 2a\}^2};$$

and that, at a point on the surface in the first line, the principal radii of curvature are given by the equation

$$\frac{1}{r^2} + \frac{2c \sin 2a}{r} \frac{u - v \cos 2a}{(c^2 + v^2 \sin^2 2a)^{\frac{3}{2}}} \frac{dv}{du} - \frac{c^2 \sin^2 2a}{(c^2 + v^2 \sin^2 2a)^2} \left(\frac{dv}{du}\right)^2 = 0.$$

12. A surface is given by the equations

$$x = \sin u\,(\cosh^2 v - \cos a \cos u \cosh v - 2\cos^2 a),$$
$$y = \sinh v\,(\cos^2 a \cos^2 u - \cos a \cos u \cosh v - 2),$$
$$z = \sin a \cos u \cosh v\,(\cosh v - \cos a \cos u).$$

Prove that the curves of reference are the lines of curvature, and that the principal radii of curvature are

$$(2\cos a \cos u - \cosh v)(\cosh v + \cos a \cos u)^2 \operatorname{cosec} a,$$
$$(2\cosh v - \cos a \cos u)(\cosh v + \cos a \cos u)^2 \operatorname{cosec} a.$$

CHAPTER III.

ORGANIC CURVES OF A SURFACE.

THE present chapter is occupied with an account of the chief curves upon a surface, with which they have organic relations as being determined, mainly or partly, by the nature of the surface itself. There will be no elaborate discussion of the properties and characteristics of any of them, though a more detailed treatment of two classes of them, viz. lines of curvature and geodesics, will be found in later chapters.

For the immediate purpose, reference may be made to the first volume of Darboux's treatise, particularly to the first three chapters of the second Book. The subject-matter is discussed in the third chapter of Bianchi's treatise, and in the second section of Knoblauch's treatise, as well as in the first section of Stahl and Kommerell's *Die Grundformeln der allgemeinen Flächentheorie*.

Orthogonal Curves.

44. We have seen (§ 25) that the angle ω between the parametric curves at a point on a surface is given by

$$\cos \omega = F(EG)^{-\frac{1}{2}}.$$

Hence the curves are perpendicular at a point if $F=0$ at the point; and they are perpendicular everywhere if $F=0$ over the surface.

In the latter case, they often are called an *orthogonal system*; and $F=0$ is the sole condition, necessary and sufficient to secure that the parametric curves form such a system.

Lines of Curvature.

45. When the surface is referred to any system of parametric curves $p=$ const., $q=$ const., and when the fundamental magnitudes of the surface of the first order and the second order are denoted by E, F, G; L, M, N; the directions of the lines of curvature through a point upon the surface are given by

$$\begin{vmatrix} Edp+Fdq, & Fdp+Gdq \\ Ldp+Mdq, & Mdp+Ndq \end{vmatrix} = 0.$$

If the parametric curves are themselves lines of curvature, the foregoing equation must (as an equation for directions) be equivalent to

$$dpdq=0.$$

Hence we must have

$$EN - GL \neq 0,$$

$$EM - FL = 0, \quad FN - GM = 0.$$

From the last two equations, we have

$$(EN - GL)M = 0, \quad (EN - GL)F = 0,$$

and therefore

$$M = 0, \quad F = 0,$$

are the conditions that the parametric curves should be lines of curvature.

When the conditions are satisfied, the radius of curvature for $p =$ constant, say α, and the radius of curvature for $q =$ constant, say β, are

$$\alpha = G/N, \quad \beta = E/L.$$

The conditions $F = 0$, $M = 0$, are necessary and sufficient to secure that the parametric curves are lines of curvature. The condition $F = 0$ makes the parametric curves an orthogonal system; the new condition $M = 0$ makes them the special orthogonal system constituted by lines of curvature.

46. Two well-known theorems can be stated in connection with the general formulæ of the preceding chapter.

Take any curve on the surface. Let $1/r$ be its circular curvature; and let $1/\rho$ be the curvature (defined as in § 31) of the normal section of the surface through the tangent to the curve. The direction-cosines of the principal normal to the curve are rx'', ry'', rz''; hence, if θ be the angle between this principal normal and the normal to the surface, we have

$$\cos\theta = X.rx'' + Y.ry'' + Z.rz''.$$

But

$$x'' = x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2 + x_1 p'' + x_2 q'',$$

and similarly for y'' and z''; hence

$$\frac{\cos\theta}{r} = Xx'' + Yy'' + Zz'' = Lp'^2 + 2Mp'q' + Nq'^2 = \frac{1}{\rho}.$$

This is Meunier's theorem.

Next, at any point take a normal section of the surface through a direction making an angle ψ with the line of curvature $p =$ constant. Let the surface be referred to the lines of curvature as parametric curves, so that $F = 0$, $M = 0$; then

$$\cos\psi = G^{\frac{1}{2}}\frac{dq}{ds}, \quad \sin\psi = E^{\frac{1}{2}}\frac{dp}{ds}.$$

The radius of curvature of the normal section is given by

$$\frac{1}{\rho} = L\left(\frac{dp}{ds}\right)^2 + N\left(\frac{dq}{ds}\right)^2 = \frac{L}{E}\sin^2\psi + \frac{N}{G}\cos^2\psi = \frac{\cos^2\psi}{\alpha} + \frac{\sin^2\psi}{\beta},$$

which is Euler's theorem on the curvature of a normal section through any direction not coinciding with a line of curvature*.

The relations of the indicatrix

$$\frac{\xi^2}{\alpha}+\frac{\eta^2}{\beta}=1$$

to the curvature are at once suggested; they are discussed in text-books on solid geometry. Only one remark need be made here, for ulterior use. If the indicatrix is an ellipse, ρ is finite for every normal section. If the indicatrix is a hyperbola, ρ is infinite for each of the directions $\psi=\tan^{-1}(-\beta/\alpha)^{\frac{1}{2}}$, which are the directions of the asymptotes of the indicatrix.

The detailed development of the analysis connected with lines of curvature and with associated properties will be deferred until the next chapter.

Conjugate Directions.

47. The familiar notion of conjugate diameters (or conjugate directions) in a central conic can be extended, through the indicatrix, so as to give rise to the notion of *conjugate directions* (and conjugate lines) on a surface†. In the case of a conic with centre C, a direction given by two points P and Q on the curve is conjugate to CR, where the tangents at P and Q intersect in R; and the definition makes an asymptote of the conic conjugate to itself.

In the case of a surface, let a line PR be drawn through a point P parallel to the intersection of the tangent plane at P with the tangent plane at a point Q; when Q tends to coincide with P (or becomes a point consecutive to P), the limiting positions of PQ and PR are called *conjugate directions* at P.

The condition that two directions dp, dq; and dp', dq'; are conjugate can be deduced from the definition. Let PQ be the direction dp, dq; and let PR be the direction dp', dq', so that PR is parallel to the ultimate intersection of the tangent planes at P and at Q. Let P be the point x, y, z; the tangent plane at P is

$$(\xi-x)X+(\eta-y)Y+(\zeta-z)Z=0,$$

where ξ, η, ζ are current coordinates. The tangent plane at Q, say $x+dx$, $y+dy$, $z+dz$, is

$$(\xi-x-dx)(X+dX)+(\eta-y-dy)(Y+dY)+(\zeta-z-dz)(Z+dZ)=0.$$

But

$$X\,dx+Y dy+Z dz=0,$$

and therefore the latter equation can be taken in the form

$$\Sigma\,\{(\xi-x)(X+dX)-dx\,dX\}=0.$$

* The result can be established from the quite general formulæ, without any special choice of parametric curves.

† The extension originated with Dupin, *Développements de géométrie* (1813), p. 91.

The quantity $dxdX$ is of the second order; hence the line of intersection of the two planes is given by

$$\left.\begin{aligned}(\xi - x) X \;\; + (\eta - y) Y \;\; + (\zeta - z) Z \;\; &= 0\\ (\xi - x)\, dX + (\eta - y)\, dY + (\zeta - z)\, dZ &= 0\end{aligned}\right\}.$$

Let a point $x + dx'$, $y + dy'$, $z + dz'$ be taken on the surface, so that it lies ultimately on the direction PR; thus dx', dy', dz' determine the direction PR, and for that direction

$$\xi - x : \eta - y : \zeta - z = dx' : dy' : dz'.$$

The first equation is satisfied identically. The second equation is

$$dx'dX + dy'dY + dz'dZ = 0,$$

which accordingly is the equation to be satisfied by the direction conjugate to PQ. Now

$$dx' = x_1 dp' + x_2 dq', \quad dX = X_1 dp + X_2 dq;$$

and therefore

$$(\Sigma x_1 X_1)\, dp dp' + (\Sigma x_2 X_1)\, dp dq' + (\Sigma x_1 X_2)\, dq dp' + (\Sigma x_2 X_2)\, dq dq' = 0,$$

that is,

$$L dp dp' + M (dp dq' + dq dp') + N dq dq' = 0.$$

This is the *condition that the two directions should be conjugate to one another.* As the analysis manifestly is reversible, the condition is seen to be sufficient as well as necessary.

The symmetry of the condition between dp, dq; and dp', dq'; justifies the assumption in the phraseology that the conjugate property is reciprocal. When written in the form

$$(L dp + M dq)\, dp' + (M dp + N dq)\, dq' = 0,$$

the condition shews that the two directions are conjugate diameters of the indicatrix; for the equation of the indicatrix is

$$\frac{L}{E}\xi^2 + \frac{2M}{(EG)^{\frac{1}{2}}}\xi\eta + \frac{N}{G}\eta^2 = 1.$$

Thus only a single condition requires to be satisfied in order that two directions may be conjugate. Hence one direction can be taken arbitrarily, and the other is then determined by the condition; and it is uniquely determined, for the condition is lineo-linear in the quantities dp/dq, dp'/dq'. Thus let a curve be

$$\phi(p, q) = \text{constant},$$

so that we have a family of curves when the constant is parametric; the direction of the curve at a point is given by

$$\frac{\partial \phi}{\partial p}\delta p + \frac{\partial \phi}{\partial q}\delta q = 0.$$

But the direction dp/dq, conjugate to $\delta p/\delta q$, is given by

$$(L dp + M dq)\, \delta p + (M dp + N dq)\, \delta q = 0;$$

that is, the conjugate direction at the point satisfies the equation

$$(Ldp + Mdq)\frac{\partial\phi}{\partial q} - (Mdp + Ndq)\frac{\partial\phi}{\partial p} = 0,$$

or, what is the same thing,

$$L\frac{\partial\phi}{\partial q} - M\frac{\partial\phi}{\partial p} + \left(M\frac{\partial\phi}{\partial q} - N\frac{\partial\phi}{\partial p}\right)\frac{dq}{dp} = 0,$$

an ordinary differential equation of the first order, the primitive of which gives a family of lines conjugate to the family $\phi(p, q) = $ constant. If this conjugate family be $\psi(p, q) = $ constant, then

$$L\frac{\partial\phi}{\partial q}\frac{\partial\psi}{\partial q} - M\left(\frac{\partial\phi}{\partial p}\frac{\partial\psi}{\partial q} + \frac{\partial\phi}{\partial q}\frac{\partial\psi}{\partial p}\right) + N\frac{\partial\phi}{\partial p}\frac{\partial\psi}{\partial p} = 0.$$

48. When two directions δp, δq; $\delta p'$, $\delta q'$; satisfy an equation

$$Adp^2 + 2Bdpdq + Cdq^2 = 0,$$

then

$$\frac{\delta p\,\delta p'}{C} = \frac{\delta p\,\delta q' + \delta q\,\delta p'}{-2B} = \frac{\delta q\,\delta q'}{A};$$

and therefore the condition that the two directions should be conjugate is

$$CL - 2BM + AN = 0.$$

The condition is sufficient, as well as necessary, to secure the conjugate character.

The parametric curves $p = a$, $q = b$, where a and b are constant, are given by

$$dpdq = 0.$$

Taking $A = 0$, $C = 0$, and B not zero, we infer that the condition necessary and sufficient to make *the parametric curves a conjugate system* is

$$M = 0.$$

In particular, the lines of curvature are conjugate to one another. This is a consequence of the fact that when the lines of curvature are made parametric curves, then one of the conditions is $M = 0$, which makes them conjugate; and it can also be verified from their general equation

$$(EM - FL)\,dp^2 + (EN - GL)\,dpdq + (FN - GM)\,dq^2 = 0,$$

by making $A, B, C = EM - FL,\ EN - GL,\ FN - GM$, respectively, in the foregoing relation.

The direction dp', dq', which is conjugate to dp, dq, is given by

$$\frac{dp'}{Mdp + Ndq} = \frac{dq'}{-(Ldp + Mdq)} = \theta,$$

say, where

$$\begin{aligned}\frac{ds'^2}{\theta^2} = (EM^2 - 2FLM + GL^2)\,dp^2 &\\ + 2(EMN - FLN - FM^2 + GLM)\,dpdq &\\ + (EN^2 - 2FMN + GM^2)\,dq^2 &= \Theta^2,\end{aligned}$$

so that

$$\theta = \frac{ds'}{\Theta}.$$

Let χ denote the angle between the two conjugate directions; then

$$dsds' \cos\chi = Edpdp' + F(dpdq' + dqdp') + Gdqdq'$$
$$= \theta\{(Edp + Fdq)(Mdp + Ndq) - (Fdp + Gdq)(Ldp + Mdq)\},$$

that is,

$$\Theta ds \cos\chi = (EM - FL)\,dp^2 + (EN - GL)\,dpdq + (FN - GM)\,dq^2.$$

Manifestly the only conjugate directions perpendicular to one another are the lines of curvature.

49. The equations relating to surfaces in general, as obtained in the preceding chapter, were constructed for any unspecified system of parametric curves. When any particular specification is introduced, some corresponding simplification in the equations may be caused. When the parametric curves are conjugate, we have

$$M = 0$$

in all the equations. This causes a special simplification in three of the partial differential equations (of § 34) satisfied by the coordinates, viz. those involving x_{12}, y_{12}, z_{12}. When the parametric curves are conjugate, these equations are

$$x_{12} = x_1\Gamma' + x_2\Delta', \quad y_{12} = y_1\Gamma' + y_2\Delta', \quad z_{12} = z_1\Gamma' + z_2\Delta';$$

that is, x, y, z are three solutions of the equation

$$\frac{\partial^2\theta}{\partial p\partial q} = \Gamma'\frac{\partial\theta}{\partial p} + \Delta'\frac{\partial\theta}{\partial q}.$$

This is a linear partial differential equation of the second order, and usually is called Laplace's equation, being written in the form (with the customary notation for partial differential equations)

$$s = ap + bq,$$

where a and b are functions of the independent variables.

The primitive* of this equation involves two arbitrary functions; hence the most general values of x, of y, and of z, obtained solely as integrals of the equations, are expressions each of which involves linearly an arbitrary function of p and an arbitrary function of q. The arbitrary functions are not unrelated; for assuming Γ' and Δ' known, we have

$$V^2\Gamma' = \tfrac{1}{2}GE_2 - \tfrac{1}{2}FG_1, \quad V^2\Delta' = \tfrac{1}{2}EG_1 - \tfrac{1}{2}FE_2,$$

as well as the Gauss characteristic equation and the Mainardi-Codazzi

* For the general theory of Laplace's linear equation, and for the special construction of the primitive when the latter can be expressed in finite terms involving the two arbitrary functions, see the author's *Theory of Differential Equations*, vol. vi, chap. xiii.

relations; and all these conditions must be satisfied when the values of x, y, z are substituted.

Even so, a large amount of arbitrary generality will survive in the complete solution which, through this stage at least, is rather an investigation in partial equations than in geometry. Geometrical applications arise by the assignment of further conditions; and illustrations of these, in connection with particular surfaces or families of surfaces, will be given from time to time.

The amplest discussion of the equation, together with many comprehensive applications to surfaces, will be found in the second volume of Darboux's treatise.

50. One particular family of surfaces, referred to conjugate lines as parametric curves, is instanced by Darboux* in the form

$$x = A(p-a)^m (q-a)^n, \quad y = B(p-b)^m (q-b)^n, \quad z = C(p-c)^m (q-c)^n,$$

where A, a, B, b, C, c, m, n are constants. It is easy to verify that

$$\left.\begin{aligned}(q-p)\,x_{12} &= nx_1 - mx_2\\ (q-p)\,y_{12} &= ny_1 - my_2\\ (q-p)\,z_{12} &= nz_1 - mz_2\end{aligned}\right\},$$

so that, multiplying these equations by X, Y, Z respectively, and adding, we have

$$M = 0;$$

and therefore the parametric curves given by $p =$ constant, $q =$ constant, are conjugate.

More generally, the direction on the surface given by δp, δq is conjugate to the direction given by dp, dq, if

$$\frac{m(m-1)\,dp\,\delta p}{(p-a)(p-b)(p-c)} = \frac{n(n-1)\,dq\,\delta q}{(q-a)(q-b(q-c)}.$$

The family includes many important sets of surfaces. When $m = n$, we have the "tetrahedral" surfaces

$$\left(\frac{x}{A}\right)^{1/m}(b-c) + \left(\frac{y}{B}\right)^{1/m}(c-a) + \left(\frac{z}{C}\right)^{1/m}(a-b) = (a-b)(b-c)(c-a),$$

special cases of which arise, in Steiner's surface for $m = n = 2$, in a well-known cubic surface for $m = n = -1$, and in the trivial plane for $m = n = 1$. When $m = \frac{3}{2}$, $n = \frac{1}{2}$, we have (as will be seen later) the surface of centres of the ellipsoid; when $m = \frac{1}{2}$, $n = \frac{1}{2}$, we have an ellipsoid; and so for other special values of m and n.

* *Théorie générale*, t. i, p. 142.

Asymptotic Lines.

51. Various definitions of *asymptotic lines* are given, according to the property selected to characterise them. They are associated simply with conjugate directions; an asymptotic line is then defined as a curve on the surface whose direction at any point is *self-conjugate*. The direction dp'/dq' must then be the same as dp/dq; that is, a self-conjugate direction is given by either of the equations

$$dxdX + dydY + dzdZ = 0,$$

$$Ldp^2 + 2Mdpdq + Ndq^2 = 0.$$

Consequently there are generally two asymptotic directions at any point of a surface. When the total curvature is positive (so that $LN > M^2$), the directions are imaginary and different; when it is negative, they are real and different; when it is zero (so that the surface is developable), there is only a single asymptotic line through a point, and it is the generator.

The curvature of the normal section of a surface through the tangent to an asymptotic line, being

$$Lp'^2 + 2Mp'q' + Nq'^2,$$

is zero. The tangent to the line then coincides with an asymptote of the indicatrix at the point; hence the name.

Consider the tangent plane at a point (x, y, z) on the surface. The distance of a neighbouring point $x + dx,\ y + dy,\ z + dz$ from that plane

$$= Xdx + Ydy + Zdz$$

$$= \tfrac{1}{2}(Ldp^2 + 2Mdpdq + Ndq^2) + \text{other terms},$$

the other terms being of the third and higher orders in dp and dq. Thus any tangent to the surface, being a straight line in the tangent plane, meets the surface in two consecutive points; but a tangent to the surface in the direction of an asymptotic line meets the surface in three consecutive points. The directions of the asymptotic lines are therefore often called the *inflexional tangents* at the point.

Another method of stating the last result is to declare that the asymptotic directions are the tangents to the curve of intersection (which has the point for a double point) of the surface by its tangent plane. On a hyperboloid of one sheet, they are of course the generators.

Let χ denote the angle between the asymptotic lines

$$Ldp^2 + 2Mdpdq + Ndq^2 = 0$$

at any point; then (§ 26)

$$\tan\chi = \frac{2V(M^2 - LN)^{\frac{1}{2}}}{EN - 2FM + GL}$$

$$= 2i\,\frac{K^{\frac{1}{2}}}{H},$$

or writing

$$K = \frac{1}{\alpha\beta}, \qquad H = \frac{1}{\alpha} + \frac{1}{\beta},$$

where α and β are the principal radii of curvature at the point, we have

$$\cos \chi = \frac{\alpha + \beta}{\alpha \sim \beta}.$$

If the asymptotic lines are everywhere perpendicular on a surface, so that $\chi = \frac{1}{2}\pi$, then

$$EN - 2FM + GL = 0,$$

that is, the surface is minimal.

52. The analytic determination of the asymptotic lines upon a surface can be made to depend upon the integration of the differential equation

$$Ldp^2 + 2Mdpdq + Ndq^2 = 0,$$

of the first order and second degree. For any surface, what is required in this mode of determination of the asymptotic lines is the preliminary construction of the quantities L, M, N.

Ex. 1. On a sphere, we can take

$$x = \cos p \cos q, \quad y = \cos p \sin q, \quad z = \sin p;$$

and then, by simple calculations, we have

$$L = -1, \quad M = 0, \quad N = -1,$$

so that the asymptotic directions (being imaginary, for the sphere is synclastic) are given by

$$dp^2 + dq^2 = 0,$$

that is, they are

$$p + iq = \text{const.}, \quad p - iq = \text{const.}$$

Ex. 2. Prove that, at the origin on any surface

$$2z = ax^2 + 2hxy + by^2 + \text{terms of higher order in } x \text{ and } y,$$

the asymptotic directions are

$$ax^2 + 2hxy + by^2 = 0.$$

Ex. 3. As a last example, consider the asymptotic lines on Fresnel's wave-surface*

$$\frac{x^2}{r^2 - a} + \frac{y^2}{r^2 - b} + \frac{z^2}{r^2 - c} = 1,$$

where $r^2 = x^2 + y^2 + z^2$. We have

$$\frac{ax^2}{r^2 - a} + \frac{by^2}{r^2 - b} + \frac{cz^2}{r^2 - c} = 0;$$

and so we take

$$r^2 = x^2 + y^2 + z^2 = q,$$

$$\frac{ax^2}{\theta - a} + \frac{by^2}{\theta - b} + \frac{cz^2}{\theta - c} = \frac{abc}{p} \frac{(\theta - p)(\theta - q)}{(\theta - a)(\theta - b)(\theta - c)},$$

as equations defining the parameters p and q, the latter equation being satisfied identically for all values of θ.

* For a full discussion, see Note VIII at the end of Darboux's fourth volume.

We then have

$$x^2 = \frac{bc}{p}\frac{(a-q)(a-p)}{(a-b)(a-c)},$$

with similar expressions for y and z; hence

$$\left.\begin{aligned} \frac{x_1}{x} &= -\frac{1}{2}\frac{a}{p(a-p)} \\ \frac{x_2}{x} &= -\frac{1}{2}\frac{1}{a-q} \end{aligned}\right\},$$

$$\left.\begin{aligned} \frac{x_{11}}{x} &= -\frac{1}{4}\frac{(4p-3a)a}{p^2(a-p)^2} \\ \frac{x_{12}}{x} &= \frac{1}{4}\frac{a}{p(a-p)(a-q)} \\ \frac{x_{22}}{x} &= -\frac{1}{4}\frac{1}{(a-q)^2} \end{aligned}\right\},$$

$$Xx = W(b-c)\{bc-(b+c)p+pq\}(a-p)(a-q),$$

where W is a multiplier, common to X, Y, Z, the value of which is immaterial at present. Then, writing

$$U = (a-b)(b-c)(c-a),$$

we have

$$\begin{aligned} L &= \Sigma X x_{11} \\ &= \Sigma Xx \,.\, \frac{x_{11}}{x} \\ &= -\tfrac{1}{4} WU \frac{p^2\{q^2-(a+b+c)q+ab+bc+ca\}+abc(q-2p)}{p(a-p)(b-p)(c-p)}, \\ M &= \Sigma X x_{12} \\ &= -\tfrac{1}{4} WU, \\ N &= \Sigma X x_{22} \\ &= -\tfrac{1}{4} WU \frac{-abc+(ab+bc+ca)p-(a+b+c)p^2-pq(q-2p)}{(a-q)(b-q)(c-q)}. \end{aligned}$$

Inserting the values of L, M, N, we at once have the differential equation for the asymptotic lines.

Darboux (*l.c.*) shews that, by introducing a parameter s in place of p, defined by the relation

$$p(p-q)(p-s) = (p-a)(p-b)(p-c),$$

the differential equation can be obtained in the form

$$\frac{ds^2}{(s-a)(s-b)(s-c)} = \frac{dq^2}{(q-a)(q-b)(q-c)},$$

the primitive of which can be expressed algebraically.

53. The conditions that the parametric curves should be asymptotic lines are easily derived from their general equation

$$L\,dp^2 + 2M\,dp\,dq + N\,dq^2 = 0.$$

If these are the parametric curves $p = a$, $q = b$, the equation must effectively be the same as

$$dp\,dq = 0,$$

and therefore we must have

$$L = 0, \quad N = 0, \quad M \neq 0.$$

(The complete difference between the condition, that the parametric curves should be two different conjugate directions, and the conditions, that they should be two self-conjugate directions, will be noticed.)

Moreover, as the analysis manifestly is reversible, the conditions are sufficient as well as necessary.

As the asymptotic lines are a couple of systems of curves distinct from one another on all surfaces which are not developable, it frequently is convenient to choose them as the parametric curves of reference. In that choice, we make $L=0$ and $N=0$ in all the general equations which have been obtained; consequently there is much simplification in the forms of those equations. Thus the Mainardi-Codazzi relations take the form

$$M_1 = (\Gamma - \Delta')M, \quad M_2 = (\Delta'' - \Gamma')M.$$

The equation of the lines of curvature (on dropping the non-zero factor M) becomes

$$E\,dp^2 - G\,dq^2 = 0.$$

We have

$$\frac{V_1}{V} = \Gamma + \Delta', \qquad \frac{V_2}{V} = \Gamma' + \Delta'',$$

always, and therefore (for this particular reference)

$$2\Gamma = \frac{M_1}{M} + \frac{V_1}{V}, \quad 2\Delta' = -\frac{M_1}{M} + \frac{V_1}{V},$$

$$2\Gamma' = -\frac{M_2}{M} + \frac{V_2}{V}, \quad 2\Delta'' = \frac{M_2}{M} + \frac{V_2}{V},$$

while, as always,

$$V^2\Gamma'' = G(F_2 - \tfrac{1}{2}G_1) - \tfrac{1}{2}FG_2, \quad V^2\Delta = E(F_1 - \tfrac{1}{2}E_2) - \tfrac{1}{2}FE_1;$$

and the measures of curvature are

$$H = -2\frac{FM}{V^2}, \quad K = -\frac{M^2}{V^2}.$$

54. To illustrate the use of asymptotic lines as curves of reference, consider surfaces of constant negative total curvature (often called *pseudo-spherical*). The asymptotic lines are real and, assuming the measure not to be zero, are distinct from one another; so they are convenient curves of reference. We have, from the definition,

$$K = -\frac{1}{\mu^2},$$

where μ is a real constant; hence

$$\frac{M}{V} = \frac{1}{\mu},$$

so that

$$\frac{M_1}{M} - \frac{V_1}{V} = 0, \quad \text{that is, } \Delta' = 0,$$

and

$$\frac{M_2}{M} - \frac{V_2}{V} = 0, \quad \text{that is, } \Gamma' = 0.$$

Reverting to the definitions of Γ' and Δ', we have

$$m'G - n'F = 0, \quad n'E - m'F = 0,$$

and therefore

$$m' = 0, \quad n' = 0,$$

that is,

$$E_2 = 0, \quad G_1 = 0.$$

Consequently E is a function of p only; by changing the variable p to p' where $E^{\frac{1}{2}} dp = dp'$, the new quantity in place of E is unity, and therefore without any loss of generality, we can take $E = 1$ for the surface. Similarly, as G is a function of q only, we can take $G = 1$ without any loss of generality for the surface.

The angle between the parametric curves has been denoted by ω; hence in the present case

$$F = \cos \omega, \quad V = \sin \omega,$$

and therefore

$$M = \frac{V}{\mu} = \frac{\sin \omega}{\mu}.$$

Now for any surface we have (§ 36)

$$-\frac{\partial^2 \omega}{\partial p \partial q} = \frac{\partial}{\partial p}\left(\frac{V\Delta'}{E}\right) + \frac{\partial}{\partial q}\left(\frac{V\Gamma'}{G}\right) + VK;$$

and therefore, in the present case, we have

$$\frac{\partial^2 \omega}{\partial p \partial q} = \frac{1}{\mu^2} \sin \omega,$$

where μ is a real constant.

Upon the integration of this equation, the determination of the most general pseudosphere rests. When any solution is found giving ω in terms of p and q, we then know

$$E(=1), \quad G(=1), \quad L(=0), \quad N(=0), \quad F(=\cos \omega), \quad M(=\mu^{-1} \sin \omega);$$

that is, by Bonnet's theorem, we have a pseudosphere completely determined save as to position and orientation. Thus, as so often happens in the differential geometry of surfaces, the solution of the problem depends upon the integration of a partial differential equation of the second order. The primitive of the equation

$$\frac{\partial^2 \omega}{\partial p \partial q} = \frac{1}{\mu^2} \sin \omega$$

has not yet been obtained.

The lines of curvature upon the pseudosphere are given by

$$dp^2 - dq^2 = 0,$$

that is, they are given by

$$p + q = u = \text{constant}, \quad p - q = v = \text{constant}.$$

When the surface is referred to the lines of curvature as parametric curves instead of the asymptotic lines, so that u and v are the variables of reference, the preceding differential equation of the surface becomes

$$\frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \frac{1}{\mu^2} \sin \omega.$$

Nul Lines.

55. The *nul lines* (or *minimal* lines) on a surface are defined in connection with arcs of zero length; they are given, as to their variables, by

$$E dp^2 + 2F dp dq + G dq^2 = 0.$$

On any real surface, the nul lines are imaginary; and their parameters are conjugate complex variables, unless V is zero, a case which usually is excluded* from consideration. But it will appear that, though the variables are imaginary, they have definite and important relations with real isothermic systems of lines upon the surface.

In order that the parametric curves may be nul lines, the equation

$$E dp^2 + 2F dp dq + G dq^2 = 0,$$

which is the defining equation of such lines, must effectively be the same as

$$dp\, dq = 0,$$

which is the general equation of parametric curves. Hence

$$E = 0, \quad G = 0, \quad F \neq 0,$$

(and therefore $V \neq 0$). The expression for any arc then becomes

$$ds^2 = 2F dp\, dq.$$

To determine the nul lines on a surface when they are not the parametric curves, we have to integrate the equation

$$E dp^2 + 2F dp dq + G dq^2 = 0,$$

that is, the equations

$$E dp + (F + iV) dq = 0, \quad E dp + (F - iV) dq = 0,$$

where we shall assume that p and q are real. Let the primitive of the first of these two equations linear in dq/dp be

$$u = \phi (p, q) = \text{constant},$$

so that u is a complex function of p and q. If v is the conjugate to u,

$$v = \psi (p, q) = \text{constant}$$

* Darboux (t. i, p. 148) shews that, when $V = 0$, the surface is a developable circumscribed to the imaginary circle at infinity.

is the primitive of the second of the equations linear in dq/dp. Then

$$du = \mu\,\{E\,dp + (F + iV)\,dq\},$$
$$dv = \nu\,\{E\,dp + (F - iV)\,dq\},$$

where μ and ν are conjugate quantities; hence

$$\begin{aligned} ds^2 &= E\,dp^2 + 2F\,dp\,dq + G\,dq^2 \\ &= \frac{1}{\mu\nu E}\,du\,dv = \lambda\,du\,dv, \end{aligned}$$

where λ is a real quantity on a real surface.

The *nul lines then are given by the equations*

$$u = a, \quad v = b.$$

These variables u and v are often called the *symmetric* variables. Later it will be seen that, while symmetric variables are not unique for a surface, the choice of such variables possessing the symmetric character is narrow.

56. When a surface is referred to nul lines as parametric curves, the various equations in the general theory are much simplified. We then have

$$\begin{array}{lll} E = 0, & G = 0, & V^2 = -F^2; \\ \Gamma = F_1/F, & \Gamma' = 0, & \Gamma'' = 0, \\ \Delta = 0, & \Delta' = 0, & \Delta'' = F_2/F. \end{array}$$

The Mainardi-Codazzi relations become

$$L_2 + \frac{F_1}{F}M = M_1, \qquad N_1 + \frac{F_2}{F}M = M_2,$$

which can be written

$$\frac{1}{M}L_2 = \frac{\partial}{\partial p}\left(\log\frac{M}{F}\right),$$
$$\frac{1}{M}N_1 = \frac{\partial}{\partial q}\left(\log\frac{M}{F}\right);$$

and the Gauss equation becomes

$$LN - M^2 = F_{12} - \frac{F_1F_2}{F}.$$

The mean curvature is

$$H = 2\frac{M}{F};$$

and the total curvature is

$$\begin{aligned} K &= \frac{LN - M^2}{-F^2} \\ &= -\frac{1}{F}\frac{\partial^2}{\partial p\,\partial q}(\log F). \end{aligned}$$

The differential equation of the lines of curvature is

$$-L\,dp^2 + N\,dq^2 = 0;$$

and the equation of the asymptotic lines is unaffected, being

$$L\,dp^2 + 2M\,dp\,dq + N\,dq^2 = 0.$$

It will appear that the discussion of the geodesics upon a surface is simplified, so far as their equations are concerned, by referring the surface to nul lines as parametric curves; for only a single fundamental magnitude (F) of the first order then occurs in the equations, instead of all three when the parametric curves are any general unspecified curves.

57. As a passing example of the use of nul lines as parametric curves, consider *surfaces with a constant mean measure of curvature*, say $2/h$, where h is a constant. Then

$$2\frac{M}{F}=\frac{2}{h};$$

so that

$$Mh=F.$$

From the Mainardi-Codazzi equations, we have

$$L_2=0, \quad N_1=0.$$

Thus L is a function of p only, if it is not zero; when we change the variable to p' where $L^{\frac{1}{2}}dp=dp'$, the new quantity L is unity, that is, we can take $L=1$ without loss of generality. Similarly we can take $N=1$ without loss of generality, when it is not zero.

The Gauss equation now becomes

$$1-M^2=F_{12}-\frac{F_1F_2}{F},$$

that is,

$$\frac{\partial^2(\log F)}{\partial p\,\partial q}=\frac{1}{F}-\frac{F}{h^2},$$

a partial equation of the second order to determine* F. Suppose that some integral of this equation is known; it gives F, and therefore also M which is equal to F/h. Thus we know E, F, G, L, M, N; that is, by Bonnet's theorem, there is a surface uniquely determined by these quantities, save as to position and orientation in space.

The lines of curvature upon the surface satisfy the equation

$$-dp^2+dq^2=0;$$

that is, they are given by

$$p+q=\text{constant}, \quad p-q=\text{constant}.$$

* Writing

$$F=he^{\theta i}, \quad h=-2\mu^2,$$

the equation becomes

$$\frac{\partial^2\theta}{\partial p\,\partial q}=\frac{1}{\mu^2}\sin\theta,$$

which is the same equation as occurred (§ 54) in the discussion of surfaces having the Gaussian measure of curvature constant.

The asymptotic lines on the surface satisfy the equation

$$dp^2 + dq^2 + 2\frac{F}{h}\,dp\,dq = 0,$$

that is, they are given by the equations

$$h\,dp + \{F \pm (F^2 - h^2)^{\frac{1}{2}}\}\,dq = 0.$$

58. It is possible to have *nul lines in space*, as well as nul lines on any given surface; and they possess the important property that, from them, it is possible to construct minimal surfaces analytically.

One method of constructing nul lines in space is given by Lie as follows. Take any plane

$$tx + uy + vz = 1,$$

subject to a specific relation

$$t^2 + u^2 + v^2 = 0$$

and to any arbitrary non-homogeneous relation

$$f(t, u, v) = 0.$$

The two relations determine (say) u and v as functions of t; thus the equation of the plane contains only a single parameter, and therefore the envelope of the plane is a developable surface. The *edge of regression of this surface is a nul line*; and so, by taking any number of different relations $f = 0$, we have any number of nul lines.

The verification of the statement is easy. A point on the edge of regression is given by

$$\left.\begin{aligned} tx + uy + vz &= 1 \\ x + u'y + v'z &= 0 \\ u''y + v''z &= 0 \end{aligned}\right\},$$

with the conditions

$$t^2 + u^2 + v^2 = 0, \qquad t + uu' + vv' = 0;$$

and these equations give x, y, z as variable functions of t, so that dx, dy, dz are not zero. But along the edge, we have

$$t\,dx + u\,dy + v\,dz + (x + u'y + v'z)\,dt = 0,$$

$$dx + u'dy + v'dz + (u''y + v''z)\,dt = 0,$$

that is,

$$t\,dx + u\,dy + v\,dz = 0,$$

$$dx + u'dy + v'dz = 0,$$

and therefore

$$\frac{dx}{uv' - u'v} = \frac{dy}{v - tv'} = \frac{dz}{tu' - u} = \mu,$$

where μ is not zero because x, y, z are variable quantities. Hence the element of arc ds is given by

$$\begin{aligned}\frac{1}{\mu^2}\,ds^2 &= (uv'-u'v)^2+(v-tv')^2+(tu'-u)^2\\ &= (1+u'^2+v'^2)\,\{u^2(1+u'^2)+v^2(1+v'^2)+2uvu'v'\},\end{aligned}$$

on substituting $-(uu'+vv')$ for t. The second factor is

$$\begin{aligned}&u^2+v^2+(uu'+vv')^2\\ &= u^2+v^2+t^2=0,\end{aligned}$$

and so

$$ds=0,$$

thus verifying* the statement that the edge of regression is a nul line in space.

59. These nul lines in space are used† by Lie to construct minimal surfaces, according to the theorem:

Let N_1 and N_2 be two nul lines in space; let P_1 be any point on N_1, and P_2 any point on N_2; then the locus of the middle point of the straight line P_1P_2 is a minimal surface.

The coordinates of P_1 can be represented by

$$x_1=2A_1(t)=2A_1,\quad y_1=2B_1(t)=2B_1,\quad z_1=2C_1(t)=2C_1,$$

where

$$A_1'^2+B_1'^2+C_1'^2=0,$$

because N_1 is a nul line. The coordinates of P_2 can be represented by

$$x_2=2A_2(\theta)=2A_2,\quad y_2=2B_2(\theta)=2B_2,\quad z_2=2C_2(\theta)=2C_2,$$

where

$$A_2'^2+B_2'^2+C_2'^2=0,$$

because N_2 is a nul line. The coordinates of the middle point of P_1P_2 are

$$\begin{aligned}x&=A_1(t)+A_2(\theta),\\ y&=B_1(t)+B_2(\theta),\\ z&=C_1(t)+C_2(\theta);\end{aligned}$$

and therefore, for its locus, we have

$$\begin{aligned}E&=\Sigma x_1^2=A_1'^2+B_1'^2+C_1'^2=0,\\ G&=\Sigma x_2^2=A_2'^2+B_2'^2+C_2'^2=0.\end{aligned}$$

* For the purpose, the Serret formulæ of § 21 can also be used; in them, we write

$$p=-\frac{t}{v},\quad q=-\frac{u}{v},$$

while the u of those formulæ now becomes $-1/v$. Then we verify that T, which is equal to

$$\{p'^2+q'^2+(pq'-p'q)^2\}^{\frac{1}{2}},$$

vanishes; and so $ds=0$.

† *Math. Ann.*, t. xiv (1879), p. 337. Lie chooses the middle point; the locus of a point, that divides P_1P_2 in any constant ratio, also is a minimal surface.

Also

$$x_{12}=0, \quad y_{12}=0, \quad z_{12}=0,$$

and therefore

$$M=0.$$

Consequently

$$EN-2FM+GL=0,$$

the equation characteristic of a minimal surface.

It thus appears that minimal surfaces can be derived from any two minimal curves.

If A_1, B_1, C_1 are algebraic functions, and also A_2, B_2, C_2 are algebraic functions, then x, y, z are algebraic functions of t and θ; that is, the minimal surfaces then are algebraic. Hence if the arbitrary relation $f(t, u, v)=0$ be chosen as an algebraic function in each case, we shall have a succession of algebraic nul lines; and thence a succession of algebraic minimal surfaces can be constructed.

In particular, if $A_1(t)$ and $A_2(\theta)$, $B_1(t)$ and $B_2(\theta)$, $C_1(t)$ and $C_2(\theta)$ are conjugate pairs of complex quantities, the minimal surface is real though both the initiating nul lines are imaginary. For instance, let

$$t=\alpha+\beta i, \qquad \theta=\alpha-\beta i,$$

where α and β are real. An algebraic minimal curve is given by

$$x_1=3t-t^3, \qquad y_1=-i(3t+t^3), \qquad z_1=3t^2,$$

and another by

$$x_2=3\theta-\theta^3, \qquad y_2=\quad i(3\theta+\theta^3), \qquad z_2=3\theta^2;$$

the minimal surface derived from these as initiating curves is Enneper's minimal surface* (also algebraical, and of order 9)

$$x=3\alpha+3\alpha\beta^2-\alpha^3, \quad y=3\beta+3\alpha^2\beta-\beta^3, \quad z=3(\alpha^2-\beta^2).$$

Isometric Lines.

60. In connection with the determination of nul lines upon a surface, we saw that the element of arc could be taken in the form

$$ds^2=\lambda\, du\, dv,$$

where λ is a real constant,

$$du=\mu\{E dp+(F+iV)dq\}, \quad dv=\nu\{E dp+(F-iV)dq\},$$

μ and ν being magnitudes independent of differential elements. On a real surface, these symmetric parameters are conjugate complex variables; so we can take

$$u=P+iQ, \quad v=P-iQ,$$

where P and Q are real variables. We now have

$$ds^2=\lambda(dP^2+dQ^2);$$

and the curves $P=$ constant, $Q=$ constant, are parametric and real.

* First obtained by Enneper, *Zeitschr. f. Math. u. Physik*, t. ix (1864), p. 108.

Because there is no term in ds^2 which involves $dPdQ$, the parametric curves are orthogonal to one another.

Now take a number of parametric curves in succession, such that the variations dP and dQ in passing from any curve to the next curve are equal to one another, the common value of all of them being κ. Then the element of arc along $Q =$ constant intercepted between two successive P-curves is equal to $\lambda^{\frac{1}{2}}\kappa$, and the element of arc along $P =$ constant intercepted between two successive Q-curves is also equal to $\lambda^{\frac{1}{2}}\kappa$; and the curves are orthogonal to one another. Thus the selected rectangle is a square; and so, by the parametric curves as chosen, the surface is divided into small squares. Such a system of parametric curves is called *isometric*, sometimes *isothermic*, sometimes orthogonal and isometric, the division of the surface into small squares being the distinguishing property.

61. Isometric variables are not unique. Take any function, say $f(P+iQ)$, of two variables given as isometric for a surface; and separate $f(P+iQ)$ into its real and imaginary parts, so that

$$f(P+iQ) = P' + iQ'.$$

Let $g(P-iQ)$ be the conjugate of $f(P+iQ)$—should all the coefficients in $f(P+iQ)$ be real, $g(P-iQ)$ is $f(P-iQ)$; then

$$g(P-iQ) = P' - iQ'.$$

Hence

$$(dP + idQ)\, f'(P+iQ) = dP' + idQ',$$

$$(dP - idQ)\, g'(P-iQ) = dP' - idQ';$$

and therefore

$$ds^2 = \lambda\,(dP^2 + dQ^2)$$

$$= \lambda'\,(dP'^2 + dQ'^2),$$

where

$$\lambda = \lambda' f'(P+iQ)\, g'(P-iQ),$$

so that λ' is a real quantity. Consequently, the new parametric curves $P' =$ constant, $Q' =$ constant, are an orthogonal isometric system.

Moreover P' and Q' constitute the aggregate of isometric variables when complete variety of form is permitted to the function f, a property which can be established as follows. Reverting to the initial symmetric variables u and v, connected with the isometric variables P and Q, we have the element of arc upon the surface in the form

$$ds^2 = \lambda\, du\, dv.$$

Let any other reduction for the arc-element, expressed by means of symmetric variables connected with other isometric variables, be represented by

$$ds^2 = \lambda' du' dv'.$$

Then u' and v' are independent functions of the first parametric variables; hence we must have

$$\lambda\, du\, dv = \lambda' \left(\frac{\partial u'}{\partial u} du + \frac{\partial u'}{\partial v} dv\right)\left(\frac{\partial v'}{\partial u} du + \frac{\partial v'}{\partial v} dv\right),$$

for all variations of u and v. Consequently

$$\frac{\partial u'}{\partial u}\frac{\partial v'}{\partial u} = 0, \quad \frac{\partial u'}{\partial v}\frac{\partial v'}{\partial v} = 0;$$

hence either

(i) $\dfrac{\partial u'}{\partial v} = 0$, so that u' is a function of u only, and then $\dfrac{\partial v'}{\partial u} = 0$, so that v' is a function of v only; or

(ii) $\dfrac{\partial u'}{\partial u} = 0$, so that u' is a function of v only, and then $\dfrac{\partial v'}{\partial v} = 0$, so that v' is a function of u only.

The two cases differ only in an interchange of variables; effectively they are only a single case, represented by

$$u' = f(u), \quad v' = g(v).$$

As u' and v' are conjugate variables, $g(v)$ is the conjugate of $f(u)$. The foregoing relations between $P + iQ$ and $P' + iQ'$ therefore produce the aggregate of isometric variables.

It will appear later, in the discussion of the representation of a surface upon other surfaces, that the relations express the *conformal representation of the surface upon itself*; and further, that the reference of a surface to an isometric surface implies the *conformal representation of the surface upon a plane*, the coordinates in the plane being the parametric variables.

62. Simple systems of isometric lines are provided by the lines of curvature upon a surface of revolution and by the lines of curvature upon a central quadric.

For a surface of revolution, the lines of curvature are the meridians and the parallels of latitude. Let r denote the distance of any point from the axis of revolution, ϕ its longitude from some meridian of reference, and let

$$z = f(r)$$

be the equation of any meridian curve; then

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 + dz^2 \\ &= (1 + f'^2)\, dr^2 + r^2 d\phi^2. \end{aligned}$$

Let

$$d\rho = (1 + f'^2)^{\frac{1}{2}} \frac{dr}{r};$$

then

$$ds^2 = r^2 (d\rho^2 + d\phi^2).$$

An isometric system is therefore provided by the curves $\phi = \text{constant}$ (which are the meridians) and $r = \text{constant}$ (which are the parallels).

In the second case, it is known, from the theory of confocal central quadrics, that the coordinates of any point on the surface

$$x^2/a + y^2/b + z^2/c = 1$$

can be expressed in the form

$$\begin{aligned} -\beta\gamma x^2 &= a(a+p)(a+q), \\ -\gamma\alpha y^2 &= b(b+p)(b+q), \\ -\alpha\beta z^2 &= c(c+p)(c+q), \end{aligned}$$

where $\alpha, \beta, \gamma = b-c,\ c-a,\ a-b$. Then

$$ds^2 = \tfrac{1}{4}(p-q)\left\{\frac{p\,dp^2}{(a+p)(b+p)(c+p)} - \frac{q\,dq^2}{(a+q)(b+q)(c+q)}\right\},$$

so that the curves $p=$ constant, $q=$ constant (that is, the lines of curvature) are an isometric system.

63. The conditions that any original parametric curves should be an isometric system are easily obtainable.

In the first place, they must be orthogonal; hence

$$F = 0.$$

Next, the element of arc must be expressible in the form

$$\lambda(dP^2 + dQ^2),$$

and $p=$ constant, $q=$ constant, are to be the same effectively as $P=$ constant, $Q=$ constant; hence

$$E = \lambda f(p), \quad G = \lambda g(q),$$

where f and g are any functions of p and q respectively. Thus

$$\frac{E}{G} = \frac{f(p)}{g(q)},$$

and therefore

$$\frac{\partial^2}{\partial p\,\partial q}\left(\log\frac{E}{G}\right) = 0,$$

which is the other necessary condition.

When an element of arc is given in the form

$$E\,dp^2 + G\,dq^2,$$

and the necessary condition is satisfied, an appropriate change in the variables leads to the form

$$\lambda(dP^2 + dQ^2);$$

and then the necessary condition is

$$E = G.$$

But it must be remembered that, for this form of the condition, one special isometric system has been chosen.

Assuming this special choice made, the conditions are

$$F=0,\quad E=G=\lambda.$$

We then have

$$\Gamma=\frac{1}{2}\frac{\lambda_1}{\lambda},\quad \Gamma'=\frac{1}{2}\frac{\lambda_2}{\lambda},\quad \Gamma''=-\frac{1}{2}\frac{\lambda_1}{\lambda}.$$

$$\Delta=-\frac{1}{2}\frac{\lambda_2}{\lambda},\quad \Delta'=\frac{1}{2}\frac{\lambda_1}{\lambda},\quad \Delta''=\frac{1}{2}\frac{\lambda_2}{\lambda}.$$

The Mainardi-Codazzi equations are

$$L_2-M_1=\frac{1}{2}\frac{\lambda_2}{\lambda}(L+N),$$

$$N_1-M_2=\frac{1}{2}\frac{\lambda_1}{\lambda}(L+N).$$

The Gauss equation is

$$LN-M^2=-\frac{1}{2}(\lambda_{11}+\lambda_{22})+\frac{1}{2\lambda}(\lambda_1^2+\lambda_2^2)$$

$$=-\frac{1}{2}\lambda\left(\frac{\partial^2\log\lambda}{\partial p^2}+\frac{\partial^2\log\lambda}{\partial q^2}\right).$$

The mean curvature is given by

$$H=\frac{1}{\lambda}(L+N);$$

and the total curvature is given by

$$K=\frac{1}{\lambda^2}(LN-M^2)$$

$$=-\frac{1}{2\lambda}\left(\frac{\partial^2\log\lambda}{\partial p^2}+\frac{\partial^2\log\lambda}{\partial q^2}\right).$$

64. The reference of any surface to isometric lines as parametric curves affects the form of the expression for the arc, and therefore affects the forms of E, F, G; but, beyond the necessity of satisfying the Gauss equation which gives a value for $LN-M^2$, no condition is thereby imposed upon the determination of L, M, N; consequently we cannot expect to have any unique determination of a surface. Any postulation of further conditions, of course, modifies the problem.

Accordingly (and especially after the two examples just given in § 62) we proceed to consider those surfaces whose isometric curves include the lines of curvature. Let the surface be referred to the special isometric lines such that we can take

$$E=G=\lambda,$$

of course with the condition $F=0$. As the parametric lines now are lines of curvature, we have (§ 45) both $F=0$, $M=0$; thus the aggregate of conditions is

$$E=G=\lambda,\quad F=0,\quad M=0.$$

What is required is the determination (as far as may be possible) of the quantities λ, L, N. The equations which they have to satisfy are

$$N_1 = \frac{1}{2}\frac{\lambda_1}{\lambda}(L+N),$$

$$L_2 = \frac{1}{2}\frac{\lambda_2}{\lambda}(L+N),$$

$$LN = -\frac{1}{2}\lambda\left(\frac{\partial^2 \log\lambda}{\partial p^2} + \frac{\partial^2 \log\lambda}{\partial q^2}\right) = \lambda\vartheta,$$

so that

$$\left.\begin{aligned} \frac{\partial}{\partial p}\left(\frac{N^2}{\lambda}\right) &= \frac{\lambda_1}{\lambda}\vartheta \\ \frac{\partial}{\partial q}\left(\frac{L^2}{\lambda}\right) &= \frac{\lambda_2}{\lambda}\vartheta \end{aligned}\right\}.$$

Writing

$$A = N^2/\lambda, \quad B = L^2/\lambda, \quad \mu = \log\lambda,$$

we have

$$\frac{\partial A}{\partial p} = \mu_1\vartheta, \quad \frac{\partial B}{\partial q} = \mu_2\vartheta, \quad AB = \vartheta^2;$$

we must eliminate A and B between these three equations. Now

$$B\frac{\partial A}{\partial q} = 2\vartheta\vartheta_2 - A\mu_2\vartheta, \quad A\frac{\partial B}{\partial p} = 2\vartheta\vartheta_1 - B\mu_1\vartheta,$$

$$B\frac{\partial^2 A}{\partial p\,\partial q} = -\frac{\partial B}{\partial p}\frac{\partial A}{\partial q} + 2\vartheta\vartheta_{12} + 2\vartheta_1\vartheta_2 - A(\mu_2\vartheta_1 + \mu_{12}\vartheta) - \mu_1\mu_2\vartheta^2.$$

But

$$\frac{\partial^2 A}{\partial p\,\partial q} = \mu_{12}\vartheta + \mu_1\vartheta_2;$$

hence, equating these values of $\frac{\partial^2 A}{\partial p\,\partial q}$, substituting for $\frac{\partial B}{\partial p}$ and $\frac{\partial A}{\partial q}$, and reducing, we find

$$0 = 2\vartheta\vartheta_{12} - 2\vartheta_1\vartheta_2 - 2\vartheta^2\mu_1\mu_2 + A(\mu_2\vartheta_1 - \mu_{12}\vartheta) + B(\mu_1\vartheta_2 - \mu_{12}\vartheta).$$

Let

$$\mu_{12}\vartheta - \mu_2\vartheta_1 = I,$$

$$\mu_{12}\vartheta - \mu_1\vartheta_2 = J,$$

$$2\vartheta\vartheta_{12} - 2\vartheta_1\vartheta_2 - 2\vartheta^2\mu_1\mu_2 = D;$$

then

$$AI + BJ = D.$$

Also

$$AB = \vartheta^2.$$

Hence taking

$$\Lambda^2 = D^2 - 4IJ\vartheta^2,$$

we have

$$2AI = D + \Lambda, \qquad 2BJ = D - \Lambda.$$

But

$$\left.\begin{aligned}\frac{\partial A}{\partial p} &= \mu_1 \vartheta \\ B\frac{\partial A}{\partial q} &= 2\vartheta\vartheta_2 - A\mu_2\vartheta\end{aligned}\right\}, \quad \left.\begin{aligned} A\frac{\partial B}{\partial p} &= 2\vartheta\vartheta_1 - B\mu_1\vartheta \\ \frac{\partial B}{\partial q} &= \mu_2\vartheta\end{aligned}\right\}.$$

Using the values of A and B just obtained, taking account of the value of Λ, and reducing, we find that only two independent relations survive, viz.

$$\left.\begin{aligned}\frac{\partial D}{\partial p}\frac{D+\Lambda}{2I\Lambda} &= \mu_1\vartheta + \frac{2J}{\Lambda}\vartheta\vartheta_1 + \frac{\vartheta^2}{\Lambda}J_1 + \frac{D+\Lambda}{D-\Lambda}\frac{J}{I\Lambda}\vartheta^2 I_1 \\ -\frac{\partial D}{\partial q}\frac{D-\Lambda}{2J\Lambda} &= \mu_2\vartheta - \frac{2I}{\Lambda}\vartheta\vartheta_2 - \frac{D-\Lambda}{D+\Lambda}\frac{I}{J\Lambda}\vartheta^2 J_2 - \frac{\vartheta^2}{\Lambda}I_2\end{aligned}\right\}.$$

From these, by the relation

$$\Lambda^2 = D^2 - 4IJ\vartheta^2,$$

we have

$$\left.\begin{aligned}\frac{\partial \Lambda}{\partial p}\frac{D+\Lambda}{2ID} &= \mu_1\vartheta - \frac{2J}{D}\vartheta\vartheta_1 - \frac{\vartheta^2}{D}J_1 + \frac{D+\Lambda}{D-\Lambda}\frac{J}{ID}\vartheta^2 I_1 \\ -\frac{\partial \Lambda}{\partial q}\frac{D-\Lambda}{2JD} &= \mu_2\vartheta - \frac{2I}{D}\vartheta\vartheta_2 + \frac{D-\Lambda}{D+\Lambda}\frac{I}{JD}\vartheta^2 J_2 - \frac{\vartheta^2}{D}I_2\end{aligned}\right\}.$$

For either pair, we thus have two simultaneous partial differential equations; they are of the fifth order in the derivatives of λ. When a value of λ is known, we find A and B from the equations

$$2AI = D + \Lambda, \quad 2BJ = D - \Lambda.$$

Then $E\,(=\lambda)$, $F\,(=0)$, $G\,(=\lambda)$, L, $M\,(=0)$, N, are known; and so, by Bonnet's theorem, the surface is determinate save as to position and orientation. Thus the solution of the problem depends upon the resolution of the two equations of the fifth order.

In connection with surfaces of this character, reference may be made to a memoir* by Weingarten and to §§ 435—437 (vol. ii) of Darboux's treatise.

As an example, we can verify that a surface of constant mean curvature has the specified character.

We have seen (§ 57) that, when a surface of constant mean curvature is referred to its nul lines as parametric curves, so that its arc-element is given by

$$ds^2 = 2F\,dp\,dq,$$

the lines of curvature are given by the equations $p \pm q = \text{constant}$. Write

$$p + q = 2u, \quad p - q = 2iv;$$

then u and v are parameters of the lines of curvature, and

$$ds^2 = 2F(du^2 + dv^2),$$

shewing that the lines of curvature are a parametric system.

* *Sitzungsb. Berl.*, t. ii (1883), p. 1163.

Geodesics.

65. Among the important lines upon a surface, one class is constituted by the curves called geodesics. Unlike lines of curvature, asymptotic lines, and nul lines, they are not determined uniquely or in pairs at a point by the surface itself. A direction taken at a point determines uniquely a geodesic having that direction at the point. The detailed consideration of their properties will be deferred until a later chapter. They are mentioned here solely because they provide a system of coordinates for the representation of the surface, corresponding in notion to the system of polar coordinates in a plane; in consequence, these are sometimes called *geodesic polar coordinates*, sometimes geodesic orthogonal coordinates.

The original definition of a geodesic on a surface is that it is the shortest distance measured along the surface between two points on its course. It is therefore a curve along which a tightly stretched string would lie at rest between the two points on a smooth surface. At any element, the internal forces due to the tensions at the two extremities lie in the osculating plane of the curve, while the external force is the pressure which acts along the normal to the surface; as these balance because the string is at rest, the osculating plane of the curve at any point contains the normal to the surface at the point. This property, characteristic of geodesics, will later be derived also from non-statical considerations.

Later (Chap. v) we shall see that there may be a limit to the range of the curve when it is to be the shortest distance from any initial point to every other along its course. When such a limit exists, each extremity of the range is called the conjugate of the other; and then, as will also be seen, it is possible to draw more than one geodesic between two points when either lies beyond the conjugate of the other. For our immediate purpose, we shall assume the domain of the surface in the vicinity of a point to be so far restricted that it shall not include the conjugate (if any) of the point along any geodesic.

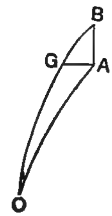
66. Without waiting for the full discussion of the general equation of geodesics, it is desirable to notice one simple and important property, viz. *if a geodesic be a plane curve (which is not merely a straight line), or if it be a line of curvature, then it is both a plane curve and a line of curvature.*

When a geodesic is a plane curve, its principal normals intersect, save only when it is a straight line. These principal normals are the normals to the surface along its course; they therefore intersect, and so the curve is a line of curvature.

Next, let a geodesic be a line of curvature. Take four consecutive points on the curve, say A, B, C, D. The normals to the surface at B and C intersect in some point, say O; these are the normals in an osculating plane of the curve at B, so that the points A, B, C, O lie in one plane. Similarly, the points B, C, D, O lie in one plane. Hence the points A, B, C, D lie in one plane, and so for points in succession; that is, the curve is a plane curve.

But the converse is not true; that is, a plane line of curvature (*e.g.* a parallel on a surface of revolution) is not necessarily a geodesic on the surface.

67. Adopting for the moment the definition relating to the shortest distance, and having regard to the statement at the end of § 65, consider two geodesics through a point O making an infinitesimal angle with one another at O. Along them measure any the same distance to points A and B, so that $OA = OB$; then* *the small rectilinear arc AB is perpendicular to both the geodesics.*



If not, take $BG = AB \sec ABG$; then BAG is a right angle, while ABG is an infinitesimal plane triangle and GA, one of the sides, is less than GB, the hypotenuse. Thus

$$\begin{aligned} OG + GA &< OG + GB \\ &< OB \\ &< OA, \end{aligned}$$

for OB and OA are equal. Then the path in the surface along OG and GA is shorter than the path along OA, in opposition to the fact that OA is the geodesic between O and A. Hence the angles OAB, OBA are right angles.

Now take any number of consecutive geodesics through O; and along them measure any the same distance, obtaining points A, B, C, ... We shall thus obtain a curve as the locus of points at a given distance from O measured in the surface along geodesics through O. The curve is sometimes called a *geodesic circle*; and sometimes, because it is orthogonal to the geodesics, an *orthogonal trajectory of the geodesics*; and sometimes a *geodesic parallel*, though the term geodesic parallels includes the orthogonal trajectories of any family of geodesics, whether concurrent or not. But it must not be assumed, and it is not in fact the case, that a geodesic circle is itself a geodesic on the surface.

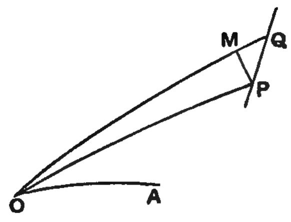
68. The property makes a point, and a geodesic distance, and the inclination of this distance to a geodesic of reference through the point, correspond to an origin, and a radius vector, and the angle between this radius vector and

* The proposition is due to Gauss.

an initial line in a plane. The associated variables (the geodesic distance and the inclination to the geodesic of reference) are called *geodesic polar coordinates.*

Accordingly, take any number of consecutive geodesics through an origin O; and let two of them meet any curve in points P and Q. Let OA, any fixed geodesic, be used for reference; let the angles at O be

$$AOP = q, \quad AOQ = q + dq, \quad POQ = dq.$$



Along OQ, measure OM from O equal to OP; then the small rectilinear arc PM is perpendicular to OQ at M. Also PM vanishes when P and Q coincide, that is, when dq vanishes; hence, as PM and dq vanish together, we can take

$$PM = Ddq,$$

where D naturally will depend upon the geodesic distance OP and may (and usually will) depend upon the variable q. Also, let

$$OP = p, \quad OQ = p + dp;$$

then

$$MQ = dp.$$

Thus the arc PQ of the curve (being any arc on the surface) is given by

$$\begin{aligned} ds^2 &= (QM)^2 + (MP)^2 \\ &= \quad dp^2 \quad + D^2dq^2; \end{aligned}$$

so that we have an expression for the elementary arc in terms of geodesic coordinates; and the magnitude D is a function of p and q. Sometimes the expression for the arc is taken in the form

$$ds^2 = dp^2 + gdq^2.$$

When we compare this expression for ds^2 with the general expression

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2,$$

the line $p =$ constant being a geodesic circle, and the line $q =$ constant being a geodesic, we see that the conditions, necessary and sufficient to secure that *the general expression for any arc should have reference to geodesic polar coordinates,* are

$$E = 1, \quad F = 0.$$

On a real surface, $G = g, = D^2$, and is therefore a positive quantity.

In establishing the expression for the arc, no account was taken of secondary magnitudes at P or of curvature properties; and so the geodesic polar coordinates do not, of themselves (as do the asymptotic lines, for example), lay any limitation upon the secondary magnitudes. But, of

course, the Gauss equation and the Mainardi-Codazzi relations must be satisfied. We have

$$\Gamma = 0, \quad \Gamma' = 0, \qquad \Gamma'' = -\tfrac{1}{2}g_1,$$
$$\Delta = 0, \quad \Delta' = \tfrac{1}{2}g_1/g, \quad \Delta'' = \tfrac{1}{2}g_2/g;$$

so that the Mainardi-Codazzi relations are

$$L_2 - M_1 = \frac{1}{2}\frac{g_1}{g} M,$$
$$N_1 - M_2 = \frac{1}{2}\left(g_1 L - \frac{g_2}{g} M + \frac{g_1}{g} N\right).$$

The Gauss equation is

$$LN - M^2 = -\frac{1}{2}g_{11} + \frac{1}{g}g_1^2;$$

and therefore the total curvature is

$$K = \frac{1}{g}(LN - M^2)$$
$$= -\frac{1}{4g^2}(\tfrac{1}{2}gg_{11} - g_1^2)$$
$$= -g^{-\frac{1}{2}}\frac{\partial^2 g^{\frac{1}{2}}}{\partial p^2}.$$

When we use the quantity D in place of g, and we write $K = \dfrac{1}{\alpha\beta}$, where α and β are the principal radii of curvature at the point, the last equation becomes

$$\frac{\partial^2 D}{\partial p^2} + \frac{D}{\alpha\beta} = 0,$$

a property first established by Gauss.

The mean curvature is given by

$$H = L + \frac{N}{g}.$$

One property of the quantity D may be noticed. When P is near O, the geodesic OP is appreciably a straight line so that $PM = pdq$; hence, when p is very small, we have

$$D = p + \text{higher powers of small quantities},$$

and so, at O, we have

$$\frac{\partial D}{\partial p} = 1.$$

As already stated, the detailed developments of the analysis, connected with geodesics and their properties, will be deferred until Chapter v.

Summary.

69. It may be convenient to make a summary statement of the different sets of conditions satisfied by the fundamental magnitudes when the parametric curves belong to one or other of the various classes of curves considered in this chapter. The curves are

(i) *orthogonal*, if $F=0$;

(ii) *lines of curvature*, if $F=0$, $M=0$;

(iii) *conjugate*, if $M=0$;

(iv) *asymptotic*, if $L=0$, $N=0$;

(v) *nul*, if $E=0$, $G=0$;

(vi) *isometric* (or *isothermic*) *orthogonal*, if $E=G$, $F=0$ (with a special selection among the variables); if the special selection is not made, then $F=0$, $\dfrac{\partial^2}{\partial p \partial q}\log(E/G)=0$;

(vii) *geodesic polar*, if $E=1$, $F=0$.

EXAMPLES.

1. Obtain an equation of asymptotic lines of the surface given by the equations
$$x^2=(b-c)(p-a)(q-a),\quad y^2=(c-a)(p-b)(q-b),\quad z^2=(a-b)(p-c)(q-c).$$

2. Shew that the asymptotic lines of the tetrahedral surface
$$(x/a)^m+(y/b)^m+(z/c)^m=1$$
are determined by the equation
$$\alpha(x/a)^{\frac{1}{2}m}+\beta(y/b)^{\frac{1}{2}m}+\gamma(z/c)^{\frac{1}{2}m}=0,$$
where α, β, γ are arbitrary constants such that $\alpha^2+\beta^2+\gamma^2=0$.

3. Prove that a conjugate system of curves on a surface remains conjugate when the surface is submitted to any projective transformation.

4. Prove that the condition, necessary and sufficient to secure that the parametric curves are conjugate, is that all the four coordinates in a homogeneous system should satisfy an equation
$$\xi_{12}+A\xi_1+B\xi_2+C=0,$$
where A, B, C are functions of p and q only.

Shew that an equation of the same form (though with different values of A, B, C) must be satisfied by each of the coordinates in a tangential system.

5. Two families of spheres are defined by the equations
$$x^2+y^2+z^2-p_1x-p_2y-p_3z-p_4=0,\quad x^2+y^2+z^2-q_1x-q_2y-q_3z-q_4=0,$$
where $p_1, \ldots, p_4$ are functions of p only and $q_1, \ldots, q_4$ are functions of q only; shew that the envelope of the radical plane of any sphere of the first system and any sphere of the second system is a surface possessing two families of conjugate plane curves.

6. A sphere of radius unity is referred to nul lines as parametric curves. Shew that the parameters u and v can be chosen so as to give

$$F=2(1+uv)^{-2}=-M, \quad L=0, \quad N=0;$$

so that the nul lines are also asymptotic, and the equation for the lines of curvature is evanescent.

Prove also that the equations of geodesics on the sphere then are

$$\frac{u''}{u'^2}=\frac{2v}{1+uv}, \quad \frac{v''}{v'^2}=\frac{2u}{1+uv},$$

where $u'=du/ds$ and $v'=dv/ds$; and obtain a primitive, other than the integral $4u'v'=(1+uv)^2$, in the form

$$au+bv=uv-1.$$

Verify that the curve so determined is part of a great circle.

7. Shew that the only developable surfaces which have isometric lines of curvature are either conical or cylindrical.

8. Shew that the variables of the nul lines satisfy the equation

$$E\left(\frac{\partial\theta}{\partial q}\right)^2-2F\frac{\partial\theta}{\partial p}\frac{\partial\theta}{\partial q}+G\left(\frac{\partial\theta}{\partial p}\right)^2=0.$$

9. A surface is given by the equation

$$ds^2=\{f(p+q)-g(p-q)\}\,dp\,dq;$$

and its Gaussian measure of curvature is constant, and not zero. Prove that either

(i) $f(p+q)=\kappa\wp(p+q), \qquad g(p-q)=\kappa\wp(p-q),$

where $\wp$ denotes the Weierstrass elliptic function; or

(ii) $f(p+q)=\kappa\operatorname{cosec}^2(p+q), \quad g(p-q)=\kappa\operatorname{cosec}^2(p-q)$; or

(iii) $f(p+q)=\kappa(p+q)^{-2}, \qquad g(p-q)=\kappa(p-q)^{-2}.$

10. Shew that the developable surfaces, given by the equation

$$ds^2=\{f(p+q)-g(p-q)\}\,dp\,dq,$$

can have one or other of the following expressions:

(i) $f(p+q)=\kappa\cosh(p+q), \quad g(p-q)=\kappa\cosh(p-q)$;

(ii) $f(p+q)=\kappa e^{p+q}, \quad g(p-q)=\kappa e^{p-q}$;

(iii) $f(p+q)=\kappa(p+q)^2, \quad g(p-q)=\kappa(p-q)^2$;

(iv) $f(p+q)=\kappa(p+q), \quad g(p-q)=\kappa(p-q)$;

and obtain the most general form.

11. Shew that the circumference of a small geodesic circle of radius p is $2\pi p(1-\frac{1}{6}K_0p^2)$, and that its area is $\pi p^2(1-\frac{1}{12}K_0p^2)$, where K_0 is the total curvature of the surface at the centre of the circle.

CHAPTER IV.

LINES OF CURVATURE.

CONCERNING the topics about to be discussed—viz., the determination of the lines of curvature on a surface, the configuration of the lines of curvature near an ordinary umbilicus on a surface, and some properties of the double-sheeted centro-surface belonging to an ordinary region of any surface,—some references are given in the course of the chapter. The student should also consult Darboux's treatise, t. iii, pp. 334—356, and Bianchi's treatise, chapter IX.

70. Some of the immediate and elementary characteristics of lines of curvature have already been given, mainly to fix them individually in the scheme of organic curves upon a surface. Among these are the properties that, along a line of curvature, consecutive normals to the surface intersect; that, at any general point, there are two lines of curvature which are perpendicular to one another, and that there is a centre of curvature for each of the lines at each point of the surface; that there is a surface of centres, being the double-sheeted locus of the centres of curvature; and so on. We now proceed to consider some developments of such results, as well as other properties of the surface which are specially controlled by the lines of curvature.

In the first place, it is important to obtain an integral equation or integral equations for their analytical expression. We know that, when the surface is referred to two parametric curves, the directions of the lines of curvature at any point satisfy the equation

$$(EM - FL)\,dp^2 + (EN - GL)\,dp\,dq + (FN - GM)\,dq^2 = 0,$$

which is definitely non-evanescent except at an umbilicus. Accordingly, it is necessary to integrate (directly or indirectly) this equation which, being of the first order and the second degree, is equivalent to the two equations

$$2(EM - FL)\,dp + \{EN - GL - V^2(H^2 - 4K)^{\frac{1}{2}}\}\,dq = 0,$$
$$2(EM - FL)\,dp + \{EN - GL + V^2(H^2 - 4K)^{\frac{1}{2}}\}\,dq = 0,$$

each being of the first order and the first degree. Let the respective primitives of these equations be

$$u = \text{constant}, \quad v = \text{constant};$$

then these primitive equations give the lines of curvature. Thus the determination depends upon the solution of a couple of ordinary equations of the first order, when we know a parametric representation of the surfaces.

The same result can otherwise be expressed in terms of partial differential equations of the first order. When a line of curvature is given by

$$u = \text{constant},$$

its direction at any point is given by

$$u_1 dp + u_2 dq = 0;$$

and the ratio of dp/dq thus determined must satisfy the general equation. Hence

$$(FN - GM) u_1^2 - (EN - GL) u_1 u_2 + (EM - FL) u_2^2 = 0.$$

The same equation is satisfied by v, when the other line of curvature is given by

$$v = \text{constant}.$$

Hence the lines of curvature are given by two functionally independent integrals of

$$(FN - GM) \theta_1^2 - (EN - GL) \theta_1 \theta_2 + (EM - FL) \theta_2^2 = 0,$$

which is a partial differential equation of the first order, in two independent variables. This has to be integrated (say) by Charpit's method; when the solution admits analytical completion, we have equations for the lines of curvature.

Umbilici.

71. The equation for the directions of the lines of curvature thus leads definitely to equations for the lines after some process of integration, always on the assumption that the equation exists. But the result cannot be inferred when the equation ceases to exist through becoming evanescent; and so this possibility must be considered further.

At such a place on a surface, called an *umbilicus*, we have

$$\frac{L}{E} = \frac{M}{F} = \frac{N}{G}, \quad = \frac{1}{\kappa},$$

say. The curvature of a normal section of the surface through any direction dp/dq, being

$$\frac{L dp^2 + 2M dp dq + N dq^2}{E dp^2 + 2F dp dq + G dq^2},$$

there becomes $1/\kappa$, and consequently is independent of the direction. Thus the two principal radii of curvature at the point become $1/\kappa$—it will be remembered that a principal radius of curvature of the surface usually is not the radius of circular curvature of the corresponding line of curvature itself—and the radius of curvature of every normal section also is $1/\kappa$. Thus there seems no specific line of curvature at the point; and so we inquire into the form of the lines of curvature in the immediate vicinity*.

* The subject was first investigated by Cayley, for the umbilicus of an ellipsoid, *Coll. Math. Papers*, vol. v, pp. 115—130: and more generally by Darboux, in Note VII, at the end of the fourth volume of his treatise. See also, in a note by the author, *Messenger of Math.*, vol. xxxii (1903), pp. 75—80.

72. At first sight, it might appear convenient to take the lines of curvature as parametric curves, because F and M would then be zero over the surface and the equations would be simplified. But the lines of curvature at and near an umbilicus have not yet been determined, and their determination is the matter at issue; we must therefore choose other parametric curves. It is equally impossible to choose asymptotic lines for the purpose; being the asymptotes of the indicatrix-conic, which is a circle at an umbilicus, they are not definite there. We might choose isometric orthogonal lines. We shall however leave the parametric curves quite general and unspecified, merely noting the simplification which would arise had isometric orthogonal lines been chosen.

Writing $dp/dq = t$, and

$$A = EM - FL, \quad B = EN - GL, \quad C = FN - GM,$$

so that A, B, C vanish at an umbilicus p_0, q_0, we have

$$At^2 + Bt + C = 0$$

as the equation of the lines of curvature. At a point $p_0 + p$, $q_0 + q$, very near an umbilicus, we have

$$\begin{aligned} A &= A_1 p + A_2 q + \tfrac{1}{2}(A_{11}p^2 + 2A_{12}pq + A_{22}q^2) + \dots, \\ B &= B_1 p + B_2 q + \tfrac{1}{2}(B_{11}p^2 + 2B_{12}pq + B_{22}q^2) + \dots, \\ C &= C_1 p + C_2 q + \tfrac{1}{2}(C_{11}p^2 + 2C_{12}pq + C_{22}q^2) + \dots, \end{aligned}$$

where the coefficients of the various powers of p and q are the values, at the umbilicus, of the derivatives of A, B, C. For the present purpose, the first derivatives are critically important. Now

$$\begin{aligned} A_1 &= E_1 M + EM_1 - F_1 L - FL_1 \\ &= EQ - FP, \end{aligned}$$

on substituting for M_1 and L_1 in terms of the derived magnitudes of the third order (§ 40) and using the umbilical relations $L/E = M/F = N/G$; and similarly

$$A_2 = ER - FQ,$$

$$B_1 = ER - GP, \quad B_2 = ES - GQ,$$

$$C_1 = FR - GQ, \quad C_2 = FS - GR.$$

We have, always,

$$EC - FB + GA = 0;$$

and therefore, at an umbilicus,

$$EC_1 - FB_1 + GA_1 = 0, \quad EC_2 - FB_2 + GA_2 = 0.$$

(Had special isometric orthogonal lines been chosen as parametric curves, we should have had $E = G$, $F = 0$; and then $A_1 + C_1 = 0$, $A_2 + C_2 = 0$. These special relations give no essential simplification.)

Owing to the form of the occurrence of t in the differential equation and to the fact that full variation of some variable is needed, we make t the independent variable. Following Darboux, we use a contact-transformation

$$qt - p = \xi,$$

so that $q = d\xi/dt$, $p = t\,d\xi/dt - \xi$; and p and q are to be expressed as functions of t. We are to have p and q small; and so ξ and $d\xi/dt$ must be small. Substituting in the differential equation, we find

$$(A_1 t^2 + B_1 t + C_1)\left(t\frac{d\xi}{dt} - \xi\right) + (A_2 t^2 + B_2 t + C_2)\frac{d\xi}{dt} + \ldots = 0,$$

where the unexpressed terms contain squares and higher powers of ξ and $d\xi/dt$; and therefore the terms of lowest order in the equation* are

$$\{A_1 t^3 + (B_1 + A_2)t^2 + (C_1 + B_2)t + C_2\}\frac{d\xi}{dt} = (A_1 t^2 + B_1 t + C_1)\,\xi + \ldots.$$

Consequently, when we require values of ξ that are small (given in the present case by having an arbitrary constant small), so that we keep in the immediate vicinity of the umbilicus, the governing part of ξ satisfies the equation

$$\{A_1 t^3 + (B_1 + A_2)t^2 + (C_1 + B_2)t + C_2\}\frac{d\xi}{dt} = (A_1 t^2 + B_1 t + C_1)\,\xi.$$

Let

$$A_1 t^3 + (B_1 + A_2)t^2 + (C_1 + B_2)t + C_2 = A_1(t - t_1)(t - t_2)(t - t_3);$$

and suppose that the quantities t_1, t_2, t_3 are unequal, and that no one of them is a root of $A_1 t^2 + B_1 t + C_1$. Then let

$$\frac{A_1 t^2 + B_1 t + C_1}{A_1 t^3 + (B_1 + A_2)t^2 + (C_1 + B_2)t + C_2} = \frac{m_1}{t - t_1} + \frac{m_2}{t - t_2} + \frac{m_3}{t - t_3};$$

we have

$$m_1 + m_2 + m_3 = 1.$$

The governing part of ξ, say ξ, is then given by

$$\xi = c\,(t - t_1)^{m_1}(t - t_2)^{m_2}(t - t_3)^{m_3},$$

where c is an arbitrary constant; and, as ξ is to be small for the investigation, we take c to be small. Then

$$\left.\begin{aligned}
q = \quad \frac{d\xi}{dt} \quad &= \left(\frac{m_1}{t - t_1} + \frac{m_2}{t - t_2} + \frac{m_3}{t - t_3}\right)\xi \\
p = t\frac{d\xi}{dt} - \xi &= \left(\frac{m_1 t_1}{t - t_1} + \frac{m_2 t_2}{t - t_2} + \frac{m_3 t_3}{t - t_3}\right)\xi \\
p - qt_1 &= \left\{\frac{m_2(t_2 - t_1)}{t - t_2} + \frac{m_3(t_3 - t_1)}{t - t_3}\right\}\xi \\
p - qt_2 &= \left\{\frac{m_1(t_1 - t_2)}{t - t_1} + \frac{m_3(t_3 - t_2)}{t - t_3}\right\}\xi \\
p - qt_3 &= \left\{\frac{m_1(t_1 - t_3)}{t - t_1} + \frac{m_2(t_2 - t_3)}{t - t_2}\right\}\xi
\end{aligned}\right\},$$

it being always remembered that c is a small quantity.

* For the general properties of such equations, see the author's *Theory of Differential Equations*, vol. ii, chap. v.

73. The configuration of the lines depends upon the values of m_1, m_2, m_3, which satisfy the relation

$$m_1 + m_2 + m_3 = 1.$$

First, suppose that the quantities m are real. Then they also satisfy the inequality

$$m_1 m_2 m_3 > 0,$$

which can be verified as follows. We have

$$EC_1 - FB_1 + GA_1 = 0, \quad EC_2 - FB_2 + GA_2 = 0,$$

so that

$$\frac{E}{A_2B_1 - A_1B_2} = \frac{-F}{C_2A_1 - C_1A_2} = \frac{G}{B_2C_1 - B_1C_2} = \lambda, \text{ say};$$

hence

$$(C_2A_1 - C_1A_2)^2 + (A_1B_2 - A_2B_1)(B_2C_1 - B_1C_2) = -\lambda^2 V^2 < 0.$$

But

$$m_1 = \frac{A_1t_1^2 + B_1t_1 + C_1}{A_1(t_1 - t_2)(t_1 - t_3)},$$

and so for m_2, m_3; hence

$$m_1 m_2 m_3 = \frac{(A_1t_1^2 + B_1t_1 + C_1)(A_1t_2^2 + B_1t_2 + C_1)(A_1t_3^2 + B_1t_3 + C_1)}{-A_1^3(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2}$$

$$= \frac{(C_2A_1 - C_1A_2)^2 + (A_1B_2 - A_2B_1)(B_2C_1 - B_1C_2)}{-A_1^4(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2},$$

and therefore $m_1 m_2 m_3$ is a positive quantity, as stated. As m_1, m_2, m_3 are real, there are two general possibilities, viz.

(α) all the quantities m_1, m_2, m_3 are positive and less than 1;

(β) one of the quantities, say m_1, is positive and greater than 1, while the other two are negative.

When a quantity m_1 is positive and greater than 1, then for values of t nearly equal to t_1, we have

$$q = \kappa_1 (t - t_1)^{m_1 - 1} + \text{higher powers of } t - t_1,$$

$$p - qt_1 = \kappa_2 (t - t_1)^{m_1} + \text{higher powers of } t - t_1,$$

and therefore p and q remain small for such values of t, while

$$p - qt_1 = \kappa q^{\frac{m_1}{m_1 - 1}}$$

near the origin. The lines of curvature therefore are as in fig. i.

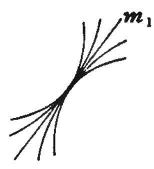
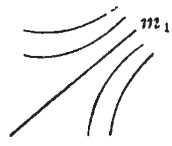


Fig. i.

When a quantity m_1 is positive and less than 1, then (in spite of the small factor c in ξ) both p and q tend to become large while $p - qt_1$ remains small; that is, the line $p - qt_1 = 0$ is an asymptote to the curves. The lines of curvature therefore are as in fig. ii.



ii. Fig.

When a quantity m_2 is negative, then (again in spite of the small factor c in ξ) both p and q, as well as $p - qt_2$, tend to become large for values of t nearly equal to t_2, while near the point

$$p - qt_2 = \kappa q^{\frac{m_2}{m_2-1}},$$

$m_2/(m_2 - 1)$ of course being positive; that is, we have a parabolic asymptote. The lines of curvature therefore are as in fig. iii.

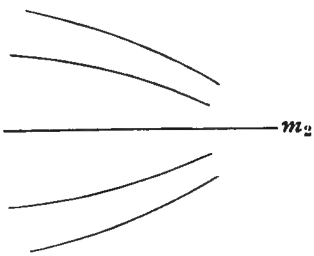


Fig. iii.

Combining these results, the whole arrangement for (α) is shewn in fig. iv, and the whole arrangement for (β) is shewn in fig. v, these giving the dispositions of the lines of curvature near the umbilicus.

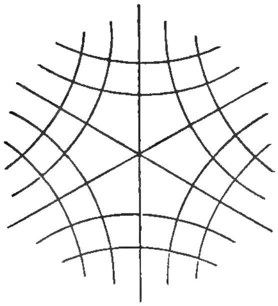


Fig. iv.

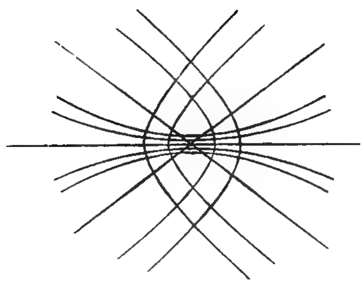


Fig. v.

The results and the diagrams were given first by Darboux.

The preceding investigation is the same as Darboux's, already cited (§ 71), in substance though it is formally different in analysis. Darboux refers the surface to the tangent plane at the umbilicus, so that its equation has the form

$$z = \tfrac{1}{2}k(x^2 + y^2) + \tfrac{1}{6}(ax^3 + 3bx^2y + 3b'xy^2 + a'y^3) + \dots .$$

The equation for the values of t is

$$b't^3 + (2b - a')t^2 + (a - 2b')t - b = 0;$$

and

$$m_1 = -\frac{(1 + t_1t_2)(1 + t_1t_3)}{(t_1 - t_2)(t_1 - t_3)},$$

with similar values for m_2 and m_3.

74. It was assumed that m_1, m_2, m_3 were all real. The alternative is that only one of them, say m_1, is real; then m_2 and m_3 are conjugate. As

$$m_1m_2m_3 = \frac{\lambda^3 V^2}{A_1^4(t_1 - t_2)^2(t_1 - t_3)^2(t_2 - t_3)^2},$$

it follows that $m_1 m_2 m_3$ is negative. Now $m_2 m_3$ is the product of two conjugate quantities and therefore is positive; hence m_1 is negative.

Then for values of t nearly equal to t_1, we have p, q, $p - t_1 q$ all large, while there is a parabolic asymptote

$$p - qt_1 = \kappa q^{\frac{m_1}{m_1 - 1}};$$

and so the configuration of the lines of curvature is as in fig. vi.

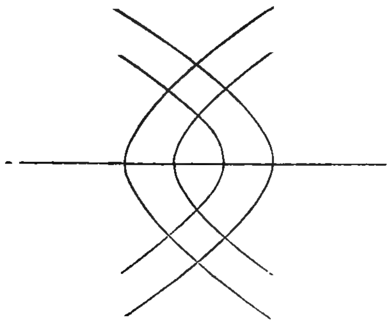


Fig. vi.

75. The suggested determination of the lines of curvature, by means of the differential equation of the first order which they satisfy, presupposes a knowledge of the fundamental magnitudes; and this may be neither given nor obtainable easily in the absence of obviously suitable parameters. Yet it may be necessary to have some integral equation giving the lines of curvature; and we can then proceed in one or other of two modes.

By the first, we make two of the Cartesian coordinates the parameters—say x and y; the parametric curves are then not orthogonal (save at special places), though their projections on the plane of z are orthogonal. Then with the usual notation for derivatives of z with respect to x and y, the fundamental magnitudes* are

$$E = 1 + p^2, \quad F = pq, \quad G = 1 + q^2,$$

$$\frac{L}{r} = \frac{M}{s} = \frac{N}{t} = (1 + p^2 + q^2)^{-\frac{1}{2}};$$

and so the differential equation of the lines becomes

$$\{(1 + p^2) s - pqr\} dx^2 + \{(1 + p^2) t - (1 + q^2) r\} dxdy + \{pqt - (1 + q^2) s\} dy^2 = 0.$$

The primitive of this equation will give the projection of the lines of curvature upon the plane $z = 0$; when it is combined with the equation of the surface, we have equations sufficient for the analytical expression of the lines of curvature.

* See Ex. 3, p. 60.

76. The other mode of proceeding depends upon the use of another form of the equation; and it is effective with some special classes of surfaces.

Let u, v, w denote three quantities proportional to the direction-cosines of the normal, so that

$$u : v : w = X : Y : Z = \frac{\partial\phi}{\partial x} : \frac{\partial\phi}{\partial y} : \frac{\partial\phi}{\partial z},$$

if the equation of the surface is $\phi(x, y, z) = 0$; then any point on the normal is given by

$$\xi = x + lu, \quad \eta = y + lv, \quad \zeta = z + lw,$$

where l is a variable parameter. Take the consecutive normal at a point along a line of curvature; denote by ξ, η, ζ the point where the two normals meet, and by $\xi + d\xi, \eta + d\eta, \zeta + d\zeta$ the point where the second normal is met by the normal at a second consecutive point along the line of curvature. Then $d\xi, d\eta, d\zeta$ is an element of the normal, so that

$$\frac{d\xi}{u} = \frac{d\eta}{v} = \frac{d\zeta}{w} = \mu,$$

say; hence

$$\mu u = dx + l\,du + u\,dl,$$

that is,

$$0 = dx + l\,du + u\,(dl - \mu);$$

and similarly

$$0 = dy + l\,dv + v\,(dl - \mu),$$

$$0 = dz + l\,dw + w\,(dl - \mu).$$

Hence

$$\begin{vmatrix} dx, & dy, & dz \\ du, & dv, & dw \\ u, & v, & w \end{vmatrix} = 0,$$

an equation satisfied in connection with variations along a line of curvature; it is a differential equation of the lines of curvature.

If, with the concurrent use of $\phi(x, y, z) = 0$ and derivatives from it, we can obtain a couple of independent integrals

$$f(x, y, z) = p, \quad g(x, y, z) = q,$$

where p and q are arbitrary quantities, these are the equations of the lines of curvature; p and q are the parametric variables of the lines. The difficulty, of course, lies in obtaining such integrals; there is no general process by which the integration is reduced to mere quadratures.

As an example, let us find equations for the lines of curvature on the cubic surface

$$xyz = 1.$$

We can take

$$u = \frac{1}{x}, \quad v = \frac{1}{y}, \quad w = \frac{1}{z}.$$

The differential equation is

$$\begin{vmatrix} dx, & dy, & dz \\ \dfrac{dx}{x^2}, & \dfrac{dy}{y^2}, & \dfrac{dz}{z^2} \\ \dfrac{1}{x}, & \dfrac{1}{y}, & \dfrac{1}{z} \end{vmatrix} = 0,$$

and so quantities A and B exist such that

$$\frac{1}{x} = \left(A + \frac{B}{x^2}\right) dx, \quad \frac{1}{y} = \left(A + \frac{B}{y^2}\right) dy, \quad \frac{1}{z} = \left(A + \frac{B}{z^2}\right) dz,$$

that is, if $B = \lambda A$,

$$A\frac{dx}{x} = \frac{1}{x^2+\lambda}, \qquad A\frac{dy}{y} = \frac{1}{y^2+\lambda}, \qquad A\frac{dz}{z} = \frac{1}{z^2+\lambda}.$$

But any direction on the surface is such that

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

Hence there are two values of λ, each associated with a line of curvature; and they are given by the equation

$$\frac{1}{x^2+\lambda} + \frac{1}{y^2+\lambda} + \frac{1}{z^2+\lambda} = 0.$$

Take the line of curvature $d'x$, $d'y$, $d'z$ which is perpendicular to dx, dy, dz, so that

$$dx\,d'x + dy\,d'y + dz\,d'z = 0;$$

then, along that line,

$$\frac{x\,d'x}{x^2+\lambda} + \frac{y\,d'y}{y^2+\lambda} + \frac{z\,d'z}{z^2+\lambda} = 0.$$

Also

$$\frac{d\lambda}{x^2+\lambda} + \frac{d\lambda}{y^2+\lambda} + \frac{d\lambda}{z^2+\lambda} = 0;$$

adding and effecting a quadrature, we have

$$(x^2+\lambda)(y^2+\lambda)(z^2+\lambda) = \text{constant}.$$

Let λ_1 and λ_2 denote the roots of the equation

$$\frac{1}{x^2+\lambda} + \frac{1}{y^2+\lambda} + \frac{1}{z^2+\lambda} = 0,$$

regarded as a quadratic in λ; then the lines of curvature on the surface $xyz = 1$ are given by

$$\left.\begin{aligned} (x^2+\lambda_1)(y^2+\lambda_1)(z^2+\lambda_1) &= p \\ (x^2+\lambda_2)(y^2+\lambda_2)(z^2+\lambda_2) &= q \end{aligned}\right\},$$

where p and q are the parametric variables.

Changing p and q into $4p$ and $4q$, Cayley shewed that these equations are equivalent to

$$\left.\begin{aligned} x^2 + \omega y^2 + \omega^2 z^2 &= 3\,(p^{\frac{1}{2}} + q^{\frac{1}{2}})^{\frac{2}{3}} \\ x^2 + \omega^2 y^2 + \omega z^2 &= 3\,(p^{\frac{1}{2}} - q^{\frac{1}{2}})^{\frac{2}{3}} \end{aligned}\right\},$$

where ω is an imaginary cube root of unity.

77. We have already seen that the conditions necessary and sufficient to make the parametric curves lines of curvature are

$$F = 0, \quad M = 0,$$

the first of which makes them perpendicular and the second of which makes them conjugate. We then have

$$\Gamma = \frac{1}{2}\frac{E_1}{E}, \quad \Gamma' = \frac{1}{2}\frac{E_2}{E}, \quad \Gamma'' = -\frac{1}{2}\frac{G_1}{E},$$

$$\Delta = -\frac{1}{2}\frac{E_2}{G}, \quad \Delta' = \frac{1}{2}\frac{G_1}{G}, \quad \Delta'' = \frac{1}{2}\frac{G_2}{G}.$$

Also, since $M = 0$, we have

$$x_{12} = x_1\Gamma' + x_2\Delta',$$

$$y_{12} = y_1\Gamma' + y_2\Delta',$$

$$z_{12} = z_1\Gamma' + z_2\Delta'.$$

Hence, if θ denote any one of the three coordinates x, y, z, the equation

$$\frac{\partial^2\theta}{\partial p\partial q} - \frac{1}{2}\frac{1}{E}\frac{\partial E}{\partial q}\frac{\partial\theta}{\partial p} - \frac{1}{2}\frac{1}{G}\frac{\partial G}{\partial p}\frac{\partial\theta}{\partial q} = 0$$

is satisfied, when the parametric variables p and q belong to the lines of curvature, a result first given by Lamé. We can verify at once that $\theta = x^2 + y^2 + z^2$ also satisfies the equation.

But the fact that x, y, z satisfy an equation

$$\frac{\partial^2\theta}{\partial p\partial q} - \lambda\frac{\partial\theta}{\partial p} - \mu\frac{\partial\theta}{\partial q} = 0$$

is not a consequence that comes only when lines of curvature are parametric curves. The equation is of the same form when the parametric curves are merely conjugate, without being perpendicular; in the latter case, however, the values of λ and μ (being Γ' and Δ') have the general form given in § 34 and not the above special form; and $\theta = x^2 + y^2 + z^2$ does not satisfy the equation.

The Mainardi-Codazzi relations, when $F = 0$ and $M = 0$, become

$$\left.\begin{aligned} L_2 &= -N\Delta + L\Gamma' = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)E_2 \\ N_1 &= -L\Gamma'' + N\Delta' = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)G_1 \end{aligned}\right\}.$$

The principal radii of curvature are α, along $p = \text{constant}$, and β, along $q = \text{constant}$; thus

$$\alpha = \frac{G}{N}, \quad \beta = \frac{E}{L},$$

so that

$$H = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{L}{E} + \frac{N}{G}, \quad K = \frac{1}{\alpha\beta} = \frac{LN}{EG}.$$

But

$$\frac{\partial}{\partial q}\left(\frac{1}{\beta}\right)=\frac{\partial}{\partial q}\left(\frac{L}{E}\right)$$
$$=\frac{L_2}{E}-\frac{L}{E^2}E_2$$
$$=\frac{1}{2}\frac{E_2}{E}\left(\frac{N}{G}-\frac{L}{E}\right)$$
$$=\frac{1}{2}\frac{E_2}{E}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right);$$

and similarly

$$\frac{\partial}{\partial p}\left(\frac{1}{\alpha}\right)=\frac{1}{2}\frac{G_1}{G}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right);$$

hence the Mainardi-Codazzi relations can be written

$$\left.\begin{aligned}\frac{\partial}{\partial p}\left(\frac{1}{\alpha}\right)&=\frac{1}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\frac{\partial \log G}{\partial p}\\ \frac{\partial}{\partial q}\left(\frac{1}{\beta}\right)&=\frac{1}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{\partial \log E}{\partial q}\end{aligned}\right\}.$$

The Gauss characteristic equation is

$$LN=-\tfrac{1}{2}(E_{22}+G_{11})+\frac{1}{4G}(E_2G_2+G_1^2)+\frac{1}{4E}(E_1G_1+E_2^2).$$

But

$$\frac{\partial}{\partial p}\left(\frac{1}{E^{\frac{1}{2}}}\frac{\partial G^{\frac{1}{2}}}{\partial p}\right)=\frac{1}{2E^{\frac{1}{2}}G^{\frac{1}{2}}}G_{11}-\frac{1}{4}\frac{G_1}{E^{\frac{1}{2}}G^{\frac{1}{2}}}\left(\frac{E_1}{E}+\frac{G_1}{G}\right),$$
$$\frac{\partial}{\partial q}\left(\frac{1}{G^{\frac{1}{2}}}\frac{\partial E^{\frac{1}{2}}}{\partial q}\right)=\frac{1}{2E^{\frac{1}{2}}G^{\frac{1}{2}}}E_{22}-\frac{1}{4}\frac{E_2}{E^{\frac{1}{2}}G^{\frac{1}{2}}}\left(\frac{E_2}{E}+\frac{G_2}{G}\right);$$

and therefore the Gauss equation becomes

$$\frac{\partial}{\partial p}\left(\frac{1}{E^{\frac{1}{2}}}\frac{\partial G^{\frac{1}{2}}}{\partial p}\right)+\frac{\partial}{\partial q}\left(\frac{1}{G^{\frac{1}{2}}}\frac{\partial E^{\frac{1}{2}}}{\partial q}\right)=-\frac{LN}{E^{\frac{1}{2}}G^{\frac{1}{2}}}$$
$$=-\frac{E^{\frac{1}{2}}G^{\frac{1}{2}}}{\alpha\beta},$$

which can also be obtained directly from Liouville's form (§ 36).

The derived magnitudes of the third order are

$$\left.\begin{aligned}P&=L_1-\frac{L}{E}E_1=E\frac{\partial}{\partial p}\left(\frac{L}{E}\right)=E\frac{\partial}{\partial p}\left(\frac{1}{\beta}\right)\\ Q&=L_2-\frac{L}{E}E_2=\tfrac{1}{2}E_2\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)=E\frac{\partial}{\partial q}\left(\frac{1}{\beta}\right)\\ R&=N_1-\frac{N}{G}G_1=\tfrac{1}{2}G_1\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)=G\frac{\partial}{\partial p}\left(\frac{1}{\alpha}\right)\\ S&=N_2-\frac{N}{G}G_2=G\frac{\partial}{\partial q}\left(\frac{N}{G}\right)=G\frac{\partial}{\partial q}\left(\frac{1}{\alpha}\right)\end{aligned}\right\}.$$

78. Consider, in particular, the formulæ for a central quadric

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1\,;$$

they will be required later in dealing with its surface of centres. We know that the lines of curvature are the intersections of the quadric with the confocal quadrics

$$\frac{x^2}{a+p}+\frac{y^2}{b+p}+\frac{z^2}{c+p}=1, \qquad \frac{x^2}{a+q}+\frac{y^2}{b+q}+\frac{z^2}{c+q}=1,$$

so that p and q are the parametric variables of the lines of curvature upon the given quadric. The coordinates of a point on the quadric are given by

$$\left.\begin{aligned} -\beta\gamma x^2 &= a\,(a+p)\,(a+q)\\ -\gamma\alpha y^2 &= b\,(b+p)\,(b+q)\\ -\alpha\beta z^2 &= c\,(c+p)\,(c+q)\end{aligned}\right\},$$

where

$$\alpha,\beta,\gamma=b-c,\ c-a,\ a-b.$$

Then, if r is the distance of a point on the quadric from the centre, and if ϖ is the perpendicular from the centre on the tangent plane at the point,

$$\left.\begin{aligned} r^2 &= x^2+y^2+z^2=a+b+c+p+q\\ \frac{1}{\varpi^2} &= \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{pq}{abc}\end{aligned}\right\}.$$

We have

$$\left.\begin{aligned} E &= \frac{1}{4}\frac{p\,(p-q)}{(a+p)\,(b+p)\,(c+p)}\\ G &= \frac{1}{4}\frac{q\,(q-p)}{(a+q)\,(b+q)\,(c+q)}\\ F &= 0\end{aligned}\right\}.$$

Also

$$\left.\begin{aligned} X &= \left\{-\frac{bc}{\beta\gamma}\frac{(a+p)\,(a+q)}{pq}\right\}^{\frac{1}{2}}\\ Y &= \left\{-\frac{ca}{\gamma\alpha}\frac{(b+p)\,(b+q)}{pq}\right\}^{\frac{1}{2}}\\ Z &= \left\{-\frac{ab}{\alpha\beta}\frac{(c+p)\,(c+q)}{pq}\right\}^{\frac{1}{2}}\end{aligned}\right\};$$

and

$$\left.\begin{aligned} L &= \frac{1}{4}\left(\frac{abc}{pq}\right)^{\frac{1}{2}}\frac{q-p}{(a+p)\,(b+p)\,(c+p)}\\ N &= \frac{1}{4}\left(\frac{abc}{pq}\right)^{\frac{1}{2}}\frac{p-q}{(a+q)\,(b+q)\,(c+q)}\\ M &= 0\end{aligned}\right\}.$$

The principal radii of curvature (say α' and β') are

$$\left.\begin{aligned}\frac{1}{\alpha'}&=\frac{N}{G}=-\left(\frac{abc}{pq}\right)^{\frac{1}{2}}\frac{1}{q}\\ \frac{1}{\beta'}&=\frac{L}{E}=-\left(\frac{abc}{pq}\right)^{\frac{1}{2}}\frac{1}{p}\end{aligned}\right\}.$$

The umbilici are given by

$$p=q;$$

but the quantities p and q are separated in general by $-a, -b, -c$, according to the values of these quantities; and so, at an umbilicus, $p=q=-a$ or $-b$ or $-c$. Thus, on an ellipsoid for which $a>b>c>0$, the umbilici are given by $p=q=-b$.

The magnitudes of the third order are

$$P=-\frac{3}{2}\frac{L}{p},\qquad Q=-\frac{1}{2}\frac{L}{q},\qquad R=-\frac{1}{2}\frac{N}{p},\qquad S=-\frac{3}{2}\frac{N}{q}.$$

79. Among the many simple properties of relation between a surface and the surfaces derived from it by inversion, we have the property* that *when a surface is inverted, its lines of curvature are transformed into lines of curvature.*

Let c be the radius of inversion, and take the centre of inversion as origin. Denoting by ξ, η, ζ the point which is the inverse of x, y, z, we have

$$\xi=c^2\frac{x}{r^2},\quad \eta=c^2\frac{y}{r^2},\quad \zeta=c^2\frac{z}{r^2},\quad r^2=x^2+y^2+z^2.$$

Then

$$\xi_1=c^2\frac{x_1}{r^2}-2\frac{c^2x}{r^3}r_1,\quad \xi_2=c^2\frac{x_2}{r^2}-2\frac{c^2x}{r^3}r_2,$$

with similar values for $\eta_1, \eta_2, \zeta_1, \zeta_2$; while

$$rr_1=xx_1+yy_1+zz_1,\quad rr_2=xx_2+yy_2+zz_2.$$

Hence, for the inverse surface,

$$E'=\xi_1^2+\eta_1^2+\zeta_1^2=\frac{c^4}{r^4}E,$$

$$F'=\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2=\frac{c^4}{r^4}F,$$

$$G'=\xi_2^2+\eta_2^2+\zeta_2^2=\frac{c^4}{r^4}G;$$

and therefore

$$V'=\frac{c^4}{r^4}V.$$

* It appears to have been noted first by Hirst, *Ann. di Mat.*, t. ii (1859), p. 164.

For the direction-cosines X', Y', Z' of the normal to the inverse surface, we have

$$X' = \frac{1}{V'}(\eta_1\zeta_2 - \eta_2\zeta_1) = \frac{2}{r^2}xW - X,$$

$$Y' = \frac{1}{V'}(\zeta_1\xi_2 - \zeta_2\xi_1) = \frac{2}{r^2}yW - Y,$$

$$Z' = \frac{1}{V'}(\xi_1\eta_2 - \xi_2\eta_1) = \frac{2}{r^2}zW - Z,$$

where

$$W = xX + yY + zZ,$$

and W is the perpendicular from the origin on the tangent plane. Also

$$W' = \frac{c^2}{r^2}W.$$

Again,

$$\xi_{11} = c^2\frac{x_{11}}{r^2} - 4c^2\frac{x_1}{r^3}r_1 + 6\frac{c^2x}{r^4}r_1^2 - 2\frac{c^2x}{r^3}r_{11},$$

with corresponding values of η_{11}, ζ_{11}, while

$$rr_{11} = xx_{11} + yy_{11} + zz_{11} + E - r_1^2;$$

hence

$$\begin{aligned} L' &= X'\xi_{11} + Y'\eta_{11} + Z'\zeta_{11} \\ &= 2\frac{W}{r^2}\Sigma x\xi_{11} - \Sigma X\xi_{11} \\ &= \frac{2W}{r^2}\left(-E\frac{c^2}{r^2} + 3\frac{c^2}{r^2}r_1^2 - \frac{c^2}{r}r_{11}\right) - \left(\frac{c^2}{r^2}L + 6\frac{c^2}{r^4}r_1^2W - 2\frac{c^2}{r^3}r_{11}W\right) \\ &= -\frac{2c^2W}{r^4}E - \frac{c^2}{r^2}L. \end{aligned}$$

Similarly

$$M' = -\frac{2c^2W}{r^4}F - \frac{c^2}{r^2}M, \qquad N' = -\frac{2c^2W}{r^4}G - \frac{c^2}{r^2}N.$$

Consequently

$$E'M' - F'L' = -\frac{c^6}{r^6}(EM - FL),$$

$$E'N' - G'L' = -\frac{c^6}{r^6}(EN - GL),$$

$$F'N' - G'M' = -\frac{c^6}{r^6}(FN - GM);$$

and therefore the lines of curvature of the inverse surface, being given by

$$(E'M' - F'L')\,dp^2 + (E'N' - G'L')\,dp\,dq + (F'N' - G'M')\,dq^2 = 0,$$

that is, by

$$(EM - FL)\,dp^2 + (EN - GL)\,dp\,dq + (FN - GM)\,dq^2 = 0,$$

are the same as the lines of curvature upon the original surface. The proposition therefore stands.

We may further note that

(i) *the angle between any two curves is unaltered by inversion ;*

(ii) *nul lines are inverted into nul lines ;*

(iii) *conjugate lines are not generally inverted into conjugate lines ;*

(iv) *asymptotic lines are not generally inverted into asymptotic lines.*

Let the principal radii of curvature of the new surface be α', along $p=$ constant, and β', along $q=$ constant. Then

$$\frac{1}{\alpha'}=\frac{N'}{G'}=-\frac{r^2}{c^2}\frac{1}{\alpha}-2\frac{W}{c^2}, \qquad \frac{1}{\beta'}=\frac{L'}{E'}=-\frac{r^2}{c^2}\frac{1}{\beta}-2\frac{W}{c^2};$$

so that

$$\frac{1}{\alpha'}-\frac{1}{\beta'}=-\frac{r^2}{c^2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right),$$

and therefore *umbilici are inverted into umbilici.* For any normal section,

$$\frac{1}{\rho'}=-\frac{r^2}{c^2}\frac{1}{\rho}-2\frac{W}{c^2}.$$

The mean measure of curvature of the new surface is

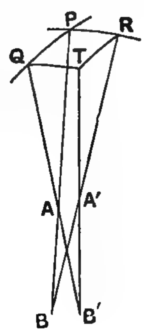
$$H'=-\frac{r^2}{c^2}H-4\frac{W}{c^2};$$

and the total curvature of the new surface is

$$K'=\frac{r^4}{c^4}K+2\frac{r^2W}{c^4}H+4\frac{W^2}{c^4}.$$

Surface of Centres.

80. We have already defined the *surface of centres* of a surface (sometimes called its *evolute*, sometimes its *centro-surface*) as the locus of the centres of curvature along the lines of curvature given by the intersection of consecutive normals. Usually it is a two-sheeted surface which may have singular lines even when the original surface is free from singularities; but in the case of a surface of revolution one of the sheets degenerates into the axis of revolution; in the case of a developable surface one of the sheets lies entirely at infinity; and similarly for some other surfaces. We proceed to consider some of the simpler results belonging to centro-surfaces.



Let P be any point on a surface, referred to its lines of curvature as parametric curves; let PQ be the line $p=$ constant, PR the line $q=$ constant; and let QT, RT be the other lines of curvature through the consecutive points Q and R. The lines PAB, QAB', $RA'B$, $TA'B'$ are the normals to the surface at P, Q, R, T, intersecting as in the figure.

Because the consecutive normals along a line of curvature intersect, the normal planes to the surface through the tangents to a line of curvature have, as their envelope, a developable surface of which the normals are generators. The edge of regression of the developable surface is the locus

of the intersection of the normals, that is, a curve upon the centro-surface. and so it is the intersection of the centro-surface and the developable surface. At the point P, we have

$$PA = \alpha = \frac{G}{N}, \quad PB = \beta = \frac{E}{L}.$$

It will be convenient to call the sheet of the centro-surface given by the centres of curvature along the lines $p=$ constant the *first* sheet, this being the locus of the points A for all the points on all these lines; and similarly the sheet for the centres of curvature along the lines $q=$ constant will be called the *second* sheet, this being the locus of the points B.

The normals at P and R, meeting at a point B on the second sheet, touch the first sheet at A and A' respectively; thus AA' is a tangent line to the first sheet. Also PAB is a tangent line to the first sheet; consequently the plane PBR is the tangent plane at A to the first sheet. The normal at A to the first sheet is perpendicular to this plane and is therefore parallel to PQ at P. Similarly the plane PAQ is the tangent plane at B to the second sheet; and the normal at B to the second sheet is parallel to PR at P. Hence the normal at any point on either sheet is parallel to the tangent to the corresponding line of curvature; and the normals to the two sheets at the two centres belonging to any point on the surface are perpendicular to one another.

But AA' is not necessarily nor generally parallel to PR, nor is BB' necessarily or generally parallel to PQ. We are not therefore in a position to assert that AA' and BB' are perpendicular to one another; as will be seen later, it is only exceptionally (in one or other of two directions on the surface at P) that associated arcs on the two sheets of the centro-surface are perpendicular.

Next, consider the osculating plane of the edge of regression at A. It contains two consecutive tangents to that edge; that is, it contains two consecutive normals to the original surface along the line of curvature and therefore it contains the tangent to its line of curvature. But at the point on the sheet, the normal to the sheet is parallel to that tangent to the line of curvature. Consequently, the osculating plane at any point of the edge of regression that lies on the first sheet contains the normal to the sheet; it is therefore (§ 65) a geodesic on the sheet. The same holds for the second sheet. Hence the edge of regression on the developable surface generated by the normals to the given surface along a line of curvature is a geodesic on the corresponding sheet of the centro-surface.

81. As regards the analytical formulæ, let ξ, η, ζ be the coordinates of A, a point on the first sheet; and let ξ', η', ζ' be the coordinates of B, a point

on the second sheet; A and B being the associated centres for P. Then we have

$$x_1 + \beta X_1 = 0, \qquad x_2 + \alpha X_2 = 0,$$
$$y_1 + \beta Y_1 = 0, \qquad y_2 + \alpha Y_2 = 0,$$
$$z_1 + \beta Z_1 = 0, \qquad z_2 + \alpha Z_2 = 0.$$

But

$$\xi = x + \alpha X, \qquad \xi' = x + \beta X;$$

hence

$$\begin{aligned} d\xi &= (x_1 + \alpha X_1)\, dp + X d\alpha \\ &= x_1 \left(1 - \frac{\alpha}{\beta}\right) dp + X d\alpha, \\ d\xi' &= (x_2 + \beta X_2)\, dq + X d\beta \\ &= x_2 \left(1 - \frac{\beta}{\alpha}\right) dq + X d\beta; \end{aligned}$$

and so

$$\left.\begin{aligned} d\xi &= x_1 \left(1 - \frac{\alpha}{\beta}\right) dp + X d\alpha \\ d\eta &= y_1 \left(1 - \frac{\alpha}{\beta}\right) dp + Y d\alpha \\ d\zeta &= z_1 \left(1 - \frac{\alpha}{\beta}\right) dp + Z d\alpha \end{aligned}\right\}, \qquad \left.\begin{aligned} d\xi' &= x_2 \left(1 - \frac{\beta}{\alpha}\right) dq + X d\beta \\ d\eta' &= y_2 \left(1 - \frac{\beta}{\alpha}\right) dq + Y d\beta \\ d\zeta' &= z_2 \left(1 - \frac{\beta}{\alpha}\right) dq + Z d\beta \end{aligned}\right\}.$$

In the first place, we notice that

$$d\xi d\xi' + d\eta d\eta' + d\zeta d\zeta' = d\alpha d\beta;$$

and therefore associated arcs on the two sheets are perpendicular only if $d\alpha$ or $d\beta$ is zero, that is, only for directions

$$P dp + Q dq = 0, \qquad R dp + S dq = 0,$$

on the original surface (§ 77).

In the second place, we notice that along the geodesic on the first sheet, which is the edge of regression of the developable surface generated by the normals along $p =$ constant, we have

$$\delta\xi = X\delta\alpha, \qquad \delta\eta = Y\delta\alpha, \qquad \delta\zeta = Z\delta\alpha.$$

If the orthogonal trajectory of this curve on the sheet be given at the point by the foregoing values of $d\xi$, $d\eta$, $d\zeta$, then the necessary and sufficient condition

$$d\xi\delta\xi + d\eta\delta\eta + d\zeta\delta\zeta = 0$$

leads to

$$d\alpha = 0;$$

that is, the orthogonal trajectories of the regressional geodesics are given by

the curves on the sheet corresponding to the loci on the surface at which the curvature of the associated line of curvature is constant.

We can at once verify from the formulæ that the normal at any point to a sheet is parallel to the tangent to the associated line of curvature. Thus, for the first sheet, we have

$$\xi_1 = x_1\left(1 - \frac{\alpha}{\beta}\right) + X\alpha_1, \qquad \xi_2 = X\alpha_2,$$

$$\eta_1 = y_1\left(1 - \frac{\alpha}{\beta}\right) + Y\alpha_1, \qquad \eta_2 = Y\alpha_2,$$

$$\zeta_1 = z_1\left(1 - \frac{\alpha}{\beta}\right) + Z\alpha_1, \qquad \zeta_2 = Z\alpha_2;$$

at the point, the direction-cosines of the normal to the sheet are proportional to

$$\left\| \begin{matrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \end{matrix} \right\|,$$

that is, to

$$\left\| \begin{matrix} x_1, & y_1, & z_1 \\ X, & Y, & Z \end{matrix} \right\|,$$

that is, to x_2, y_2, z_2, which are proportional to the direction-cosines of the line of curvature $p =$ constant. Similarly, for the other sheet. Let these direction-cosines be A, B, C for the first sheet, and be A', B', C' for the second sheet; then

$$\left.\begin{aligned} A &= G^{-\frac{1}{2}} x_2, \qquad B = G^{-\frac{1}{2}} y_2, \qquad C = G^{-\frac{1}{2}} z_2 \\ A' &= E^{-\frac{1}{2}} x_1, \qquad B' = E^{-\frac{1}{2}} y_1, \qquad C' = E^{-\frac{1}{2}} z_1 \end{aligned}\right\}.$$

Let $d\sigma$ denote an elementary arc on the first sheet, and $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ denote the fundamental magnitudes of the first order for that sheet; and let $d\sigma'$, $\mathbf{E}'$, $\mathbf{F}'$, $\mathbf{G}'$ have the similar significance for the second sheet. Then

$$d\sigma^2 = d\xi^2 + d\eta^2 + d\zeta^2 = E\left(1 - \frac{\alpha}{\beta}\right)^2 dp^2 + d\alpha^2,$$

$$d\sigma'^2 = d\xi'^2 + d\eta'^2 + d\zeta'^2 = d\beta^2 + G\left(1 - \frac{\beta}{\alpha}\right)^2 dq^2,$$

so that

$$\left.\begin{aligned} \mathbf{E} &= \alpha_1^2 + E\left(1 - \frac{\alpha}{\beta}\right)^2 \\ \mathbf{F} &= \alpha_1\alpha_2 \\ \mathbf{G} &= \alpha_2^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \mathbf{G}' &= \beta_2^2 + G\left(1 - \frac{\beta}{\alpha}\right)^2 \\ \mathbf{F}' &= \beta_1\beta_2 \\ \mathbf{E}' &= \beta_1^2 \end{aligned}\right\}.$$

It is to be noted, from the form of the expression for $d\sigma$, that the curves $p =$ constant are geodesics upon the first sheet, while the curves $\alpha =$ constant are their orthogonal trajectories—in agreement with former results; and similarly for the curves $q =$ constant and $\beta =$ constant on the second sheet.

Let $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$ denote the fundamental magnitudes of the second order for the first sheet; and let $\mathbf{L}'$, $\mathbf{M}'$, $\mathbf{N}'$ have the similar significance for the second sheet. Then

$$\mathbf{L} = A\xi_{11} + B\eta_{11} + C\zeta_{11} = -\tfrac{1}{2}G^{-\frac{1}{2}}\left(1 - \frac{\alpha}{\beta}\right)E_2 = \frac{E}{G^{\frac{1}{2}}}\frac{\alpha}{\beta^2}\beta_2,$$

$$\mathbf{M} = A\xi_{12} + B\eta_{12} + C\zeta_{12} = 0,$$

$$\mathbf{N} = A\xi_{22} + B\eta_{22} + C\zeta_{22} = -G^{\frac{1}{2}}\frac{\alpha_2}{\alpha};$$

and similarly for $\mathbf{L}'$, $\mathbf{M}'$, $\mathbf{N}'$. The whole set of values is

$$\left.\begin{aligned} \mathbf{L} &= \frac{E}{G^{\frac{1}{2}}}\frac{\alpha}{\beta^2}\beta_2 \\ \mathbf{M} &= 0 \\ \mathbf{N} &= -G^{\frac{1}{2}}\frac{\alpha_2}{\alpha} \end{aligned}\right\}, \qquad \left.\begin{aligned} \mathbf{N}' &= \frac{G}{E^{\frac{1}{2}}}\frac{\beta}{\alpha^2}\alpha_1 \\ \mathbf{M}' &= 0 \\ \mathbf{L}' &= -E^{\frac{1}{2}}\frac{\beta_1}{\beta} \end{aligned}\right\}.$$

The values of the fundamental magnitudes for either sheet are deduced from the set for the other sheet by interchanging simultaneously p and q, E and G, L and N, α and β.

Manifestly, neither $\mathbf{F} = 0$ nor $\mathbf{F}' = 0$ save in the special circumstances that a principal radius of curvature is a function of one of the parameters only. But $\mathbf{M} = 0$ and $\mathbf{M}' = 0$; hence *the two curves on either sheet, that correspond to lines of curvature on the original surface, are conjugate to one another, but in general are not lines of curvature on the sheet.*

The total curvatures for the two sheets are

$$\mathbf{K} = -\frac{1}{(\alpha - \beta)^2}\frac{\beta_2}{\alpha_2}, \qquad \mathbf{K}' = -\frac{1}{(\alpha - \beta)^2}\frac{\alpha_1}{\beta_1};$$

and the measures of mean curvature are

$$\mathbf{H} = \frac{1}{G^{\frac{1}{2}}}\frac{\alpha\beta_2}{(\alpha - \beta)^2} - \frac{G^{\frac{1}{2}}\alpha_1^2\beta^2}{E\alpha\alpha_2(\alpha - \beta)^2} - \frac{G^{\frac{1}{2}}}{\alpha\alpha_2},$$

$$\mathbf{H}' = \frac{1}{E^{\frac{1}{2}}}\frac{\beta\alpha_1}{(\alpha - \beta)^2} - \frac{E^{\frac{1}{2}}\beta_2^2\alpha^2}{G\beta\beta_1(\alpha - \beta)^2} - \frac{E^{\frac{1}{2}}}{\beta\beta_1}.$$

82. The lines of curvature on the first sheet, being in general

$$(\mathbf{EM} - \mathbf{FL})\,dp^2 + (\mathbf{EN} - \mathbf{GL})\,dp\,dq + (\mathbf{FN} - \mathbf{GM})\,dq^2 = 0,$$

are, on substitution, given by the equation

$$\alpha_1\alpha_2\frac{E}{G^{\frac{1}{2}}}\frac{\alpha\beta_2}{\beta^2}dp^2 + \alpha_1\alpha_2 G^{\frac{1}{2}}\frac{\alpha_2}{\alpha}dq^2 + \left[\left\{\alpha_1^2 + E\left(1 - \frac{\alpha}{\beta}\right)^2\right\}G^{\frac{1}{2}}\frac{\alpha_2}{\alpha} + \frac{E}{G^{\frac{1}{2}}}\frac{\alpha}{\beta^2}\alpha_2^2\beta_2\right]dp\,dq = 0.$$

The lines of curvature on the second sheet similarly are given by the equation

$$\beta_1\beta_2 E^{\frac{1}{2}}\frac{\beta_1}{\beta}dp^2+\beta_1\beta_2\frac{G}{E^{\frac{1}{2}}}\frac{\beta\alpha_1}{\alpha^2}dq^2$$
$$+\left[\left\{\beta_2{}^2+G\left(1-\frac{\beta}{\alpha}\right)^2\right\}E^{\frac{1}{2}}\frac{\beta_1}{\beta}+\frac{G}{E^{\frac{1}{2}}}\frac{\beta}{\alpha^2}\alpha_1\beta_1{}^2\right]dpdq=0.$$

These two equations are the same if the coefficients are proportional to one another. Writing them momentarily in the form

$$a\,dp^2+b\,dpdq+c\,dq^2=0,$$
$$a'dp^2+b'dpdq+c'dq^2=0,$$

we have the necessary conditions given by

$$\frac{c'}{c}=\frac{b'}{b}=\frac{a'}{a}.$$

The condition $\frac{c'}{c}=\frac{a'}{a}$ leads to a relation

$$\frac{\beta_2}{\alpha_2}=\frac{\beta_1}{\alpha_1};$$

and, when this relation is used, the condition $\frac{b'}{b}=\frac{a'}{a}$ leads to a relation

$$\alpha_1=\beta_1;$$

that is, we have

$$\alpha_1-\beta_1=0,\quad \alpha_2-\beta_2=0.$$

Consequently we must have

$$\alpha-\beta=\text{constant},$$

and so we have the theorem due to Ribaucour*:

When the lines of curvature on one sheet of the centro-surface correspond to the lines of curvature on the other sheet (that is, when they are determined by the same analytical relation for the two sheets), the difference of the principal radii of curvature of the original surface is constant.

Moreover, we then have

$$\mathbf{K}=\mathbf{K}'=-\frac{1}{(\alpha-\beta)^2};$$

that is, the Gauss measure of curvature is constant and negative and the same everywhere on each of the sheets.

83. The asymptotic lines on the first sheet, being in general

$$\mathbf{L}dp^2+2\mathbf{M}dpdq+\mathbf{N}dq^2=0,$$

are, on substitution, given by the equation

$$E\alpha^2\beta_2dp^2-G\beta^2\alpha_2dq^2=0.$$

Those on the second sheet are given by

$$E\alpha^2\beta_1dp^2-G\beta^2\alpha_1dq^2=0.$$

* *Comptes Rendus*, t. lxxiv (1872), p. 1399.

These two equations are the same if

$$\alpha_1\beta_2 - \alpha_2\beta_1 = 0,$$

that is, if a relation

$$f(\alpha, \beta) = 0$$

subsists between the principal radii of curvature of the original surface without the occurrence of other variable quantities. Such surfaces are called Weingarten surfaces (§ 42) and are to be discussed later. Meanwhile, we have the result* that *the asymptotic lines on the two sheets of the centro-surface of a Weingarten surface correspond to one another; and, conversely, if the asymptotic lines on the two sheets of a centro-surface correspond to one another, the original surface is a Weingarten surface.*

84. As an example of the general theory, consider the centro-surface of an ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

and suppose that $a>b>c$, all three quantities being positive. The expressions for x, y, z, X, Y, Z, and for the principal radii of curvature have already (§ 78) been obtained. The radii of curvature are positive on the concave side of the surface (§ 31); hence the centres of curvature are

$$\left.\begin{aligned} \xi &= x - \beta' X = \left\{-\frac{1}{a\beta\gamma}(a+p)^3(a+q)\right\}^{\frac{1}{2}} \\ \eta &= y - \beta' Y = \left\{-\frac{1}{b\gamma\alpha}(b+p)^3(b+q)\right\}^{\frac{1}{2}} \\ \zeta &= z - \beta' Z = \left\{-\frac{1}{c\alpha\beta}(c+p)^3(c+q)\right\}^{\frac{1}{2}} \end{aligned}\right\},$$

$$\left.\begin{aligned} \xi' &= x - \alpha' X = \left\{-\frac{1}{a\beta\gamma}(a+p)(a+q)^3\right\}^{\frac{1}{2}} \\ \eta' &= y - \alpha' Y = \left\{-\frac{1}{b\gamma\alpha}(b+p)(b+q)^3\right\}^{\frac{1}{2}} \\ \zeta' &= z - \alpha' Z = \left\{-\frac{1}{c\alpha\beta}(c+p)(c+q)^3\right\}^{\frac{1}{2}} \end{aligned}\right\}.$$

Elimination of p and q among the values of ξ, η, ζ leads to a relation between ξ, η, ζ which is the equation of the first sheet. But elimination of p and q among the values of ξ', η', ζ' manifestly leads to the same equation; that is, there is a single equation representing the two sheets of the surface.

Now we have

$$\frac{a\xi^2}{(a+p)^3} + \frac{b\eta^2}{(b+p)^3} + \frac{c\zeta^2}{(c+p)^3} = 0,$$

$$\frac{a\xi^2}{(a+p)^2} + \frac{b\eta^2}{(b+p)^2} + \frac{c\zeta^2}{(c+p)^2} = 1;$$

* The result is also due to Ribaucour (*l.c.*).

and the equation would be obtainable by the elimination of p between these two relations. The elimination can be effected as follows. Take the equation

$$\frac{t\xi^2}{a+t}+\frac{t\eta^2}{b+t}+\frac{t\zeta^2}{c+t}=t+\theta,$$

where θ is a disposable parameter. It is a quartic in t; and the two preceding equations express the two conditions that the quartic should have a triple root. Write the quartic in the form

$$t^4+4k_1t^3+6k_2t^2+4k_3t+k_4=0,$$

where

$$\begin{aligned} 4k_1 &= \theta \quad + A - X,\\ 6k_2 &= \theta A + B - Y,\\ 4k_3 &= \theta B + C - Z,\\ k_4 &= \theta C, \end{aligned}$$

$$\begin{aligned} A &= a+b+c, & X &= \xi^2+\eta^2+\zeta^2,\\ B &= bc+ca+ab, & Y &= (b+c)\,\xi^2+(c+a)\,\eta^2+(a+b)\,\zeta^2,\\ C &= abc, & Z &= bc\xi^2+ca\eta^2+ab\zeta^2. \end{aligned}$$

The conditions that the quartic should have a triple root are that the quadrinvariant and the cubinvariant should vanish; hence

$$k_4-4k_1k_3+3k_2^2=0,$$

$$\begin{vmatrix} 1, & k_1, & k_2\\ k_1, & k_2, & k_3\\ k_2, & k_3, & k_4 \end{vmatrix}=0.$$

The former gives

$$\lambda_0\theta^2+\lambda_2\theta+\lambda_4=0,$$

and the latter gives

$$\mu_0\theta^3+\mu_2\theta^2+\mu_4\theta+\mu_6=0,$$

where λ_0 and μ_0 are of degree 0, λ_2 and μ_2 of degree 2, λ_4 and μ_4 of degree 4, and μ_6 of degree 6, in the variables, each of them being an even function in its own degree. Eliminating θ, we have

$$\begin{vmatrix} \lambda_0, & \lambda_2, & \lambda_4, & 0\,, & 0\\ 0\,, & \lambda_0, & \lambda_2, & \lambda_4, & 0\\ 0\,, & 0\,, & \lambda_0, & \lambda_2, & \lambda_4\\ \mu_0, & \mu_2, & \mu_4, & \mu_6, & 0\\ 0\,, & \mu_0, & \mu_2, & \mu_4, & \mu_6 \end{vmatrix}=0$$

as the equation of the centro-surface. Manifestly it is a surface of the twelfth order.

To obtain a notion of the form of the centro-surface, consider it near the plane $\zeta = 0$ and, in particular, its section by that plane. In that plane we must have $-p = c$, or $-q = c$. When $-p = c$, then

$$-\alpha\beta\gamma\xi^2 = (a-c)^3(a+q) = -\beta^3(a+q),$$
$$-b\gamma\alpha\eta^2 = (b-c)^3(b+q) = \alpha^3(b+q),$$

so that

$$\frac{a\xi^2}{\beta^2} + \frac{b\eta^2}{\alpha^2} = 1,$$

an ellipse, being the locus of points where the normal to the ellipsoid along the principal section is met by the normal at an adjacent point on the other line of curvature. For small values of ζ near that ellipse, we take $c + p = -P$ where P is small; then, approximately,

$$\xi = \left\{\frac{(a+q)\beta^2}{a\gamma}\right\}^{\frac{1}{2}}\left(1 - \frac{3}{2}\frac{P}{\beta}\right),$$
$$\eta = \left\{-\frac{(b+q)\alpha^2}{b\gamma}\right\}^{\frac{1}{2}}\left(1 - \frac{3}{2}\frac{P}{\alpha}\right),$$
$$\zeta = \left\{\frac{c+q}{c\alpha\beta}\right\}^{\frac{1}{2}} P^{\frac{3}{2}},$$

so that there is a cuspidal edge of the centro-surface at the ellipse. When $-q = c$, then

$$(a\gamma\xi^2)^{\frac{1}{3}} = a + p, \quad (-b\gamma\eta^2)^{\frac{1}{3}} = b + p,$$

so that

$$(a\xi^2)^{\frac{1}{3}} + (b\eta^2)^{\frac{1}{3}} = \gamma^{\frac{2}{3}},$$

the evolute of the principal section of the ellipsoid. For small values of ζ near that evolute, we take $c + q = -Q$ where Q is small; then

$$-\alpha\beta\gamma\xi^2 = (a+p)^3(-\beta - Q), \quad -b\gamma\alpha\eta^2 = (b+p)^3(\alpha - Q), \quad -c\alpha\beta\zeta^2 = -(c+p)^3 Q,$$

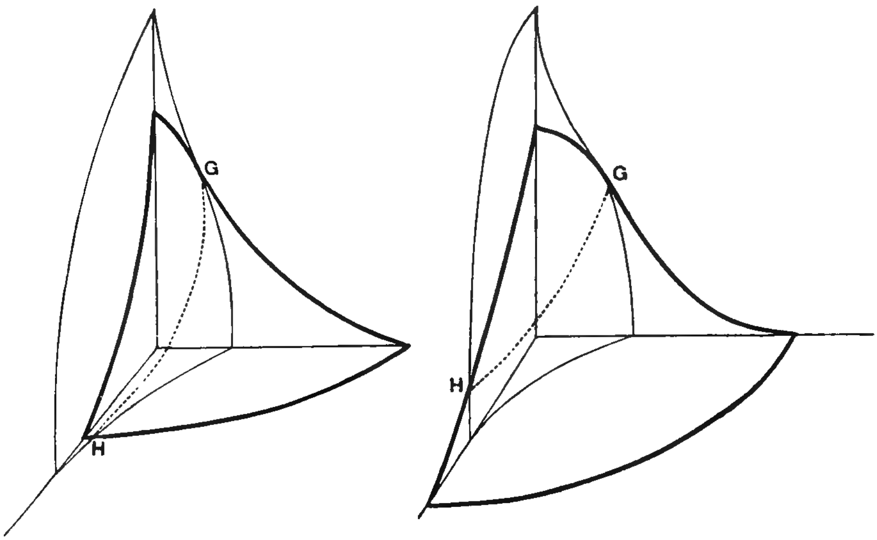
so that the plane $\zeta = 0$ is normal to the surface and the evolute section is an ordinary curve upon the surface. As regards the degree of the intersection, the degree of the ellipse must be counted thrice because of the cuspidal edge, and the degree of the evolute is six; hence the degree of the intersection is twelve, as is to be expected.

Similarly for the other coordinate planes. The sections are:

$$\text{in } \zeta = 0, \qquad \frac{a\xi^2}{\beta^2} + \frac{b\eta^2}{\alpha^2} = 1, \qquad (a\xi^2)^{\frac{1}{3}} + (b\eta^2)^{\frac{1}{3}} = \gamma^{\frac{2}{3}},$$
$$\text{in } \xi = 0, \qquad \frac{b\eta^2}{\gamma^2} + \frac{c\zeta^2}{\beta^2} = 1, \qquad (b\eta^2)^{\frac{1}{3}} + (c\zeta^2)^{\frac{1}{3}} = \alpha^{\frac{2}{3}},$$
$$\text{in } \eta = 0, \qquad \frac{c\zeta^2}{\alpha^2} + \frac{a\xi^2}{\gamma^2} = 1, \qquad (c\zeta^2)^{\frac{1}{3}} + (a\xi^2)^{\frac{1}{3}} = \beta^{\frac{2}{3}}.$$

The form of the two sheets of the surface in the positive octant is as shewn in the figures. In the left, $a + c > 2b$; in the right, $a + c < 2b$. The point G

corresponds to an umbilicus on the ellipsoid. The dotted line GH is a nodal curve where the sheets intersect; and they touch one another at G.



The nodal curve is given by the equality of ξ, η, ζ for one set of values of p and q to ξ', η', ζ' for another set of values p' and q'. Thus

$$(a+p)^3(a+q) = (a+p')(a+q')^3,$$
$$(b+p)^3(b+q) = (b+p')(b+q')^3,$$
$$(c+p)^3(c+q) = (c+p')(c+q')^3;$$

and so p, q, p' could be expressed in terms of q'; or all four quantities p, q, p', q' could be expressed in terms of one parameter. As q and p' occur linearly in the equations, they can be eliminated at once; we have

$$\begin{vmatrix} a(a+q')^3 - a(a+p)^3, & (a+q')^3, & (a+p)^3 \\ b(b+q')^3 - b(b+p)^3, & (b+q')^3, & (b+p)^3 \\ c(c+q')^3 - c(c+p)^3, & (c+q')^3, & (c+p)^3 \end{vmatrix} = 0.$$

Removing a non-vanishing factor $(a-b)(b-c)(c-a)(p-q')^4$, and writing

$$A = a+b+c, \quad B = ab+bc+ca, \quad C = abc,$$

we find

$$3pq'(p+q') + A\{(p+q')^2 + 2pq'\} + 3B(p+q') + 2C = 0,$$

as a relation giving p in terms of q'. To express them in terms of a single parameter σ, we take

$$3pq' + A(p+q') + B = 2\sigma,$$

and then

$$Apq' + B(p+q') + C = -\sigma(p+q')$$

that is,

$$Apq' + (B+\sigma)(p+q') + C = 0.$$

Hence

$$\frac{pq'}{AC - B^2 + B\sigma - 2\sigma^2} = \frac{p+q'}{AB - 2A\sigma - 3C} = \frac{1}{3B + 3\sigma - A^2};$$

thus p and q' are the roots of the equation in P

$$(3B - A^2 + 3\sigma) P^2 + (3C - AB + 2A\sigma) P + AC - B^2 + B\sigma - 2\sigma^2 = 0.$$

The values of q and p' are then given linearly by the original equations; they are

$$\frac{q}{C(2q'-p) - (A+p+q')q'^3} = \frac{p'}{C(2p-q') - (A+p+q')p^3}$$
$$= \frac{3B - A^2 + 3\sigma}{(3C - AB + 2A\sigma)\sigma}.$$

There are two nodal lines, each closed, each passing through the centres of curvature at two umbilici; they are symmetrical with reference to the surface; and they do not intersect upon the centro-surface unless $a + c = 2b$, in which case they touch at two points.

For further details and properties of the centro-surface of an ellipsoid, a memoir* by Cayley may be consulted.

Derived Surfaces.

85. In addition to the surface of centres, there are various surfaces specially connected with the centres of curvature which suggest themselves for consideration. Thus there is the *middle evolute*, which is the locus of the point midway between the two centres. There are also the *parallel* surfaces, being the loci of points taken at a constant distance along the normal from the given surface.

Consider, generally, a derived surface obtained by measuring along the normal a variable distance l from the surface; thus l will be a function of p and q and, unless the surface be a Weingarten surface, l can be regarded as a function of α and β. Let the point thus associated with x, y, z be denoted by $\bar{\xi}, \bar{\eta}, \bar{\zeta}$; then

$$\bar{\xi} = x + lX, \quad \bar{\eta} = y + lY, \quad \bar{\zeta} = z + lZ.$$

We have

$$\bar{\xi}_1 = x_1 + lX_1 + Xl_1 = -(l-\beta)\frac{x_1}{\beta} + Xl_1,$$

$$\bar{\xi}_2 = x_2 + lX_2 + Xl_2 = -(l-\alpha)\frac{x_2}{\alpha} + Xl_2,$$

* *Coll. Math. Papers*, vol. viii, paper 520, where other references are given.

and similarly for derivatives of $\bar{\eta}$ and $\bar{\zeta}$. Let $\bar{E}$, $\bar{F}$, $\bar{G}$ denote the fundamental magnitudes of the first order; then

$$\left.\begin{aligned}\bar{E} &= \left(\frac{l-\beta}{\beta}\right)^2 E + l_1^2\\ \bar{F} &= l_1 l_2\\ \bar{G} &= \left(\frac{l-\alpha}{\alpha}\right)^2 G + l_2^2\end{aligned}\right\}.$$

Also let $\bar{X}$, $\bar{Y}$, $\bar{Z}$ be the direction-cosines of the normal to the new surface, and write

$$\bar{V}^2 = \bar{E}\bar{G} - \bar{F}^2 = \frac{EG}{\alpha^2\beta^2}(l-\alpha)^2(l-\beta)^2 + l_1^2\left(\frac{l-\alpha}{\alpha}\right)^2 G + l_2^2\left(\frac{l-\beta}{\beta}\right)^2 E;$$

then

$$\begin{aligned}\bar{V}\bar{X} &= \bar{\eta}_1\bar{\zeta}_2 - \bar{\zeta}_1\bar{\eta}_2\\ &= \left(\frac{l-\beta}{\beta}y_1 - Yl_1\right)\left(\frac{l-\alpha}{\alpha}z_2 - Zl_2\right) - \left(\frac{l-\beta}{\beta}z_1 - Zl_1\right)\left(\frac{l-\alpha}{\alpha}y_2 - Yl_2\right)\end{aligned}$$

$$\left.\begin{aligned}&= \frac{V}{\alpha\beta}(l-\alpha)(l-\beta)X + l_1\frac{l-\alpha}{\alpha}\frac{G}{V}x_1 + l_2\frac{l-\beta}{\beta}\frac{E}{V}x_2\\ \bar{V}\bar{Y} &= \frac{V}{\alpha\beta}(l-\alpha)(l-\beta)Y + l_1\frac{l-\alpha}{\alpha}\frac{G}{V}y_1 + l_2\frac{l-\beta}{\beta}\frac{E}{V}y_2\\ \bar{V}\bar{Z} &= \frac{V}{\alpha\beta}(l-\alpha)(l-\beta)Z + l_1\frac{l-\alpha}{\alpha}\frac{G}{V}z_1 + l_2\frac{l-\beta}{\beta}\frac{E}{V}z_2\end{aligned}\right\}.$$

When the normal to the derived surface coincides with the normal to the original surface, we have $\bar{X} = X$, $\bar{Y} = Y$, $\bar{Z} = Z$; and so

$$l_1 = 0, \quad l_2 = 0,$$

except for special surfaces and special values of l such that

$$l = \alpha, \quad l_2 = 0; \quad \text{or} \quad l_1 = 0, \quad l = \beta.$$

Thus, in general, $l =$ constant; or a surface parallel to the given surface is the only surface, which is derived by taking points along the normal and has its tangent plane parallel to the original tangent plane.

When the normal to the derived surface is perpendicular to the normal to the original surface, we have $X\bar{X} + Y\bar{Y} + Z\bar{Z} = 0$; and so

$$(l-\alpha)(l-\beta) = 0.$$

The centro-surface is therefore the only surface thus derived, which has its tangent plane perpendicular to the original tangent plane.

And generally, the inclination ϕ of the two normals to one another is given by

$$\bar{V}\cos\phi = \frac{V}{\alpha\beta}(l-\alpha)(l-\beta).$$

For the fundamental magnitudes of the second kind, say $\overline{L}$, $\overline{M}$, $\overline{N}$, we have

$$\bar{\xi}_{11} = -\frac{l-\beta}{\beta} x_{11} - x_1 \frac{\partial}{\partial p}\left(\frac{l}{\beta}\right) + Xl_{11} + X_1 l_1$$

$$= X\left\{l_{11} - \frac{E}{\beta^2}(l-\beta)\right\} - x_1\left\{\frac{E_1}{2E}\,\frac{l-\beta}{\beta} + 2\frac{l_1}{\beta} - \frac{l}{\beta^2}\beta_1\right\} + x_2 \frac{E_2}{2G}\,\frac{l-\beta}{\beta},$$

on reduction (§§ 34, 77); also

$$\bar{\xi}_{12} = Xl_{12} - x_1\left(\frac{E_2}{2E}\,\frac{l-\alpha}{\alpha} + \frac{l_2}{\beta}\right) - x_2\left(\frac{G_1}{2G}\,\frac{l-\beta}{\beta} + \frac{l_1}{\alpha}\right),$$

$$\bar{\xi}_{22} = X\left\{l_{22} - \frac{G}{\alpha^2}(l-\alpha)\right\} + x_1 \frac{G_1}{2E}\,\frac{l-\alpha}{\alpha} - x_2\left\{\frac{G_2}{2G}\,\frac{l-\alpha}{\alpha} + 2\frac{l_2}{\alpha} - \frac{l}{\alpha^2}\alpha_2\right\},$$

with corresponding values for the second derivatives of $\bar{\eta}$, $\bar{\zeta}$. Thus

$$\left.\begin{aligned}
\overline{V}\overline{L} &= \overline{V}\overline{X}\bar{\xi}_{11} + \overline{V}\overline{Y}\bar{\eta}_{11} + \overline{V}\overline{Z}\bar{\zeta}_{11}\\
&= l_{11}\frac{V}{\alpha\beta}(l-\alpha)(l-\beta) - l_1^2\frac{V}{\alpha\beta}(2l-2\alpha) - \frac{EV}{\alpha\beta^3}(l-\alpha)(l-\beta)^2\\
&\quad + l_1\left(\frac{l\beta_1}{\beta^2} - \frac{E_1}{2E}\,\frac{l-\beta}{\beta}\right)\frac{l-\alpha}{\alpha}V + l_2\left(\frac{l-\beta}{\beta}\right)^2\frac{VE_2}{2G}\\
\overline{V}\overline{M} &= l_{12}\frac{V}{\alpha\beta}(l-\alpha)(l-\beta) - l_1 l_2\frac{V}{\alpha\beta}(2l-\alpha-\beta)\\
&\quad - l_1\left(\frac{l-\alpha}{\alpha}\right)^2\frac{VE_2}{2E} - l_2\left(\frac{l-\beta}{\beta}\right)^2\frac{VG_1}{2G}\\
\overline{V}\overline{N} &= l_{22}\frac{V}{\alpha\beta}(l-\alpha)(l-\beta) - l_2^2\frac{V}{\alpha\beta}(2l-2\beta) - \frac{GV}{\alpha^3\beta}(l-\alpha)^2(l-\beta)\\
&\quad + l_1\left(\frac{l-\alpha}{\alpha}\right)^2\frac{VG_1}{2E} + l_2\left(\frac{l\alpha_2}{\alpha^2} - \frac{G_2}{2G}\,\frac{l-\alpha}{\alpha}\right)\frac{l-\beta}{\beta}V
\end{aligned}\right\}.$$

It is clear that, in general, the lines on a surface thus derived, which correspond to the lines of curvature, are not perpendicular, for $\overline{F}$ is not zero; and they are not conjugate, for $\overline{M}$ is not zero. If however l is a function of only one of the two parametric variables, then $\overline{F}$ is zero while $\overline{M}$ is not zero; then the lines, which correspond to the lines of curvature on the original surface, are perpendicular to one another. And if l be constant, so that the derived surface is a parallel surface, then both $\overline{F}$ and $\overline{M}$ are zero; that is, the lines of curvature on all parallel surfaces correspond to one another.

86. Various simplifications arise in regard to these derived surfaces when special values are assigned to the quantity l.

We have already considered the cases when l is made equal to one or other of the principal radii of curvature; the derived surface is the centro-surface.

When $l=\frac{1}{2}(\alpha+\beta)$, the derived surface is the middle evolute; it was apparently first considered by Ribaucour* and discussed subsequently as to its properties by Appell and by Goursat. The latter, in particular, discussed the inverse problem of determining a surface or surfaces which have an assigned surface as their middle evolute; the resolution of the problem† depends, as many inverse problems in differential geometry depend, upon a knowledge of the most general integral of a partial differential equation of the second order.

For a Weingarten surface such that $\alpha+\beta=$ constant, the middle evolute is a parallel surface. A minimal surface is its own middle evolute. But for surfaces not of special character such as Weingarten surfaces, the properties of the middle evolute are not of conspicuous importance.

87. One property of parallel surfaces—the persistence of the lines of curvature through all the surfaces which are parallel to a given surface—has already been indicated. The normals at associated points coincide; and the lines of curvature on the two surfaces correspond. But the asymptotic lines do not correspond, nor do the nul lines.

As regards the measures of curvature, we have

$$\bar{E}=\left(\frac{l-\beta}{\beta}\right)^2 E,\quad \bar{F}=0,\quad \bar{G}=\left(\frac{l-\alpha}{\alpha}\right)^2 G,$$

$$\bar{L}=-\frac{l-\beta}{\beta^2}E,\quad \bar{M}=0,\quad \bar{N}=-\frac{l-\alpha}{\alpha^2}G.$$

The principal radii of curvature, being $\bar{G}/\bar{N}$ and $\bar{E}/\bar{L}$, are $\alpha-l$ and $\beta-l$, as is to be expected. Also

$$\bar{H}=\frac{1}{\alpha-l}+\frac{1}{\beta-l}=\frac{H-2lK}{1-lH+l^2K},$$

$$\bar{K}=\frac{1}{(\alpha-l)(\beta-l)}=\frac{K}{1-lH+l^2K}.$$

In particular, if K be constant and equal to $1/a^2$, and we take $l=\pm a$, then

$$\bar{H}=\mp\frac{1}{a};$$

that is, *a surface parallel to a surface of constant total curvature* $1/a^2$ *at a distance* a *is a surface of constant mean curvature.* This result, one more link between surfaces having constant measures of curvature, is due to Bonnet.

* *Liouville's Journal*, 4$^{\text{me}}$ Sér. t. vii (1891), pp. 5—108, 219—270. Other references are given by Darboux, *Théorie générale*, t. iv, p. 327.

† Goursat, *Amer. Journ. Math.*, vol. x (1888), p. 187.

EXAMPLES.

1. Obtain the lines of curvature on the surface $xy=az$, where a is constant, in the form

$$p=(z^2+x^2)^{\frac{1}{2}}+(z^2+y^2)^{\frac{1}{2}}, \quad q=(z^2+x^2)^{\frac{1}{2}}-(z^2+y^2)^{\frac{1}{2}}.$$

2. Shew that the lines of curvature upon the surface

$$x^l y^m z^n=a,$$

where l, m, n are constant, and a is a parameter, are given by the equations

$$\left(\frac{x^2}{l}+\lambda\right)^l\left(\frac{y^2}{m}+\lambda\right)^m\left(\frac{z^2}{n}+\lambda\right)^n=p, \quad \left(\frac{x^2}{l}+\mu\right)^l\left(\frac{y^2}{m}+\mu\right)^m\left(\frac{z^2}{n}+\mu\right)^n=q,$$

where λ and μ are the roots of the equation

$$\frac{l}{\frac{x^2}{l}+\theta}+\frac{m}{\frac{y^2}{m}+\theta}+\frac{n}{\frac{z^2}{n}+\theta}=0,$$

regarded as a quadratic in θ, the parametric variables of the lines of curvature being p and q; and prove that the principal radii of curvature are pk and qk, where $k=(l^2/x^2+m^2/y^2+n^2/z^2)^{\frac{1}{2}}$. Prove also that the surfaces $a=$ constant, $p=$ constant, $q=$ constant, are orthogonal to one another.

3. Shew that the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ has real umbilici at the points

$$y=0, \quad x^2=\frac{a^2(a^2-b^2)}{a^2-c^2}, \quad z^2=\frac{c^2(b^2-c^2)}{a^2-c^2},$$

on the assumption that $a>b>c$. Discuss the lines of curvature in the vicinity of any one of these points; and shew that they have the configuration in fig. vi (p. 99).

4. Verify that the lines of curvature on the quadric

$$\frac{y^2}{a}+\frac{z^2}{c}=4x$$

are the intersections by the confocal quadrics

$$\frac{y^2}{a-p}+\frac{z^2}{c-p}=4(x-p).$$

Trace their course upon the surface; and find the principal radii of curvature in terms of the parameters of the confocal quadrics.

5. A surface is inverted with respect to any centre. Shew that the quantity $r/\rho+p/r$ is unaltered, save as to sign, where r is the distance of a point from the centre of inversion, p is the perpendicular upon the tangent plane, and $1/\rho$ is the curvature of the normal section at the point in any direction.

6. A surface is referred to its lines of curvature as the parametric curves; shew that

$$\left.\begin{aligned} EX_{11}&=-L^2X-(P+L\Gamma)x_1-L\Delta x_2 \\ EX_{12}&=\qquad\quad -(Q+L\Gamma')x_1-L\Delta' x_2 \\ GX_{12}&=\qquad\quad -N\Gamma' x_1-(R+N\Delta')x_2 \\ GX_{22}&=-N^2X-N\Gamma'' x_1-(S+N\Delta'')x_2 \end{aligned}\right\},$$

with corresponding formulæ for derivatives of Y and Z.

7. A sphere of diameter a rolls on the outside of a closed oval surface of volume V and area S; and the parallel surface, which is its outer envelope, has volume V' and area S'. Shew that

$$V'-V=\tfrac{1}{2}a(S'+S)-\tfrac{2}{3}\pi a^3.$$

8. A surface, parallel to a given surface, is generated as the envelope of a sphere of constant diameter a rolling on the surface. With the customary notation for magnitudes on the given surface, shew that the fundamental magnitudes for the parallel surface are

$$\left.\begin{aligned} E'&=(1-a^2K)\,E+(a^2H-2a)\,L\\ F'&=(1-a^2K)\,F+(a^2H-2a)\,M\\ G'&=(1-a^2K)\,G+(a^2H-2a)\,N \end{aligned}\right\},\qquad \left.\begin{aligned} L'&=(1-aH)\,L+aKE\\ M'&=(1-aH)\,M+aKF\\ N'&=(1-aH)\,N+aKG \end{aligned}\right\}.$$

9. A distance l, equal to the harmonic mean of the principal radii of curvature, is measured along the normal to the surface; and $d\sigma$ denotes an elementary arc on the surface, which is the locus of the point so obtained. Prove that

$$d\sigma^2=dl^2+\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 ds^2;$$

and give a geometric interpretation of the result.

10. Prove that each sheet of the evolute of a pseudo-sphere is applicable to a catenoid.

CHAPTER V

GEODESICS.

THE literature connected with geodesics is very copious. Only a few important references will be given here; fuller references will be found in the authorities quoted.

For geodesics on surfaces of revolution in particular, a full treatment will be found in the first chapter of the sixth book in vol. iii of Darboux's treatise; and reference should also be made to Halphen's *Fonctions elliptiques*, ch. vi.

For the general properties of geodesics and the use of the notion of geodesic curvature, the fundamental memoir is that of Gauss*; and full treatment is given, as usual, by Darboux†.

The method of determining geodesics by means of the solution of partial differential equations of the first order is expounded by Darboux in his third volume, pp. 1—39 and pp. 66—85. At the end of the fourth volume he has appended a Note (II) by Kœnigs, dealing specially with geodesics which can be obtained through quadratic integrals and summarising a number of results deduced in another memoir‡.

One portion of the subject-matter has been omitted deliberately—the analogy between theoretical dynamics and the theory of geodesics. It was developed first by Jacobi§; and an excellent account is given by Darboux in the last two chapters of the second volume of his treatise.

Reference may also be made throughout to the sixth chapter of Bianchi's treatise.

At the beginning of this chapter, various propositions from the calculus of variations are stated. In their application to the theory of geodesics, they are used especially in connection with the range along which a geodesic is actually the shortest distance on the surface.

88. The definition and a few elementary properties of geodesics have already been given; these curves will now be discussed in fuller detail, and three main methods of discussion will be indicated.

A geodesic upon a surface has been defined as a curve of shortest length measured in the surface between two points; and a descriptive property was deduced to the effect that the osculating plane of the curve contains the normal to the surface. The curve may be produced to any length on the

* *Disquisitiones generales circa superficies curvas*, Ges. Werke, t. iv, pp. 217—258.

† See, in particular, the second volume of his treatise, pp. 402—437; and the third volume, pp. 113—192.

‡ *Mém. des Sav. Etr.*, t. xxxi, No. 6, (1894).

§ See his *Vorlesungen über Dynamik*.

surface, and the deduced descriptive property will be possessed at every point; but the curve is not necessarily the shortest distance between any two points however far it is produced. Thus on a sphere a great circle is a geodesic curve; the shortest distance on the sphere between two points is the smaller arc of the great circle through the points, and not the greater arc, though the latter everywhere possesses the deduced property. Hence we must possess some method of determining the limits, if any, between which a geodesic curve is actually the shortest distance, and outside which it may cease to be the shortest distance, though it possesses everywhere the deduced property. For this purpose (as for other connected purposes) the calculus of variations will be used; fortunately, the expression of an arc involves only derivatives of the first order, and so only the simplest propositions will be required.

In a second range of investigation, the property (which will sometimes be called the *geodesic property*) that the osculating plane of the curve contains the normal to the surface is used, initially to obtain equations for the geodesic, and later to determine their properties, especially when the geodesics are drawn as tangents to non-geodesic curves. For this purpose (as also for other connected purposes), the Gaussian analysis for surfaces will be used.

In a third range of investigation, the analytical association with theoretical dynamics is used. Thus, to take only the simplest instance, we know that a particle, moving on the concave side of a smooth surface under the influence of no forces other than the pressure, describes a geodesic. More generally, the Lagrangian equations of motion of a particle in a conservative field have the form characteristic of the equations of a geodesic as deduced by the calculus of variations. The theory of partial differential equations of the first order is much used in the developments of those Lagrangian equations; and so it may be expected that the theory will be useful in deriving some properties of geodesics. Some illustrations, especially as connected with the actual determination of the curves, will be given in due course.

Application of the Calculus of Variations.

89. Without pretending to give a full summary of the results obtained in the calculus of variations for problems of the first order, it will be sufficient for our purpose to state the essentially useful propositions, as they can be applied to geodesics on a surface*. It is the length of the arc between two

* The proofs will be found, with varying elaboration, in any one of the more modern text-books on the Calculus of Variations, such as those by Bolza, Hadamard, Hancock, Kneser. Some of the propositions are of old standing. Thus the simplest case of the first (or, what is the same thing, the fifth) is due to Euler. The second is due to Legendre, and the third to Jacobi. The fourth is due to Weierstrass, who reconstructed the subject and whose lectures, in authoritative form, are not yet published.

points on the surface which is to be made a minimum; so we have to consider integrals

$$\int (Ep'^2 + 2Fp'q' + Gq'^2)^{\frac{1}{2}}\, dt, \quad \int (E + 2F\theta + G\theta^2)^{\frac{1}{2}}\, dp,$$

where, in the first, p and q are to be made appropriate functions of t, while $p' = dp/dt$ and $q' = dq/dt$; and, in the second, $\theta = dq/dp$, and q is to be made an appropriate function of p; always so as to secure the minimum. The propositions are as follows.

I. When the quantity to be made a minimum is

$$\int f(p, q, p', q')\, dt,$$

where the function f is homogeneous, and of the first order, in p' and q', the quantities p and q must satisfy the equations

$$\frac{\partial f}{\partial p} - \frac{d}{dt}\left(\frac{\partial f}{\partial p'}\right) = 0, \quad \frac{\partial f}{\partial q} - \frac{d}{dt}\left(\frac{\partial f}{\partial q'}\right) = 0.$$

Because f is homogeneous and of the first order in p' and q', these two equations are equivalent to the single equation

$$U = \frac{\partial^2 f}{\partial p \partial q'} - \frac{\partial^2 f}{\partial q \partial p'} + (p'q'' - q'p'')f_1 = 0,$$

owing to the relations

$$\frac{\partial f}{\partial p} - \frac{d}{dt}\left(\frac{\partial f}{\partial p'}\right) = Uq', \quad \frac{\partial f}{\partial q} - \frac{d}{dt}\left(\frac{\partial f}{\partial q'}\right) = -Up',$$

the quantity f_1 being

$$f_1 = -\frac{1}{p'q'}\frac{\partial^2 f}{\partial p' \partial q'}.$$

Thus either equation can be treated alone. In any of the forms, it is the characteristic equation; the primitive gives a possible minimum.

It is an addition to the proposition that, even if the curve should suffer a sudden change of direction at a free (and not fixed) point in its course, the values of $\partial f/\partial p'$ and $\partial f/\partial q'$ are continuous in the passage through the free point.

II. The preceding quantity f_1 must be positive everywhere along the curve if a minimum is to exist.

This condition is necessary, though not sufficient, to make the second variation positive. The preceding condition is necessary and sufficient to make the first variation zero.

III (i). When the primitive of the characteristic equation can be determined, let it be denoted by

$$p = \phi(t, a, b), \quad q = \psi(t, a, b),$$

where a and b are arbitrary constants; and write

$$\frac{\partial \phi}{\partial t} = \phi'(t), \quad \frac{\partial \phi}{\partial a} = \phi_1(t), \quad \frac{\partial \phi}{\partial b} = \phi_2(t),$$

$$\frac{\partial \psi}{\partial t} = \psi'(t), \quad \frac{\partial \psi}{\partial a} = \psi_1(t), \quad \frac{\partial \psi}{\partial b} = \psi_2(t).$$

Construct the functions

$$u_1 = \psi'(t)\,\phi_1(t) - \phi'(t)\,\psi_1(t) = \chi_1(t),$$

$$u_2 = \psi'(t)\,\phi_2(t) - \phi'(t)\,\psi_2(t) = \chi_2(t),$$

$$\Theta(t, t_0) = \chi_1(t)\,\chi_2(t_0) - \chi_2(t)\,\chi_1(t_0);$$

and take the independent variable t as increasing throughout the range of integration. Then a range of integration, beginning at t_0, must not extend so far as the root of $\Theta(t, t_0) = 0$ which is next greater than t_0.

A geometric expression of the condition is due to Jacobi. Take a curve satisfying the characteristic equation and passing through the lower limit of the integral represented by t_0; and take a consecutive curve (that is, one which makes an infinitesimal angle at t_0 with the preceding curve) also satisfying the characteristic equation and passing through the same initial point. Let the first point after the initial point at which these two curves ultimately intersect (if they do intersect) be called the *conjugate* of the initial point. Then the range must not extend as far as the conjugate of the initial point.

III (ii). When the primitive of the characteristic equation is not known, it may happen that some special integral is known. In that case, the critical function $\Theta(t, t_0)$ must be obtained by another process. Let

$$f_2 = \frac{1}{q''^2}\left\{\frac{\partial^2 f}{\partial p^2} - q''^2 f_1 - \frac{d}{dt}\left(\frac{\partial^2 f}{\partial p \partial p'} - q'q''f_1\right)\right\},$$

and form the equation

$$\frac{d}{dt}\left(f_1 \frac{du}{dt}\right) - uf_2 = 0,$$

where u is the dependent variable, inserting the values of p', q', p'', q'' derived from the special integral. This linear equation in u of the second order has to be completely integrated; its primitive is

$$u = cu_1 + c'u_2$$

$$= c\chi_1(t) + c'\chi_2(t),$$

where c and c' are arbitrary constants. The critical function is $\Theta(t, t_0)$, where

$$\Theta(t, t_0) = \chi_1(t)\,\chi_2(t_0) - \chi_2(t)\,\chi_1(t_0);$$

the condition, as regards the range of integration, has already been stated.

IV. Let

$$\frac{\partial f}{\partial p'} = g_1(p, q, p', q'), \quad \frac{\partial f}{\partial q'} = g_2(p, q, p', q');$$

and construct the function $\mathfrak{E}$ such that

$$\mathfrak{E} = \{g_1(p, q, P', Q') - g_1(p, q, p', q')\} P' \\ + \{g_2(p, q, P', Q') - g_2(p, q, p', q')\} Q'.$$

This function $\mathfrak{E}$ must be positive everywhere along the geodesic curve for all directions given by P' and Q', other than $P' = p'$ and $Q' = q'$. The functions g_1 and g_2 are homogeneous of order zero in p' and q'; for the function $\mathfrak{E}$, the independent variable can be taken to be s, the arc of the curve.

These tests are sufficient and necessary to secure that the curve provides a minimum; that is, the integral receives a positive increment for small variations of p and q. These variations are called weak, when δp, δq, $\delta p'$, $\delta q'$ are small and tend to zero; they are called strong when $\delta p'$ and $\delta q'$ are not small, though δp and δq are small and tend to zero. The first three tests are sufficient to secure the minimum property for weak variations; the additional fourth test (the excess-function test) is necessary and sufficient to secure the minimum property also for strong variations.

V. When the integral, which has to be made a minimum, has the form

$$\int W(p, q, \theta)\, dp,$$

where $\theta = dq/dp$, the first three tests have a simpler form; and they represent the older stage of the calculus of variations, when the variations considered admissible were of the type called weak.

The characteristic equation in (I) is

$$\frac{d}{dp}\left(\frac{\partial W}{\partial \theta}\right) - \frac{\partial W}{\partial q} = 0.$$

The test contained in (II) is that the quantity

$$\frac{\partial^2 W}{\partial \theta^2}$$

must be positive everywhere.

For the test in (III), let the primitive of the characteristic equation be

$$q = g(p, a, b).$$

Then the quantity

$$A\frac{\partial g}{\partial a} + B\frac{\partial g}{\partial b},$$

where A and B are arbitrary constants, must not again acquire in the course of the range the value that it has at the beginning; so that the range is thus

limited. But if the primitive is not known, while some special integral is known, then the equation

$$\left\{\frac{\partial^2 W}{\partial q^2} - \frac{d}{dp}\left(\frac{\partial^2 W}{\partial q \partial \theta}\right)\right\} u - \frac{d}{dp}\left(\frac{\partial^2 W}{\partial \theta^2}\frac{du}{dp}\right) = 0$$

(when for q and θ their values derived from the special integral are substituted) must be completely integrated. Let the primitive be

$$u = Au_1 + Bu_2.$$

Then the quantity $Au_1 + Bu_2$ must not again acquire in the course of the range the value that it has at the beginning.

Such are the tests needed for our purpose. We proceed to apply them, first, in general to all geodesics as far as possible and then, later, to some particular geodesics when they can be applied only upon a knowledge of details.

90. The element of arc upon the surface is, as usual,

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2.$$

When the curve is a geodesic, some relation must exist between p and q so as to define the curve; or, what is the same thing, p and q must be expressible in terms of a single parameter, say t. Then if $p' = dp/dt$, $q' = dq/dt$, $\theta = dq/dp$, the arc is given in either of two forms, viz.

$$ds^2 = (Ep'^2 + 2Fp'q' + Gq'^2)\,dt^2, \quad ds^2 = (E + 2F\theta + G\theta^2)\,dp^2;$$

and therefore, when the arc on the surface between two points has a minimum length, the integrals

$$\int (Ep'^2 + 2Fp'q' + Gq'^2)^{\frac{1}{2}}\,dt, \qquad \int (E + 2F\theta + G\theta^2)^{\frac{1}{2}}\,dp,$$

must satisfy the minimum tests provided by the calculus of variations.

Two of the tests are satisfied for all geodesics on all surfaces, it being remembered that we are dealing with portions of surfaces which are devoid of singularities.

Consider the test in (II). When we write

$$f = f(p, q, p', q') = (Ep'^2 + 2Fp'q' + Gq'^2)^{\frac{1}{2}},$$

where we naturally take the positive sign for the real radical, we have

$$\begin{aligned} f_1 &= -\frac{1}{p'q'}\frac{\partial^2 f}{\partial p' \partial q'} \\ &= \frac{V^2}{f^3}, \end{aligned}$$

on reduction. This is always positive on a real surface; and so the necessary condition is satisfied.

When we write

$$W = W(p, q, \theta) = (E + 2F\theta + G\theta^2)^{\frac{1}{2}},$$

again taking the positive sign for the real radical, we have

$$\frac{\partial^2 W}{\partial \theta^2} = \frac{V^2}{W^3},$$

which always is positive; so that the condition is satisfied for this form also, as is to be expected when it is satisfied for the other form. It follows that, in the discussion of geodesics, we need pay no further attention to the test in (II).

Next, consider the excess-function test in (IV). We have

$$g_1 = \frac{Ep' + Fq'}{f} = E\frac{dp}{ds} + F\frac{dq}{ds},$$

$$g_2 = \frac{Fp' + Gq'}{f} = F\frac{dp}{ds} + G\frac{dq}{ds};$$

and therefore

$$\begin{aligned}\mathfrak{E} &= \{EP' + FQ' - (Ep' + Fq')\}\,P' + \{FP' + GQ' - (Fp' + Gq')\}\,Q' \\ &= 1 - \cos\Omega,\end{aligned}$$

where Ω is the angle between the direction p', q' and the direction P', Q'. Thus the excess-function is positive for all directions given by P' and Q', other than $P' = p'$ and $Q' = q'$. The test is satisfied for all geodesics on all surfaces; and therefore we need pay no further attention to the test in (IV).

Accordingly, we now have only to consider the characteristic equation and the determination of conjugate points.

91. When we develop the characteristic equation

$$\frac{d}{dt}\left(\frac{\partial f}{\partial p'}\right) - \frac{\partial f}{\partial p} = 0,$$

where $f = (Ep'^2 + 2Fp'q' + Gq'^2)^{\frac{1}{2}}$, we have

$$\frac{d}{dt}\left\{\frac{1}{f}(Ep' + Fq')\right\} - \frac{1}{2f}(E_1p'^2 + 2F_1p'q' + G_1q'^2) = 0,$$

that is,

$$2\frac{d}{ds}\left(E\frac{dp}{ds} + F\frac{dq}{ds}\right) - E_1\left(\frac{dp}{ds}\right)^2 - 2F_1\frac{dp}{ds}\frac{dq}{ds} - G_1\left(\frac{dq}{ds}\right)^2 = 0;$$

and so

$$2E\frac{d^2p}{ds^2} + 2F\frac{d^2q}{ds^2} + E_1\left(\frac{dp}{ds}\right)^2 + 2E_2\frac{dp}{ds}\frac{dq}{ds} + (2F_2 - G_1)\left(\frac{dq}{ds}\right)^2 = 0,$$

that is,

$$E\frac{d^2p}{ds^2} + F\frac{d^2q}{ds^2} + m\left(\frac{dp}{ds}\right)^2 + 2m'\frac{dp}{ds}\frac{dq}{ds} + m''\left(\frac{dq}{ds}\right)^2 = 0.$$

Similarly, when we develop the characteristic equation

$$\frac{d}{dt}\left(\frac{\partial f}{\partial q'}\right) - \frac{\partial f}{\partial q} = 0,$$

we find

$$F\frac{d^2p}{ds^2} + G\frac{d^2q}{ds^2} + n\left(\frac{dp}{ds}\right)^2 + 2n'\frac{dp}{ds}\frac{dq}{ds} + n''\left(\frac{dq}{ds}\right)^2 = 0.$$

These two equations are *the equations of geodesics in general.* They serve to determine p and q in terms of s, taken as a parameter; when s is eliminated between the determinate values of p and q, a single relation survives which is the integral equation of geodesics on the surface.

As regards the significance of the equations, we note that

$$\frac{d^2x}{ds^2} = x_{11}\left(\frac{dp}{ds}\right)^2 + 2x_{12}\frac{dp}{ds}\frac{dq}{ds} + x_{22}\left(\frac{dq}{ds}\right)^2 + x_1\frac{d^2p}{ds^2} + x_2\frac{d^2q}{ds^2},$$

with similar expressions for d^2y/ds^2 and d^2z/ds^2, these relations holding for any curve. Hence

$$\begin{aligned} x_1\frac{d^2x}{ds^2} + y_1\frac{d^2y}{ds^2} + z_1\frac{d^2z}{ds^2} & \\ &= m\left(\frac{dp}{ds}\right)^2 + 2m'\frac{dp}{ds}\frac{dq}{ds} + m''\left(\frac{dq}{ds}\right)^2 + E\frac{d^2p}{ds^2} + F\frac{d^2q}{ds^2} \\ &= 0, \end{aligned}$$

when the curve is a geodesic; and similarly

$$x_2\frac{d^2x}{ds^2} + y_2\frac{d^2y}{ds^2} + z_2\frac{d^2z}{ds^2} = 0.$$

Hence

$$\frac{1}{y_1z_2 - z_1y_2}\frac{d^2x}{ds^2} = \frac{1}{z_1x_2 - x_1z_2}\frac{d^2y}{ds^2} = \frac{1}{x_1y_2 - y_1x_2}\frac{d^2z}{ds^2},$$

that is,

$$\frac{1}{X}\frac{d^2x}{ds^2} = \frac{1}{Y}\frac{d^2y}{ds^2} = \frac{1}{Z}\frac{d^2z}{ds^2},$$

so that the principal normal of the curve coincides with the normal to the surface, in accordance with the earlier inference (§ 65) that the osculating plane of the curve contains the normal to the surface.

We may remark here that this property is sometimes made the basis of a definition of a geodesic.

92. Other forms can be given to the general equations. In their first form, they are

$$E\frac{d^2p}{ds^2} + F\frac{d^2q}{ds^2} = -m\left(\frac{dp}{ds}\right)^2 - 2m'\frac{dp}{ds}\frac{dq}{ds} - m''\left(\frac{dq}{ds}\right)^2,$$

$$F\frac{d^2p}{ds^2} + G\frac{d^2q}{ds^2} = -n\left(\frac{dp}{ds}\right)^2 - 2n'\frac{dp}{ds}\frac{dq}{ds} - n''\left(\frac{dq}{ds}\right)^2;$$

when they are resolved for $\frac{d^2p}{ds^2}$ and $\frac{d^2q}{ds^2}$, we find

$$\left.\begin{aligned}\frac{d^2p}{ds^2}+\Gamma\left(\frac{dp}{ds}\right)^2+2\Gamma'\frac{dp}{ds}\frac{dq}{ds}+\Gamma''\left(\frac{dq}{ds}\right)^2=0\\ \frac{d^2q}{ds^2}+\Delta\left(\frac{dp}{ds}\right)^2+2\Delta'\frac{dp}{ds}\frac{dq}{ds}+\Delta''\left(\frac{dq}{ds}\right)^2=0\end{aligned}\right\}.$$

These also are therefore general equations of a geodesic, and they prove more useful than the general equations in their initial form.

Moreover, we are to expect that the two characteristic equations are equivalent to one only; and we know that the integral equation of a geodesic is a single relation between p and q, so that the single characteristic equation ought to be a relation between p and q which (owing to the form of the general equations) should be an ordinary differential equation of the second order. Now

$$\frac{dp}{ds}\frac{dq}{dp}=\frac{dq}{ds}, \qquad \left(\frac{dp}{ds}\right)^3\frac{d^2q}{dp^2}=\frac{dp}{ds}\frac{d^2q}{ds^2}-\frac{dq}{ds}\frac{d^2p}{ds^2};$$

hence

$$\frac{d^2q}{dp^2}=\Gamma''\left(\frac{dq}{dp}\right)^3+(2\Gamma'-\Delta'')\left(\frac{dq}{dp}\right)^2+(\Gamma-2\Delta')\frac{dq}{dp}-\Delta,$$

which is the (single) *general equation of geodesics* on a surface.

One important inference can be made from this form of the equation. Consider a region of the surface devoid of singularities; then the quantities Γ, Γ', Γ'', Δ, Δ', Δ'' are finite and (even when they are not uniform functions of p and q) have regular branches in that region. It is known* that a unique solution of an ordinary differential equation of the foregoing form exists, which gives q as a uniform function of p and is such that, for an assigned value of p, both q and dq/dp have arbitrarily assigned values; in other words, *a geodesic through any ordinary point on a surface is uniquely determined by its direction through the point.* Thus we have a justification (among other things) for the use of geodesic polar coordinates.

It is to be noted that all the forms of the general equations of geodesics involve, among the fundamental magnitudes of the surface, only those of the first order and their derivatives. Hence when a surface is deformed in any way, without stretching and without tearing, so that the arc-element is unaltered, the geodesics remain geodesics on the deformed surface; for the quantities E, F, G are unaltered during any such process. And the result is essentially contained in the deformations of the type indicated.

Further, it is to be expected that the nul lines on a surface will possess analytically the geodesic property of being the shortest distance between two points on a surface; thus the relation

$$f=(Ep'^2+2Fp'q'+Gq'^2)^{\frac{1}{2}}=0$$

* See the author's *Theory of Differential Equations*, vol. iii, § 209.

should satisfy the characteristic equations. It is easy to verify that, from the first form of the characteristic equations, we have

$$(Ep' + Fq')\frac{df}{dt} = \{Ep'' + Fq'' + \tfrac{1}{2}E_1p'^2 + E_2p'q' + (F_2 - \tfrac{1}{2}G_1)q'^2\}f,$$

$$(Fp' + Gq')\frac{df}{dt} = \{Fp'' + Gq'' + (F_1 - \tfrac{1}{2}E_2)p'^2 + G_1p'q' + \tfrac{1}{2}G_2q'^2\}f,$$

which clearly are satisfied by

$$f = 0,$$

that is, by the nul lines on the surface.

Geodesics on Surfaces of Revolution.

93. The general equation of geodesics does not appear to admit of integration in finite terms for all surfaces. But it is possible to integrate, wholly or partially, the equation for many classes of surfaces; and special methods, sometimes individual to a class of surfaces, sometimes general in scope and effective in particular cases, are used to obtain the primitive. If by any method we can obtain an integral equation containing two independent arbitrary constants, it is effectively the primitive of the general characteristic equation. All that then remains, in order to complete the process at present under consideration, is the determination of the range between conjugate points.

Among surfaces which thus admit integral expression for their geodesics, one conspicuous class is constituted by surfaces of revolution. We proceed to consider them briefly in this regard.

Take the axis of z as the axis of revolution; and let the equation of the surface be

$$r^2 = x^2 + y^2 = 2u(z) = 2u.$$

Let $x = r\cos\phi$, $y = r\sin\phi$, so that ϕ is the azimuth of a point on the surface; and let the geodesic cut the meridian at an angle ψ; then

$$\sin\psi = r\frac{d\phi}{ds}.$$

Also, we have

$$r\,dr = u'dz,$$

so that

$$\begin{aligned} ds^2 &= dr^2 + r^2d\phi^2 + dz^2 \\ &= r^2d\phi^2 + \left(\frac{u'^2}{2u} + 1\right)dz^2. \end{aligned}$$

Thus, for the characteristic equations in ϕ and z, we have

$$f = \left\{r^2\phi'^2 + \left(\frac{u'^2}{2u} + 1\right)z'^2\right\}^{\frac{1}{2}}.$$

This quantity f does not involve ϕ explicitly, so that $\partial f/\partial\phi = 0$; thus the characteristic equation in ϕ becomes

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \phi'}\right) = 0;$$

hence

$$\frac{1}{f} r^2 \phi' = h,$$

that is,

$$r^2 \frac{d\phi}{ds} = h,$$

where h is an arbitrary constant. (When we are dealing with the motion of a particle upon the surface, as indicated in § 88, the quantity h is a constant multiple of the moment of its momentum round the axis.) We thus have a first integral of the equations; it can also be written

$$r \sin \psi = h.$$

Further, we have

$$ds^2 = \frac{h^2}{2u} ds^2 + \left(\frac{u'^2}{2u} + 1\right) dz^2,$$

and therefore

$$ds = \left(\frac{2u + u'^2}{2u - h^2}\right)^{\frac{1}{2}} dz = \left(\frac{r^2 + u'^2}{r^2 - h^2}\right)^{\frac{1}{2}} dz.$$

Hence

$$\begin{aligned} d\phi &= \frac{h}{r^2} ds \\ &= \left(\frac{r^2 + u'^2}{r^2 - h^2}\right)^{\frac{1}{2}} \frac{h\,dz}{r^2}, \end{aligned}$$

so that

$$\begin{aligned} \phi - \gamma &= h \int \left(\frac{r^2 + u'^2}{r^2 - h^2}\right)^{\frac{1}{2}} \frac{dz}{r^2}, \\ &= Z, \end{aligned}$$

say, where γ is an arbitrary constant of integration. We now have an integral equation containing two independent arbitrary constants h and γ; it is the general integral equation of geodesics on surfaces of revolution.

When a geodesic curve between two given points on the surface is required, the constants h and γ for the curve are obtained from the conditions which result from substituting the coordinates of the points in the integral equation.

In order that the curve may be real, we must have

$$r \geqslant h.$$

If and when $r = h$, we have

$$\sin \psi = 1;$$

that is, the geodesic touches the parallel at the point which thus is a highest point or a lowest point on the geodesic.

Moreover as r may not be less than h, it is necessary to take account of the range of values of h.

94. First, consider the vicinity of a parallel of minimum radius c; we there have a neck of the surface, the parallel itself being a geodesic.

I. Let $h = c$. Near the neck, let the surface be

$$r^2 = c^2 + \lambda_2 z^2 + \ldots,$$

there being no first power of z because of the neck; then

$$u' = \lambda_2 z + \ldots,$$

$$r^2 + u'^2 = c^2 + (\lambda_2 + \lambda_2^2) z^2 + \ldots,$$

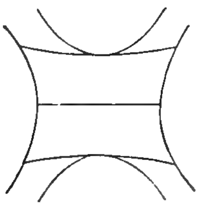
and so

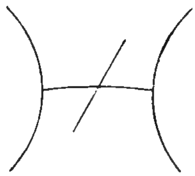
$$\phi - \gamma = \int \frac{c}{c^2 + \ldots} \left(\frac{c^2 + \ldots}{\lambda_2 z^2 + \ldots} \right)^{\frac{1}{2}} dz$$

$$= \int \frac{1}{\lambda_2^{\frac{1}{2}}} \left(\frac{1}{z} + \text{positive powers} \right) dz,$$

so that ϕ becomes large; that is, the geodesic on such a surface near the neck-circle is asymptotic to that circle. Such is the fact at the neck of a hyperboloid of one sheet.

II. Let $h > c$. Then as we are to have $r \geqslant h$ for reality, we must have $r > c$, so that the geodesic never meets the neck-circle. It touches the parallels given by $r = h$; and otherwise lies above the upper parallel or below the lower parallel as in the figure.





III. Let $h < c$. Then as $r \geqslant c$, we have $r > h$, and so $\sin \psi$ is never unity; thus the geodesic crosses the neck-circle, cutting it at a finite (non-zero) angle.

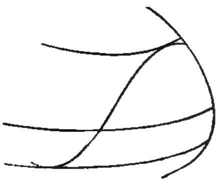
Hence near the neck of a surface there are three kinds of possible geodesics. The first of the classes indicated is a boundary between the second and the third of the classes.

95. Next, consider the vicinity of a parallel of maximum radius a; when the surface is symmetrical with respect to the plane of the parallel, we have an equator.

As $\sin \psi = h/r$, and r cannot be greater than a, it follows that h cannot be greater than a.

I. Let $h = a$. Then a is the only possible value of r, in order that the curve may be real; we have the parallel of maximum radius a, which is itself a geodesic.

II. Let $h < a$. Then ψ is real so long as r is not less than h; it is $\frac{1}{2}\pi$ when $r = h$, that is, the geodesic touches the parallel or parallels given by $r = h$; and it is $\sin^{-1}(h/a)$ at the parallel of greatest radius. Also, as r changes continuously from h to a, ψ decreases continuously from $\frac{1}{2}\pi$ to $\sin^{-1}(h/a)$; and as r then changes continuously from a to h, ψ increases continuously from $\sin^{-1}(h/a)$ to $\frac{1}{2}\pi$. Thus the geodesic undulates between the two parallels, which are given by $r = h$, nearest to the parallel of greatest radius.



Take the plane of the parallel $r = a$ as the plane $z = 0$. Above the plane $r^2 = 2u(z)$, and below the plane $r^2 = 2u(-z)$; hence, as

$$\phi - \gamma = h\int\left(\frac{r^2 + u'^2}{r^2 - h^2}\right)^{\frac{1}{2}} \frac{dz}{r^2},$$

the difference of longitude, say D, between a place of highest latitude and the nearest place of lowest latitude is

$$D = h\int_h^a \left[\left\{\frac{r^2 + u'^2(z)}{r^2 - h^2}\right\}^{\frac{1}{2}} \frac{1}{ru'(z)} + \left\{\frac{r^2 + u'^2(-z)}{r^2 - h^2}\right\}^{\frac{1}{2}} \frac{1}{ru'(-z)}\right] dr.$$

If the parallel $r = a$ is an equator, so that the surface is symmetrical with respect to its plane, then

$$D = 2h\int_h^a \left\{\frac{r^2 + u'^2(z)}{r^2 - h^2}\right\}^{\frac{1}{2}} \frac{dr}{ru'(z)}.$$

Such a geodesic is not usually a closed curve; but it is a closed curve* if D is commensurable with π, that is, if

$$D = m\pi,$$

where m is a commensurable number. Take the latter symmetrical case. Let a new variable t for integration and a new constant g for a limit of integration be defined by relations

$$\frac{1}{r^2} = t + \frac{1}{a^2}, \quad \frac{1}{h^2} = g + \frac{1}{a^2},$$

and write

$$\frac{1}{r^2} + \frac{1}{u'^2(z)} = \chi^2(t);$$

then

$$m\pi = D = \int_0^h \frac{r^2\chi(t)}{(h - t)^{\frac{1}{2}}} dt.$$

* For this investigation, see Darboux's treatise, t. iii, § 582.

Now m is purely numerical; consequently* we must have

$$r^2\chi(t) = mt^{-\frac{1}{2}},$$

that is,

$$r^4\left\{\frac{1}{r^2} + \frac{1}{u'^2(z)}\right\} = m^2\frac{a^2r^2}{a^2 - r^2},$$

and therefore

$$\left(\frac{dz}{dr}\right)^2 = \frac{(m^2-1)\,a^2 + r^2}{a^2 - r^2}.$$

But z is a maximum or minimum (so that $dz = 0$) when $r = h$; hence *m is less than unity*, and

$$\left(\frac{dz}{dr}\right)^2 = \frac{r^2 - h^2}{a^2 - r^2},$$

which is the equation defining the surface of revolution that possesses closed geodesics undulating across the equator.

As regards this surface, its element of arc ds is

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 + dz^2 \\ &= \frac{a^2 - h^2}{a^2 - r^2}dr^2 + r^2 d\phi^2. \end{aligned}$$

Now $h^2 = a^2(1 - m^2)$; let

$$r = a\sin u, \quad \phi = m\phi';$$

then

$$ds^2 = m^2a^2(du^2 + \sin^2 u\, d\phi'^2).$$

But the last expression is the square of the arc-element on a sphere of radius ma; hence *the surface of revolution in question is deformable into a sphere,* which is Darboux's result.

96. But it may happen that, in the vicinity of the parallel of maximum radius a, there is no parallel given by $r = h$; as our only condition is that $h < a$, it might happen that on the whole surface there is no parallel given by $r = h$. In either case, the geodesic crosses the parallel given by $r = a$ at a finite non-zero angle; in its march away from that parallel across parallels of decreasing radius, it crosses the meridians at a constantly increasing angle, which however remains less than a right angle unless and until it reaches a parallel given by $r = h$.

97. It now becomes necessary to investigate the range along the geodesic curve for which the curve is actually the shortest distance between the extreme points, or, what is the same thing, to determine the conjugate of a given point.

* The result can easily be established. It is an example of a theorem given by Abel (*Œuvres complètes*, 1881, vol. i, pp. 14, 15) in a memoir now regarded as a pioneer in the subject which, under the name *integral equations*, has attracted many investigators in recent years.

There are two cases to consider. In the first, h is not zero and the curve is not a meridian; in the second, h is zero and the curve is a meridian.

In the former case, we have

$$\phi - \gamma = h \int \left(\frac{r^2 + u'^2}{r^2 - h^2} \right)^{\frac{1}{2}} \frac{dz}{r^2},$$

provided that the geodesic curve is not given by the very special case $r = h$, $dz = 0$, that is, provided it is not a parallel of maximum or minimum radius (in which event, the method of treatment is similar to that adopted for the case of geodesic meridians). Then

$$\frac{\partial \phi}{\partial \gamma} = 1, \quad \frac{\partial \phi}{\partial h} = \int \frac{(r^2 + u'^2)^{\frac{1}{2}}}{(r^2 - h^2)^{\frac{3}{2}}} dz;$$

and the condition is that, in the range, the quantity

$$A \frac{\partial \phi}{\partial \gamma} + B \frac{\partial \phi}{\partial h}$$

shall not again attain the value which it has at the beginning of the range; that is, the quantity

$$\frac{\partial \phi}{\partial h}$$

must not again attain its initial value.

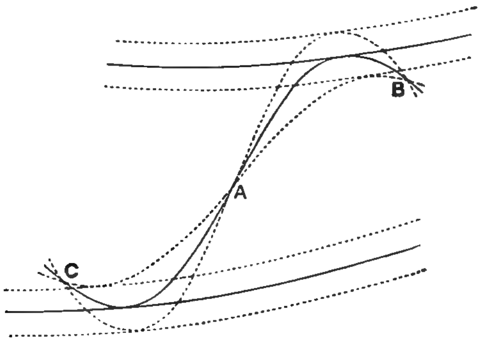
Suppose that there is no parallel given by $r = h$ (so that every point on the surface is at a distance from the axis greater than h). Then the subject of integration in $\partial\phi/\partial h$ is always finite and positive, and dz has the same sign along the curve; thus $\partial\phi/\partial h$ is always increasing or always decreasing along the geodesic, and so it cannot again acquire its initial value. There is no finite limit to the range of shortest distance along the curve; no point on the geodesic has a conjugate at a finite distance.

Suppose that there is a parallel given by $r = h$. Then from the initial point of the range until the parallel is neared, the subject of integration is finite and of the same positive sign while dz is of uniform sign. In passing through contact with the parallel, the relations

$$d\phi = h \left(\frac{r^2 + u'^2}{r^2 - h^2} \right)^{\frac{1}{2}} \frac{dz}{r^2}, \quad d \left(\frac{\partial \phi}{\partial h} \right) = \frac{(r^2 + u'^2)^{\frac{1}{2}}}{(r^2 - h^2)^{\frac{3}{2}}} dz,$$

shew that $\dfrac{\partial \phi}{\partial h}$ passes through an infinite value and always increases as ϕ increases in passing through the contact; that is, $\dfrac{\partial \phi}{\partial h}$ changes its sign in passing through the infinite value and begins to increase from $-\infty$. After some stage it will increase to its initial value; at that stage, we have the conjugate of the initial point. But the actual analytical determination of the conjugate in precise expression depends upon the particular surface.

The same result can be obtained by regarding the conjugate of the initial point as the ultimate position of the next intersection with a consecutive curve through the initial point. In order to have such a consecutive curve, we need values $h+dh$, $\gamma+d\gamma$ of the arbitrary constants; in the figure, let this curve be represented by the dotted lines (for positive and



negative values of dh respectively), while the original curve is represented by the continuous line. Then the point C is the conjugate of A for the direction AC; and a range along the geodesic, beginning at A, is the shortest distance for all points from A to C short of C. Similarly B is the conjugate of A for the direction AB.

Note. In dealing with the critical function

$$\int \frac{(r^2+u'^2)^{\frac{1}{2}}}{(r^2-h^2)^{\frac{3}{2}}}\,dz,$$

it proves necessary to exercise care in the choice of the current variable for the integral, so that it shall admit of continuous increase (or continuous decrease) throughout the range of integration that corresponds to the continuous range of the curve.

98. Consider, for example, the non-meridian geodesics on an oblate spheroid*. The surface is

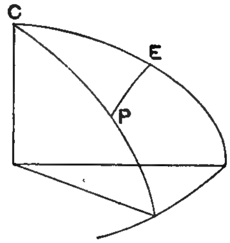
$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1,$$

so that we can take

$$z=c\cos\theta,\quad r=a\sin\theta,\quad x=a\sin\theta\cos\phi,\quad y=a\sin\theta\sin\phi.$$

* See two notes by the author, *Messenger of Math.*, vol. xxv (1896), p. 84, p. 161. References to Jacobi, Halphen, and Cayley are given on pp. 94, 95, (*l.c.*).

We know that a non-meridian geodesic undulates between two parallels; so let E be the highest point of our geodesic EP, and let CE, CP be the meridians through E and P. We thus have a geodesic triangle CEP, right-angled at E; the angle ECP is ϕ. Let $CPE = \psi$; and let α be the value of θ at E. Then (§§ 93, 95)



$$\frac{a^2}{f}\sin^2\theta \,.\, \phi' = h;$$

and therefore

$$a\sin^2\theta\frac{d\phi}{ds} = \sin\alpha,$$

which is a first integral of the characteristic equation.

This leads to

$$d\phi = \frac{(1-e^2\sin^2\theta)^{\frac{1}{2}}\sin\alpha}{(\sin^2\theta - \sin^2\alpha)^{\frac{1}{2}}\sin\theta}\,d\theta,$$

which can be regarded as the differential equation of the geodesic. The explicit integration requires elliptic functions and can be effected as follows. Let

$$\cos\theta = \cos\alpha \operatorname{cn} u,$$

where u is a new variable vanishing when $\theta = \alpha$, and where the modulus k of the elliptic functions is given by

$$k^2 = \frac{e^2\cos^2\alpha}{1-e^2\sin^2\alpha}.$$

Then

$$d\phi = \frac{(1-e^2\sin^2\alpha)^{\frac{1}{2}}}{\sin\alpha}\,\frac{\operatorname{dn}^2 u}{1+\cot^2\alpha\operatorname{sn}^2 u}\,du,$$

so that

$$\begin{aligned}\phi &= \frac{(1-e^2\sin^2\alpha)^{\frac{1}{2}}}{\sin\alpha}\int_0^u \frac{\operatorname{dn}^2 u}{1+\cot^2\alpha\operatorname{sn}^2 u}\,du \\ &= U,\end{aligned}$$

say.

The general equation of geodesics, without the initial choice of the meridian of reference, would be

$$\phi - \gamma = U,$$

containing two arbitrary constants α and γ.

As regards the arc of the geodesic, we have

$$\begin{aligned}\frac{ds}{a} &= \frac{\sin^2\theta}{\sin\alpha}\,d\phi \\ &= (1-e^2\sin^2\alpha)^{\frac{1}{2}}\operatorname{dn}^2 u\,du,\end{aligned}$$

and therefore

$$s = a(1 - e^2 \sin^2 \alpha)^{\frac{1}{2}} E(u),$$

where E is the second elliptic integral; the arc being measured from the place $u = 0$, that is, from the point E.

As $CPE = \psi$, we have (§ 93)

$$r \sin \psi = h = a \sin \alpha;$$

hence

$$\sin \theta \sin \psi = \sin \alpha.$$

Hence

$$\tan \psi = \frac{\sin \alpha}{(\sin^2 \theta - \sin^2 \alpha)^{\frac{1}{2}}} = \frac{\tan \alpha}{\operatorname{sn} u},$$

and

$$\tan \theta \cos \psi = \operatorname{tn} u.$$

Manifestly, some of the relations for the parts of the geodesic triangle CPE on the spheroid are similar to those of a right-angled triangle on a sphere; for the respective surfaces they are

$$\left.\begin{aligned} \sin \theta \sin \psi &= \sin \alpha \\ \operatorname{cn} u \cos \alpha &= \cos \theta \\ \tan \psi \operatorname{sn} u &= \tan \alpha \\ \tan \theta \cos \psi &= \operatorname{tn} u \\ s(1 - e^2 \sin^2 \alpha)^{-\frac{1}{2}} &= aE(u) \end{aligned}\right\}. \qquad \left.\begin{aligned} \sin \theta \sin \psi &= \sin \alpha \\ \cos u \cos \alpha &= \cos \theta \\ \tan \psi \sin u &= \tan \alpha \\ \tan \theta \cos \psi &= \tan u \\ s &= au \end{aligned}\right\}.$$

But it should be noted that, on the spheroid, θ is not the angle subtended by CP at the centre, as it is on a sphere; nor is α the angle subtended by CE at the centre, as it is on a sphere.

On the auxiliary sphere of the spheroid (that is, a sphere having the same equator), take the projection of the spheroid orthogonal to the equator. Let C', E', P' be the projections of C, E, P; the great circles $C'P'$ and $C'E'$ are the projections of the meridians CP and CE; while $E'P'$, the projection of the geodesic EP, is not a great circle. The angles subtended by $C'E'$ and $C'P'$ at the centre are α and θ; also $E'C'P'$ is ϕ, and $C'E'P'$ is a right angle. Let the angle $E'P'C'$ be denoted by ψ', and the arc $E'P'$ by s'; then we have the equations

$$s' = a(1 - e^2 \sin^2 \alpha)^{\frac{1}{2}} \operatorname{am} u,$$

$$\sin \theta \sin \psi' = \sin \alpha \operatorname{dn} u,$$

$$\tan \psi' \operatorname{sn} u = (1 - e^2 \sin^2 \alpha)^{\frac{1}{2}} \operatorname{dn} u \tan \alpha,$$

$$\tan \theta \cos \psi' = (1 - e^2 \sin^2 \alpha)^{-\frac{1}{2}} \operatorname{tn} u.$$

The establishment of these relations is left as an exercise.

Also, we have

$$\phi(1-e^2\sin^2\alpha)^{-\frac{1}{2}}=\tan^{-1}\left(\frac{\operatorname{tn}u}{\sin\alpha}\right)-\frac{1}{\sin\alpha}\int_0^u\frac{\operatorname{dn}u-\operatorname{dn}^2u}{1+\cot^2\alpha\operatorname{sn}^2u}\,du$$

on the spheroid, and

$$\phi=\tan^{-1}\left(\frac{\tan u}{\sin\alpha}\right)$$

on the sphere.

The geodesic undulates between the two parallels determined by $\theta=\alpha$, $\theta=\pi-\alpha$. Where it cuts a parallel determined by $\theta=\beta$, we have

$$\cos\beta=\cos\alpha\operatorname{cn}u;$$

thus the successive points are $u=u_1$, $u=4K-u_1$, $u=4K+u_1$, and so on. The difference in longitude between the highest point and the nearest lowest point of the geodesic is

$$(1-e^2\sin^2\alpha)^{\frac{1}{2}}\left\{\pi-\frac{1}{\sin\alpha}\int_0^{2K}\frac{\operatorname{dn}u-\operatorname{dn}^2u}{1+\cot^2\alpha\operatorname{sn}^2u}\,du\right\},$$

and is therefore less than π.

To find the conjugate of any point on the geodesic, we take the general integral equation in the form

$$\phi-\gamma=\int\frac{(1-e^2\sin^2\theta)^{\frac{1}{2}}\sin\alpha}{(\sin^2\theta-\sin^2\alpha)^{\frac{1}{2}}\sin\theta}\,d\theta=I.$$

Then

$$\frac{\partial\phi}{\partial\alpha}=\frac{\partial I}{\partial\alpha},\quad\frac{\partial\phi}{\partial\gamma}=1.$$

The critical function $A\dfrac{\partial\phi}{\partial\alpha}+B\dfrac{\partial\phi}{\partial\gamma}$ is not again to acquire its initial value, that is, $\dfrac{\partial I}{\partial\alpha}$ is not again to acquire its initial value. Now

$$\frac{\partial I}{\partial\alpha}=\int\frac{(1-e^2\sin^2\theta)^{\frac{1}{2}}\sin\theta\cos\alpha}{(\sin^2\theta-\sin^2\alpha)^{\frac{3}{2}}}\,d\theta$$

$$=\frac{(1-e^2\sin^2\alpha)^{\frac{1}{2}}}{\cos\alpha}\int\frac{\operatorname{dn}^2u}{\operatorname{sn}^2u}\,du,$$

on introducing the elliptic functions; and

$$\int\frac{\operatorname{dn}^2u}{\operatorname{sn}^2u}\,du=k'^2u-E(u)-\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u}.$$

Hence, if u_1 be the conjugate of u_0, we have

$$k'^2u_1-E(u_1)-\frac{\operatorname{cn}u_1\operatorname{dn}u_1}{\operatorname{sn}u_1}=k'^2u_0-E(u_0)-\frac{\operatorname{cn}u_0\operatorname{dn}u_0}{\operatorname{sn}u_0},$$

so that

$$E(u_1-u_0)-k'^2(u_1-u_0)=\frac{\operatorname{sn}(u_1-u_0)}{\operatorname{sn}u_1\operatorname{sn}u_0}.$$

The required value of u_1 is the root of this equation, regarded as an equation in u_1, which is next greater than u_0. By tracing the curves

$$y = \frac{\operatorname{sn}(x - u_0)}{\operatorname{sn} x \operatorname{sn} u_0}, \quad y = E(x - u_0) - k'^2 (x - u_0),$$

it is easy to verify that, when $0 < u_0 < 2K$, then

$$4K > u_1 > u_0 + 2K.$$

99. In the second case mentioned in § 97, when the curve is a meridian, h is zero, and ϕ is constant. We cannot deduce the critical function from the value of ϕ, and must proceed to obtain it as the primitive of the linear equation of the second order given in § 89, III (ii). Returning to the general surface of revolution, we have always

$$f = \left\{ r^2 \left(\frac{d\phi}{dz}\right)^2 + \frac{u'^2}{2u} + 1 \right\}^{\frac{1}{2}},$$

the arc being $\int f\,dz$; hence, denoting $d\phi/dz$ by ϕ', we have

$$\frac{\partial^2 f}{\partial \phi^2} = 0, \quad \frac{\partial^2 f}{\partial \phi \partial \phi'} = 0,$$

for f does not explicitly involve ϕ; and

$$\frac{\partial^2 f}{\partial \phi'^2} = \frac{r^3}{(u'^2 + r^2)^{\frac{1}{2}}}$$

when we insert the zero value of ϕ'. The equation for the critical function U is

$$\frac{d}{dz}\left\{ \frac{r^3}{(u'^2 + r^2)^{\frac{1}{2}}} \frac{dU}{dz} \right\} = 0,$$

and therefore

$$U = B + A \int (u'^2 + r^2)^{\frac{1}{2}} \frac{dz}{r^3}.$$

The range is limited by the condition that U must not again acquire the value which it has at the beginning of the range; in other words, the quantity

$$\int (u'^2 + r^2)^{\frac{1}{2}} \frac{dz}{r^3}$$

must not again in the range acquire its initial value.

Another form of the function is

$$\int (u'^2 + r^2)^{\frac{1}{2}} \frac{dr}{r^2 u'}.$$

It will be noticed that the form of the function coincides with the critical function in the earlier case when h is made zero therein.

Note. As before, so here, in dealing with the critical function

$$\int (u'^2 + r^2)^{\frac{1}{2}} \frac{dz}{r^3},$$

it proves necessary to exercise care in the choice of the current variable for the range of the integral, so that it may increase continuously (or decrease continuously) throughout the range.

Ex. 1. In the case of a circular cylinder $r =$ constant, $u' = 0$; the critical function is $\frac{z}{a^2}$, and along a (rectilinear) meridian this function never resumes its initial value. There is no limit to the shortest-distance property.

Similarly for a circular cone.

Ex. 2. In the case of a sphere, we have

$$r^2 = a^2 - z^2, \quad u' = -z,$$

so that the critical function is

$$\int \frac{a\,dz}{(a^2 - z^2)^{\frac{3}{2}}},$$

that is, $\frac{1}{a}\tan^{-1}\frac{z}{a}$. Hence the conjugate of any point z_0 on the meridian is given by $z_0 + a\pi$, that is, the diametrically opposite point on the meridian; and therefore (as is to be expected) a great circle is the shortest distance for any length less than half the circle.

Ex. 3. In the case of a paraboloid of revolution, the axis of the parabola being the axis of revolution, we have

$$r^2 = 2lz, \quad u' = l,$$

so that the critical function is

$$\int (l^2 + r^2)^{\frac{1}{2}} \frac{dr}{r^2 l},$$

that is,

$$-\frac{(l^2 + r^2)^{\frac{1}{2}}}{rl} + \frac{1}{l}\sinh^{-1}\frac{r}{l}.$$

Hence any arc of the meridian, whatever its length when it does not include the vertex, is a shortest distance. When an arc of a meridian does include the vertex, and r_1 is the distance of any point from the axis, then the shortest distance along the arc must not extend so far as a point distant r_2 from the axis, where

$$\sinh^{-1}\frac{r_1}{l} + \sinh^{-1}\frac{r_2}{l} = \frac{1}{r_1}(l^2 + r_1^2)^{\frac{1}{2}} + \frac{1}{r_2}(l^2 + r_2^2)^{\frac{1}{2}}.$$

Ex. 4. In the case of an anchor-ring, we have

$$z = a\sin\theta, \quad r = c + a\cos\theta.$$

The critical function becomes $\int \frac{a\,d\theta}{(c + a\cos\theta)^2}$, which is

$$-\frac{a}{(c^2 - a^2)^{\frac{3}{2}}}(c\psi - a\sin\psi),$$

where

$$\tan\tfrac{1}{2}\psi = \left(\frac{c-a}{c+a}\right)^{\frac{1}{2}} \tan\tfrac{1}{2}\theta,$$

and, in the function, ψ has to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. In the march of the function with the increase of θ, the angle ψ increases; when it increases beyond $\frac{1}{2}\pi$, we must take $\psi-\pi$ in its place.

Thus the conjugate of a point on the meridian is the point half-way round the meridian.

Ex. 5. A meridian is drawn on an oblate spheroid; and any point on it is denoted by $r=a\sin\theta$, $z=c\cos\theta$. Shew that, if θ_1 be the conjugate of θ_0, then

$$E(u_1-u_0)=(1-e^2)(u_1-u_0)+\frac{\operatorname{sn}(u_1-u_0)}{\sin\theta_1\sin\theta_0},$$

where

$$\operatorname{am} u_1=\theta_1, \quad \operatorname{am} u_0=\theta_0,$$

and where e, $=(a^2-c^2)^{\frac{1}{2}}/a$, is the modulus of the elliptic functions.

The Gauss Theory of Geodesics.

100. We proceed now to the discussion of geodesics upon a surface and their relation to other curves on the surface, without any special regard to the range within which they are the shortest distance between two points. The fundamental property is that the principal normal of the curve coincides with the normal to the surface at any point. This property is sometimes used as the explicit definition of the curve (§ 91); it has been derived (§ 65) from statical considerations; it has been shewn to be a consequence (§ 91), under the calculus of variations, of the definition by the shortest arc-distance.

Under the last method, it was deduced from the characteristic equations in the calculus of variations. It is important, however, that the establishment of these characteristic equations should not be based solely upon that method; so, accepting the geodesic property (whether as a definition, or as derived from statical considerations), we can establish the general equations as follows. Now

$$x''=x_{11}p'^2+2x_{12}p'q'+x_{22}q'^2+x_1p''+x_2q'',$$

where dashes now denote differentiation with regard to s; and therefore, on the assumption of the characteristic property, and taking the sign of the radius of curvature of a normal section as in § 31, we have

$$\frac{X}{\rho}=x_{11}p'^2+2x_{12}p'q'+x_{22}q'^2+x_1p''+x_2q'',$$

and similarly

$$\frac{Y}{\rho}=y_{11}p'^2+2y_{12}p'q'+y_{22}q'^2+y_1p''+y_2q'',$$

$$\frac{Z}{\rho}=z_{11}p'^2+2z_{12}p'q'+z_{22}q'^2+z_1p''+z_2q''.$$

Multiplying by X, Y, Z, and adding, we have

$$\frac{1}{\rho}=Lp'^2+2Mp'q'+Nq'^2,$$

the customary formula for the curvature of a normal section; it therefore

gives the circular curvature of the geodesic at the point. Again, multiplying by x_1, y_1, z_1, and adding; and multiplying by x_2, y_2, z_2, and adding; we have

$$\left.\begin{aligned} Ep''+Fq''+mp'^2+2m'p'q'+m''q'^2&=0\\ Fp''+Gq''+np'^2+2n'p'q'+n''q'^2&=0\end{aligned}\right\},$$

which are the general equations of a geodesic on a surface.

From these, as before (§§ 91, 92), we have

$$\frac{d^2q}{dp^2}=\Gamma''\left(\frac{dq}{dp}\right)^3+(2\Gamma'-\Delta'')\left(\frac{dq}{dp}\right)^2+(\Gamma-2\Delta')\frac{dq}{dp}-\Delta,$$

as the single general equation of geodesics on the surface.

In all the forms of the equations as given in § 92, the parametric variables are general and the parametric curves are completely unrestricted. Some simplification arises when the parametric curves are specialised.

101. As an example, consider the geodesics upon the quadric*

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1.$$

Let the quadric be referred to its lines of curvature as usual; then (§ 78) we have

$$E=\tfrac{1}{4}(p-q)P,\quad F=0,\quad G=\tfrac{1}{4}(q-p)Q,$$

where

$$P=\frac{p}{(a+p)(b+p)(c+p)},\quad Q=\frac{q}{(a+q)(b+q)(c+q)}.$$

Hence (§ 77)

$$\begin{aligned} \Gamma&=\ \frac{1}{2}\left(\frac{1}{p-q}+\frac{P'}{P}\right), & \Delta&=\ \frac{1}{2}\frac{P}{(q-p)Q},\\ \Gamma'&=-\frac{1}{2}\frac{1}{p-q}, & \Delta'&=-\frac{1}{2}\frac{1}{q-p},\\ \Gamma''&=\ \frac{1}{2}\frac{Q}{(p-q)P}, & \Delta''&=\ \frac{1}{2}\left(\frac{1}{q-p}+\frac{Q'}{Q}\right),\end{aligned}$$

where $P'=dP/dp$, $Q'=dQ/dq$. Writing

$$q'=\frac{dq}{dp},\quad q''=\frac{d^2q}{dp^2},$$

we have the equation of the geodesics in the form

$$2(p-q)PQq''=(Qq'^2-P)(Qq'-P)-(p-q)PQq'\left(\frac{Q'q'}{Q}-\frac{P'}{P}\right).$$

To integrate this equation, introduce a new quantity u such that

$$\frac{P}{u+p}=\frac{Qq'^2}{u+q}=\frac{Qq'^2-P}{q-p}.$$

* For a discussion of geodesics upon quadrics not of revolution, see a paper by the author, *Proc. Lond. Math. Soc.*, vol. xxvii (1896), pp. 250—280.

Then

$$\frac{P'}{P}-\frac{u'+1}{u+p}-\frac{Q'q'}{Q}-2\frac{q''}{q'}+\frac{u'+q'}{u+q}=0.$$

Substituting for u, and inserting the value of q'' from the differential equation, we have

$$\frac{u'}{u+q}-\frac{u'}{u+p}=0,$$

that is,

$$u'=0,$$

and so

$$u=\text{constant}=\theta.$$

Thus a first integral of the equation of geodesics on the quadric is

$$\frac{Pdp^2}{\theta+p}=\frac{Qdq^2}{\theta+q}.$$

Let

$$R(p)=p(a+p)(b+p)(c+p)(\theta+p),$$
$$R(q)=q(a+q)(b+q)(c+q)(\theta+q);$$

then the equation is

$$\frac{pdp}{\{R(p)\}^{\frac{1}{2}}}\pm\frac{qdq}{\{R(q)\}^{\frac{1}{2}}}=0,$$

where the lower sign is chosen for a geodesic along which p and q increase together or decrease together, and the upper sign is chosen in the alternative cases. Now

$$\frac{\dfrac{pdp^2}{(a+p)(b+p)(c+p)}}{\theta+p}=\frac{\dfrac{qdq^2}{(a+q)(b+q)(c+q)}}{\theta+q}=\frac{\dfrac{4ds^2}{p-q}}{p-q},$$

and therefore

$$\frac{2ds}{p-q}=\frac{pdp}{\{R(p)\}^{\frac{1}{2}}}=\mp\frac{qdq}{\{R(q)\}^{\frac{1}{2}}}.$$

Let the upper sign be chosen; then

$$\frac{dp}{\{R(p)\}^{\frac{1}{2}}}+\frac{dq}{\{R(q)\}^{\frac{1}{2}}}=2\frac{ds}{p-q}\left(\frac{1}{p}-\frac{1}{q}\right)=-2\frac{ds}{pq}.$$

Accordingly, the first integral of the equation of the geodesics can be taken in the form

$$\left.\begin{aligned}\frac{dp}{2\{R(p)\}^{\frac{1}{2}}}+\frac{dq}{2\{R(q)\}^{\frac{1}{2}}}&=dv\\ \frac{pdp}{2\{R(p)\}^{\frac{1}{2}}}+\frac{qdq}{2\{R(q)\}^{\frac{1}{2}}}&=0\end{aligned}\right\},$$

being the canonical equations for hyperelliptic integrals; and*

$$ds = -pq\,dv.$$

Also we have

$$d\Upsilon = \frac{(b+p)(c+p)}{2\{R(p)\}^{\frac{1}{2}}}dp + \frac{(b+q)(c+q)}{2\{R(q)\}^{\frac{1}{2}}}dq$$
$$= (bc - pq)\,dv,$$

so that

$$ds = d\Upsilon - bc\,dv,$$

that is,

$$s - s_0 = \Upsilon - bcv.$$

The integration of the equations thus requires the use of hyperelliptic functions of the simplest class†, just as elliptic functions are required for the oblate spheroid, and circular functions are required for the sphere.

Let the quadric be an ellipsoid, so that a, b, c are positive; and suppose that

$$a > b > c.$$

Then we have

$$a > -p > b > -q > c.$$

The curves, p = constant, are the intersection of the ellipsoid with the confocal hyperboloids of two sheets; and the curves, q = constant, are the intersection of the ellipsoid with the confocal hyperboloids of one sheet. To secure real values, we must have $R(p) \geqslant 0$, $R(q) \geqslant 0$; hence

$$-p > \theta > -q,$$

and so

$$a > -p > \theta,\ b > -q > c.$$

Because θ and b lie between $-p$ and $-q$, there are three cases according as $\theta = b$, $\theta < b$, $\theta > b$.

(i) When $\theta = b$, the geodesic passes through an umbilicus and, when continued, through the diametrally opposite umbilicus.

(ii) When $\theta < b$, the geodesic touches (but does not cross) a line of curvature given by $\theta = -q$; it undulates between the two lines of curvature given by $q = -\theta$, and these are lines upon the confocal hyperboloid of one sheet.

(iii) When $\theta > b$, the geodesic touches (but does not cross) a line of curvature given by $\theta = -p$; it undulates between the two lines of curvature given by $p = -\theta$, and these are lines upon the confocal hyperboloid of two sheets.

* This agrees with the form given by Weierstrass in 1861; *Ges. Werke*, t. i, p. 262. See also Cayley, *Coll. Math. Papers*, vol. vii, p. 493, vol. viii, p. 156, p. 188.

† The expressions for the coordinates and the length of the arc in terms of the current parameter v are given in the author's paper, already quoted. It may be added that only elliptic functions are required for the equations of umbilical geodesics.

Ex. Let ϖ denote the perpendicular from the centre of the ellipsoid upon the tangent plane at any point; and let D', D'' denote the semi-diameters of the ellipsoid, parallel to the respective directions of any geodesic through the point and of a line of curvature through the point. Prove that $\varpi D'$ is constant along the geodesic and that $\varpi D''$ is constant along the line of curvature. Is the converse true for either line or for both?

Discuss the configuration of the curves

$$\varpi D = k^2,$$

where k is a parametric constant, and D is the semi-diameter of the ellipsoid parallel to the tangent to the curve at the point.

102. Returning now to the general equation of geodesics, let i denote the angle, made by a geodesic with the parametric curve $q=b$ through the point, and measured towards the curve $p=a$. Then (§ 26, θ' now being denoted by i, and θ later by j), we have

$$E^{\frac{1}{2}} \cos i = Ep' + Fq',$$

and therefore

$$\frac{d}{ds}(Ep' + Fq') = \tfrac{1}{2} E^{-\frac{1}{2}} (E_1 p' + E_2 q') \cos i - E^{\frac{1}{2}} \frac{di}{ds} \sin i$$

$$= \frac{1}{2E}(E_1 p' + E_2 q')(Ep' + Fq') - Vq' \frac{di}{ds};$$

consequently, along a geodesic, we have

$$\tfrac{1}{2}(E_1 p'^2 + 2F_1 p'q' + G_1 q'^2) = \frac{1}{2E}(E_1 p' + E_2 q')(Ep' + Fq') - Vq' \frac{di}{ds}.$$

Hence

$$V\frac{di}{ds} = \left(\frac{1}{2}\frac{F}{E}E_1 + \frac{1}{2}E_2 - F_1\right)p' + \left(\frac{1}{2}\frac{F}{E}E_2 - \frac{1}{2}G_1\right)q'$$

$$= -\frac{V^2\Delta}{E}p' - \frac{V^2\Delta'}{E}q'.$$

Thus, along a geodesic, we have

$$\frac{E}{V}di = -\Delta dp - \Delta' dq,$$

together with

$$ds \cos i = E^{-\frac{1}{2}}(E dp + F dq), \quad ds \sin i = E^{-\frac{1}{2}} V dq,$$

where i is the inclination of the geodesic to the parametric curve $q=b$. Similarly, if j is the inclination of the geodesic to the parametric curve $p=a$ through the point, measured towards the curve $q=b$, we have

$$\frac{G}{V}dj = -\Gamma' dp - \Gamma'' dq,$$

together with

$$ds \cos j = G^{-\frac{1}{2}}(F dp + G dq), \quad ds \sin j = G^{-\frac{1}{2}} V dp.$$

These results are in accord with the relation (§ 36)

$$\frac{d\omega}{V} = -\left(\frac{\Delta}{E} + \frac{\Gamma'}{G}\right) dp - \left(\frac{\Delta'}{E} + \frac{\Gamma''}{G}\right) dq,$$

for the variation of the angle between the parametric curves.

Geodesic Curvature of Curves.

103. We now are in a position to develope a notion as to another curvature of curves on a surface. We have already considered the circular curvature and the torsion of any twisted curve, and therefore of any curve upon a surface. In the case of a plane curve, we regard the curvature in connection with the deviation of the curve from its tangent. Geodesics on a surface have much analogy with straight lines in a plane, even when the surface is not developable; and so it is natural to consider a curvature of a curve upon the surface in connection with the deviation of the curve from its geodesic tangent.

Accordingly, let a curve at any point cut the parametric curve $q = b$ at an angle i; and at the point draw the geodesic tangent to the curve. At a consecutive point on the curve, let $i + di$ be the inclination of the curve to the parametric curve, $q =$ constant, through the point; and let $i + \delta i$ be the inclination of the consecutive geodesic tangent at the consecutive point to that consecutive parametric curve. Then the angular deviation (measured from the parametric curve $q = b$ towards the curve $p = a$) of the curve from its geodesic tangent is $di - \delta i$, that is,

$$di + \frac{V\Delta}{E} dp + \frac{V\Delta'}{E} dq.$$

This is called the angle of *geodesic contingence* of the curve; the rate of arc-variation of this angle is called its *geodesic curvature.* Denoting* the latter by $1/\gamma$, we have

$$\frac{ds}{\gamma} = di + \frac{V\Delta}{E} dp + \frac{V\Delta'}{E} dq.$$

Similarly, we have

$$\frac{ds}{\gamma} = -\left(dj + \frac{V\Gamma'}{G} dp + \frac{V\Gamma''}{G} dq\right),$$

where j is the inclination to the parametric curve $p = a$, and the geodesic contingence as measured from that curve is $\delta j - dj$.

Of course, when the curve itself is a geodesic, its geodesic curvature is zero; hence

$$\frac{di}{ds} + \frac{V\Delta}{E}\frac{dp}{ds} + \frac{V\Delta'}{E}\frac{dq}{ds} = 0$$

* Sometimes the symbol ρ_g is used, instead of γ.

is the equation of a geodesic. When the value of i is inserted, this equation reduces to the earlier general equation of geodesics.

Let $1/\gamma''$ be the geodesic curvature of $p=a$, and $1/\gamma'$ be the geodesic curvature of $q=b$. The element of arc along $p=a$ is $G^{\frac{1}{2}}\,dq$; hence

$$\frac{G^{\frac{1}{2}}\,dq}{\gamma''}=d\omega+\frac{V\Delta'}{E}\,dq,$$

$$\frac{G^{\frac{1}{2}}\,dp}{\gamma''}=\quad-\frac{V\Gamma''}{G}\,dq,$$

that is,

$$\left.\begin{aligned}\frac{1}{\gamma''}&=-V\Gamma''G^{-\frac{3}{2}}\\ \frac{1}{\gamma''}&=G^{-\frac{1}{2}}\frac{\partial\omega}{\partial q}+\frac{V\Delta'}{EG^{\frac{1}{2}}}\end{aligned}\right\}.$$

Similarly,

$$\left.\begin{aligned}\frac{1}{\gamma'}&=V\Delta E^{-\frac{3}{2}}\\ \frac{1}{\gamma'}&=-E^{-\frac{1}{2}}\frac{\partial\omega}{\partial p}-\frac{V\Gamma'}{E^{\frac{1}{2}}G}\end{aligned}\right\}.$$

Now, in general, we have

$$\begin{aligned}\frac{1}{\gamma}&=\frac{di}{ds}+\frac{V\Delta}{E}\frac{dp}{ds}+\frac{V\Delta'}{E}\frac{dq}{ds}\\ &=\frac{di}{ds}+\frac{G^{\frac{1}{2}}\Delta}{E}\sin j+\frac{\Delta'}{E^{\frac{1}{2}}}\sin i.\end{aligned}$$

The simplest case arises when the parametric curves are orthogonal. Then

$$i+j=\omega=\tfrac{1}{2}\pi,\quad V^2=EG,$$

$$\frac{1}{\gamma''}=\frac{\Delta'}{E^{\frac{1}{2}}}=\frac{G_1}{2GE^{\frac{1}{2}}},\quad \frac{1}{\gamma'}=\frac{G^{\frac{1}{2}}\Delta}{E}=-\frac{E_2}{2EG^{\frac{1}{2}}};$$

and so

$$\frac{1}{\gamma}=\frac{di}{ds}+\frac{\cos i}{\gamma'}+\frac{\sin i}{\gamma''},$$

which effectively is Liouville's formula*.

Further simplification would occur if we could choose the parametric curves so that $1/\gamma'$ and $1/\gamma''$ are zero, that is, if there were two orthogonal families of geodesics on the surface. In that case $G_1=0$, $E_2=0$; that is, E

* In his edition of Monge's *Application de l'Analyse à la Géométrie*, p. 575. The signs of γ, γ', γ'' in the text differ from those in Liouville's formula; they agree, when the geodesic contingence is taken to be $\delta i-di$ instead of $di-\delta i$.

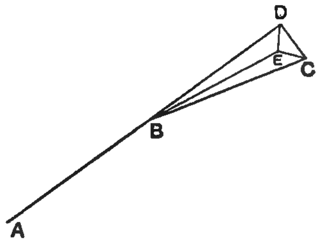
is a function of p only which can be absorbed into dp^2, while G is a function of q only which can be absorbed into dq^2; thus

$$ds^2 = dp^2 + dq^2,$$

and so the surface is a developable surface*. Thus the suggested simplification cannot be effected in general.

104. We can obtain another expression for the geodesic curvature by a different method, due to Liouville also.

Let AB, BC be two consecutive equal elements of arc, which can be taken as infinitesimal straight lines. Produce the line AB to D so that $BD = AB$, and let DE be the normal to the surface at E; join BE, EC.



Then ABE is the geodesic having the arc-element AB in common with the curve; that is, it is the geodesic tangent to the curve. Also $ABDC$ is the osculating plane of the curve; and $ABDE$ is the normal plane to the surface. Hence

angle EBC = angle of geodesic contingence,

...... DBC = angle of circular contingence,

...... DBE = angle of contingence for normal section.

Let ϖ denote the angle between the principal normal of the curve and the normal to the surface, measured from the former towards the latter, so that ϖ can range from $+\pi$ to $-\pi$; in the figure†, the angle $CDE = \varpi$. Also, now let

ρ = radius of circular curvature of the curve, taken positive as in § 4;

γ = ... geodesic ....;

ρ' = ... curvature of normal section of the surface, taken as in § 31.

Then

$$DE = BD \,.\, \text{angle } DBE = \frac{1}{2}\frac{AB^2}{\rho'},$$

$$DC = BD \,.\, \text{angle } DBC = \frac{1}{2}\frac{AB^2}{\rho},$$

$$EC = BE \,.\, \text{angle } EBC = \frac{1}{2}\frac{AB^2}{\gamma}.$$

* See also § 114.

† The figure obviously assumes that the normal section is such as to give ρ' positive and that $0 < \varpi < \frac{1}{2}\pi$. The figures for the possible alternatives are easily constructed; they lead to the same analytical results, regard being had to the sign of ρ' and the range of ϖ.

But

$$DE = DC\cos\varpi,\quad EC = DC\sin\varpi;$$

hence

$$\frac{1}{\rho'} = \frac{\cos\varpi}{\rho},\quad \frac{1}{\gamma} = \frac{\sin\varpi}{\rho}.$$

The first of these two results is Meunier's theorem. The second is a new expression for the geodesic curvature.

The direction-cosines of the binormal of the curve are

$$\rho(y'z'' - z'y''),\quad \rho(z'x'' - x'z''),\quad \rho(x'y'' - y'x''),$$

and the direction-cosines of the principal normal are $\rho x''$, $\rho y''$, $\rho z''$; hence

$$\cos\varpi = X\rho x'' + Y\rho y'' + Z\rho z'',$$

so that

$$\begin{aligned}\frac{1}{\rho'} &= \frac{\cos\varpi}{\rho}\\ &= Xx'' + Yy'' + Zz''\\ &= Lp'^2 + 2Mp'q' + Nq'^2,\end{aligned}$$

the familiar result. Also

$$\sin\varpi = X\rho(y'z'' - z'y'') + Y\rho(z'x'' - x'z'') + Z\rho(x'y'' - y'x''),$$

so that

$$\begin{aligned}\frac{1}{\gamma} &= \frac{\sin\varpi}{\rho}\\ &= \begin{vmatrix} X, & Y, & Z\\ x', & y', & z'\\ x'', & y'', & z''\end{vmatrix}\\ &= V\{p'q'' - q'p'' + \Delta p'^3 + (2\Delta' - \Gamma)p'^2q' + (\Delta'' - 2\Gamma')p'q'^2 - \Gamma''q'^3\},\end{aligned}$$

on substitution and reduction.

The expression on the right-hand side is

$$p'(q'' + \Delta p'^2 + 2\Delta'p'q' + \Delta''q'^2) - q'(p'' + \Gamma p'^2 + 2\Gamma'p'q' + \Gamma''q'^2),$$

and so vanishes when the curve is a geodesic. Also $\varpi = 0$ or π, when the curve is a geodesic. Thus all the forms are verified in connection with the necessary property that $1/\gamma$ is zero for a geodesic.

105. Two expressions for the geodesic curvature have been obtained, viz.

$$\frac{di}{ds} + \frac{V\Delta}{E}\frac{dp}{ds} + \frac{V\Delta'}{E}\frac{dq}{ds},$$

$$V\{p'q'' - q'p'' + \Delta p'^3 + (2\Delta' - \Gamma)p'^2q' + (\Delta'' - 2\Gamma')p'q'^2 - \Gamma''q'^3\},$$

and these should be equal to one another. Now

$$E^{\frac{1}{2}}\cos i = Ep' + Fq', \quad E^{\frac{1}{2}}\sin i = Vq';$$

hence

$$Ep'' + Fq'' = -E_1p'^2 - (E_2 + F_1)\,p'q' - F_2q'^2 + \tfrac{1}{2}E^{-\frac{1}{2}}(E_1p' + E_2q')\cos i - E^{\frac{1}{2}}\sin i\,\frac{di}{ds},$$

$$Vq'' = -V_1p'q' - V_2q'^2 + \tfrac{1}{2}E^{-\frac{1}{2}}(E_1p' + E_2q')\sin i + E^{\frac{1}{2}}\cos i\,\frac{di}{ds}.$$

Multiply the latter by $E^{\frac{1}{2}}\cos i$, the former by $E^{\frac{1}{2}}\sin i$, and subtract; then

$$EV(p'q'' - q'p'') = E\frac{di}{ds} - (V_1p'q' + V_2q'^2)(Ep' + Fq') + \{E_1p'^2 + (E_2 + F_1)\,p'q' + F_2q'^2\}\,Vq'.$$

Now (§ 34)

$$V_1 = V(\Gamma + \Delta'), \qquad V_2 = V(\Gamma' + \Delta''),$$

$$E_1 = 2(E\Gamma + F\Delta), \quad F_1 = E\Gamma' + F(\Gamma + \Delta') + G\Delta,$$

$$E_2 = 2(E\Gamma' + F\Delta'), \quad F_2 = E\Gamma'' + F(\Gamma' + \Delta'') + G\Delta';$$

substituting and reducing, we find

$$EV(p'q'' - q'p'') = E\frac{di}{ds} + V\Delta p'(2Fp'q' + Gq'^2) + V\Delta'q'(Ep'^2 + 2Fp'q' + Gq'^2) - EV(2\Delta' - \Gamma)\,p'^2q' - EV(\Delta'' - 2\Gamma')\,p'q'^2 + EV\Gamma''q'^3,$$

which proves the equality of the two expressions for the geodesic curvature.

Another expression, due to Bonnet, for the geodesic curvature of a curve is required when the equation of the curve is given in the form $\phi(p, q) = 0$.

Let Θ denote the positive square root of $E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2$; then, as

$$\phi_1p' + \phi_2q' = 0,$$

we take

$$\frac{p'}{\phi_2} = \frac{q'}{-\phi_1} = \frac{1}{\Theta},$$

thus assigning the direction along the curve that is positive. Also

$$\phi_1p'' + \phi_2q'' + \phi_{11}p'^2 + 2\phi_{12}p'q' + \phi_{22}q'^2 = 0,$$

and therefore

$$\Theta(p'q'' - q'p'') + \phi_{11}p'^2 + 2\phi_{12}p'q' + \phi_{22}q'^2 = 0.$$

Now

$$\frac{\partial}{\partial p}\left(\frac{F\phi_2 - G\phi_1}{\Theta}\right) + \frac{\partial}{\partial q}\left(\frac{F\phi_1 - E\phi_2}{\Theta}\right) = V^2\{p'q'' - q'p'' + \Delta p'^3 + (2\Delta' - \Gamma)\,p'^2q' + (\Delta'' - 2\Gamma')\,p'q'^2 - \Gamma''q'^3\},$$

on using the preceding relations; and therefore

$$\frac{1}{\gamma} = \frac{1}{V}\left\{\frac{\partial}{\partial p}\left(\frac{F\phi_2 - G\phi_1}{\Theta}\right) + \frac{\partial}{\partial q}\left(\frac{F\phi_1 - E\phi_2}{\Theta}\right)\right\},$$

which is Bonnet's expression for the geodesic curvature* of the curve

$$\phi(p, q) = 0.$$

106. From the two relations

$$\frac{1}{\rho'} = \frac{\cos\varpi}{\rho}, \qquad \frac{1}{\gamma} = \frac{\sin\varpi}{\rho},$$

we have

$$\frac{1}{\rho^2} = \frac{1}{\rho'^2} + \frac{1}{\gamma^2};$$

consequently, inserting the values of ρ' and γ that have been obtained, we have an expression for the circular curvature of any curve on the surface. Thus, for the curve $p = a$, we have

$$\left[\frac{1}{\rho^2}\right]_{p=a} = \frac{N^2}{G^2} + \frac{V^2\Gamma''^2}{G^3};$$

and for the curve $q = b$, we have

$$\left[\frac{1}{\rho^2}\right]_{q=b} = \frac{L^2}{E^2} + \frac{V^2\Delta^2}{E^3}.$$

The curvature of torsion of the curve could be derived from the formula

$$\frac{1}{\rho^2\sigma} = \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix};$$

but the expression is complicated, requiring the use of the derived magnitudes of the third order. Another and more convenient (but equivalent) expression for the torsion, connecting it with the torsion of the geodesic tangent to the curve, can be obtained as follows. Let $d\tau'$ be the angle of torsion for the geodesic tangent; so that, in passing to a consecutive point, its osculating plane (being the normal plane to the surface) turns through an angle $d\tau'$ about the tangent. The inclination of the osculating plane of the curve at the point to the osculating plane of the geodesic is ϖ; the inclination at a consecutive point is $\varpi + d\varpi$, so that the osculating plane of the curve has turned, round the tangent, through an angle $d\varpi$ relative to the osculating plane of the geodesic; and these rotations are in the same sense. Hence the angle through which the osculating plane of the curve has turned in space round the tangent is $d\tau' - d\varpi$; and therefore

$$\frac{1}{\sigma} = \frac{d\tau'}{ds} - \frac{d\varpi}{ds},$$

the expression in question.

* See also §§ 135, 142.

107. The quantity $\frac{d\tau'}{ds}$ is the *torsion of the geodesic*; sometimes (but less often than formerly) it is called the geodesic torsion of the curve. The analogy of this name with the geodesic curvature of a curve (which is the arc-rate of deviation of the curve from its geodesic tangent) is not justified by any intrinsic property of the magnitude; so we shall not use this descriptive name which implies that the magnitude specifically belongs to the curve.

The actual magnitude of the torsion of the geodesic can be expressed analytically in a simple form as follows. At a point on the surface, let the configuration be referred to the indicatrix with the lines of curvature as the directions at the point of the axes of reference; and suppose (as in § 46) that the geodesic makes an angle ψ with the line of curvature associated with the principal radius α. The circular curvature of the geodesic (being the curvature of the normal section through the tangent) is given by

$$\frac{1}{\rho'} = \frac{\cos^2\psi}{\alpha} + \frac{\sin^2\psi}{\beta}.$$

The equation of the surface in the vicinity of P is

$$2z = \frac{x^2}{\alpha} + \frac{y^2}{\beta} + \text{higher powers.}$$

The direction-cosines of the tangent to the geodesic at P are $\cos\psi$, $\sin\psi$, 0; the direction-cosines of its principal normal (being the normal to the surface) are $0, 0, 1$; hence the direction-cosines of the binormal are $\sin\psi$, $-\cos\psi$, 0. The direction-cosines of its principal normal at a neighbouring point, distant ds from P, are

$$-\frac{ds}{\alpha}\cos\psi, \quad -\frac{ds}{\beta}\sin\psi, \quad 1;$$

hence the direction-cosines of the consecutive binormal (which, of course, is perpendicular to the first tangent) are

$$\sin\psi, \quad -\cos\psi, \quad ds\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\sin\psi\cos\psi.$$

The last of these direction-cosines is $\cos(\frac{1}{2}\pi + d\tau')$, when we take the tangent to the curve, the positive direction (§ 31) of the normal to the surface, and the binormal to the geodesic as a set of lines similar to the customary rectangular configuration. Hence

$$\frac{d\tau'}{ds} = \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\cos\psi\sin\psi,$$

which is the expression for the torsion of the geodesic.

Manifestly the torsion vanishes at a point on a geodesic where the geodesic touches a line of curvature; and it vanishes at an umbilicus for every geodesic through the umbilicus. Manifestly also two geodesics at right angles have equal and opposite torsions.

Families of Geodesics and Geodesic Parallels.

108. We have seen that, when geodesic polar coordinates are used upon a surface, the element of arc on the surface can be expressed in the form

$$ds^2 = dp^2 + D^2 dq^2;$$

in this form, the parametric lines $q = b$ are a family of geodesics. But it so happens that, in the deduction of this form, the geodesics are a family of concurrent curves; and it might be desirable to have one set of parametric curves composed of a family of non-concurrent geodesics.

Accordingly, consider generally the possibility of having one set of parametric curves, say $q =$ constant, constituted by geodesics. Then the relations

$$q = b, \quad p' = E^{-\frac{1}{2}},$$

where b is an arbitrary constant, must satisfy the general equations, which are characteristic of geodesics, viz.,

$$2\frac{d}{ds}(Ep' + Fq') = (E_1, F_1, G_1 \between p', q')^2,$$

$$2\frac{d}{ds}(Fp' + Gq') = (E_2, F_2, G_2 \between p', q')^2.$$

The former becomes an identity. The second equation gives

$$\frac{\partial}{\partial p}\left(\frac{F}{E^{\frac{1}{2}}}\right) = \tfrac{1}{2} E_2 E^{-\frac{1}{2}} = \frac{\partial}{\partial q}(E^{\frac{1}{2}});$$

and therefore some function θ of p and q exists such that

$$E^{\frac{1}{2}} = \frac{\partial \theta}{\partial p}, \qquad \frac{F}{E^{\frac{1}{2}}} = \frac{\partial \theta}{\partial q}.$$

Thus, for the element of arc on the surface, we have

$$\begin{aligned} ds^2 &= E\,dp^2 + 2F\,dp\,dq + G\,dq^2 \\ &= \left(\frac{\partial \theta}{\partial p}\right)^2 dp^2 + 2\frac{\partial \theta}{\partial p}\frac{\partial \theta}{\partial q}\,dp\,dq + G\,dq^2 \\ &= d\theta^2 + g\,dq^2, \end{aligned}$$

where

$$g = G - \left(\frac{\partial \theta}{\partial q}\right)^2.$$

It is manifest, from the form of the expression for the arc, that the curves

$$\theta = \text{constant}, \quad q = \text{constant},$$

are perpendicular to one another. The curves $q = b$ are geodesics; the curves $\theta = a$ are the orthogonal trajectories of the geodesics. But, further, the element of arc along any geodesic $q = b$ is given by

$$ds = d\theta;$$

that is, the geodesic distance between two θ-curves, given by

$$\theta = \theta_1, \quad \theta = \theta_0,$$

is $\theta_1 - \theta_0$, and so is the same for all geodesics $q =$ constant (which, of course, cut the θ-curves orthogonally). The curves $\theta = a$ are called a *family of geodesic parallels.* The members of the family are given by the parametric values of a; and the geodesic distance between two members of the family is the difference between the values of their parameters.

The equations are thus the same as when we use geodesic polar coordinates. In other words, the arc-element and everything that depends upon the expression for the arc-element are the same whether the geodesics are concurrent or not concurrent; and the orthogonals of the geodesics are, in both cases, geodesic parallels.

Note. The question as to whether the orthogonal geodesics of any family of geodesic parallels are, or are not, concurrent, can be settled by proceeding to form their envelope, if any. They are concurrent, if the envelope is a point. Thus it is found that, on the surface

$$ds^2 = 4f(p-q)\,dp\,dq,$$

geodesic parallels are given by

$$a(p+q) - \int \{a^2 - f(t)\}^{\frac{1}{2}}\,dt = \text{constant},$$

where a is an arbitrary constant; the orthogonal geodesics are

$$\frac{1}{a}(p+q) - \int \{a^2 - f(t)\}^{-\frac{1}{2}}\,dt = \text{constant};$$

where, in both equations, t denotes $p - q$.

Along the geodesics, we have

$$dp + dq - \left(\frac{a^2}{a^2 - f}\right)^{\frac{1}{2}}(dp - dq) = 0,$$

so that, if $\xi = dq/dp$, we have

$$\frac{a^2 - f}{a^2} = \left(\frac{1-\xi}{1+\xi}\right)^2$$

as the differential equation of the first order, satisfied by geodesics. The envelope (if any) of the curves is obtained by assigning equal roots to ξ; hence it is given by

$$f = a^2,$$

which in general is a curve (real or imaginary) and not a point. Thus the geodesics in the family indicated are not concurrent in general; when they happen to be concurrent, we have geodesic polar coordinates.

The meridians on a surface of revolution are a family of concurrent geodesics when the axis of revolution meets the surface in real points.

109. One remark, partly in connection with the general notion of parallel curves on a surface, may be made here. It is not possible to take any arbitrarily assigned family of curves $\theta(p, q) = a$, where a is the parameter, as

a family of geodesic parallels; and the reason is simple. Measure a small distance δn along the surface normal to any curve of the family $\theta(p, q) = c$; as the tangential direction along the curve is given by $\theta_1 dp + \theta_2 dq = 0$, the direction of the normal distance δn is given by

$$\frac{\delta p}{G\theta_1 - F\theta_2} = \frac{\delta q}{E\theta_2 - F\theta_1} = \frac{\delta n}{V(E\theta_2^2 - 2F\theta_1\theta_2 + G\theta_1^2)^{\frac{1}{2}}}.$$

If the other extremity of this normal distance lies on a curve of the same family, then, as

$$\begin{aligned}\theta(p + \delta p, q + \delta q) &= \theta(p, q) + \theta_1\delta p + \theta_2\delta q \\ &= c + \frac{\delta n}{V}(E\theta_2^2 - 2F\theta_1\theta_2 + G\theta_1^2)^{\frac{1}{2}},\end{aligned}$$

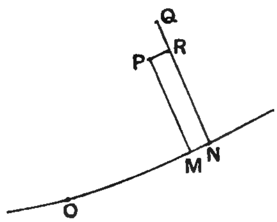
we must have

$$\frac{1}{V}(E\theta_2^2 - 2F\theta_1\theta_2 + G\theta_1^2)^{\frac{1}{2}} = \text{function of } \theta,$$

in order that it may belong to the same family. This condition is not generally satisfied, either by the equation of a family of curves, or by the equation of any member of the family taken in the foregoing form.

The matter however suggests the general idea of curves, parallel to any assigned curve of the family; but the parallel curves, thus derived from any curve, form another distinct family which, as will be seen, are geodesic parallels.

110. Take any curve; and through successive points on the curve draw the geodesics which cut it orthogonally. When we measure a length t along the curve from a fixed point O, say to M, and take a length l along the geodesic normal at M, say to P, we have a uniquely determined point P on the surface. The locus of P, for a constant length l measured along the geodesic normals, is said to be *parallel* to the original curve; and, by taking any number of different lengths l, we obtain any number of curves parallel to the original arbitrarily assumed curve.



All these parallel curves cut orthogonally the geodesic lines drawn as normals to the original curve; and so the parallel curves form a family of geodesic parallels. The property can be established as follows.

Let a consecutive point N be taken; and along the geodesic normal at N, let another length $l + dl$ be measured, so that $MN = dt$, $QN = l + dl$. Taking $RN = l$, we have $QR = dl$. Denote the angle QRP by ω; and let PR, which is not necessarily equal to MN, be denoted by rdt, where the variable quantity r is equal to unity when $l = 0$. Then the arc PQ on the surface is given by

$$ds^2 = dl^2 - 2r\,dt\,dl\cos\omega + r^2dt^2.$$

Now the equation $t=$ constant, under our construction, gives a geodesic; hence as $E=1$, the condition in § 108 becomes

$$\frac{\partial}{\partial l}(r\cos\omega)=0,$$

so that $r\cos\omega$ is a function of t alone. To find this function of t, consider the position at M; we there have $\omega=\frac{1}{2}\pi$, $r=l$, so that the function of t is zero. Thus $\omega=\frac{1}{2}\pi$, and the locus of P is normal to the geodesics MP. Moreover,

$$ds^2=dl^2+r^2dt^2;$$

thus the curves $l=$ constant, being the curves parallel to the original curve, are a family of geodesic parallels.

Ex. Consider a sphero-conic given by

$$x^2+y^2+z^2=r^2, \quad ax^2+by^2+cz^2=0,$$

assuming that no one of the quantities α, β, γ $(=b-c,\ c-a,\ a-b$ respectively) vanishes. On the sphere, draw great circles orthogonal to the sphero-conic; and along the great circles measure any the same distance subtending an angle ψ at the centre of the sphere. A new curve is thus obtained, parallel to the sphero-conic; let x', y', z' denote the point on the parallel curve corresponding to the point x, y, z on the sphero-conic; prove that

$$x'\frac{\alpha}{x}+y'\frac{\beta}{y}+z'\frac{\gamma}{z}=0, \quad x'x+y'y+z'z=r^2\cos\psi,$$

and verify from these equations that the locus of x', y', z' cuts orthogonally the great circles orthogonal to the sphero-conic. Shew also that the locus of x', y', z' is not a sphero-conic; so that the family of curves on the sphere, parallel to the sphero-conic, are not a family of sphero-conics.

It thus appears that any assigned curve can be taken as initiating a family of geodesic parallels. The result is not in contradiction with the result of § 109; for the locus of P is given by an equation of the family of parallels, while the equation of the original curve is not generally, as given initially, some particular case of this equation. Let the equation of the original curve be $\theta(p, q)=0$; and let the equation of the family of parallels be $\phi(p, q)=l$, where l has the same significance as before. The equations $\theta=0$, $\phi=0$, can be simultaneously satisfied, though there is no functional relation between θ and ϕ alone; the condition of § 109 is satisfied for $\phi=l$ or 0, while it usually is not satisfied for $\theta=0$.

Ex. In a plane, $E=1$, $F=0$, $G=1$. The equation of a parabola $\theta=y^2-4x=0$ does not satisfy the condition of § 109, for $\theta_1{}^2+\theta_2{}^2$ is not a function of θ alone.

Let the curves parallel to the parabola be drawn; the curve at a distance c is given by the equations

$$\mu^3+\mu(2-x)-y=0, \quad \mu^2(2-x)-3\mu y+x^2+y^2-c^2=0,$$

μ being a parameter, and also by the equation

$$c^6-u_1c^4+u_2c^2-(y^2-4x)^2\{y^2+(x-1)^2\}=0,$$

where the values of u_1 and u_2, polynomials in x and y, are not immediately important.

The equation of the family of parallels is $c=\phi(x, y)$. It can be verified directly (with less labour from the two equations than from the single equation) that

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=1,$$

for all values of c. Manifestly the original parabola is given by $c=0$; we then have $\phi(x, y)=0$, which is satisfied solely through the real curve $\theta=y^2-4x=0$, though there is no functional relation between ϕ and θ alone.

It will appear later (§ 115) that the necessary and sufficient condition, in order that a family of curves $\theta(p, q)=a$ may be geodesic parallels, is that

$$E\theta_2^2-2F\theta_1\theta_2+G\theta_1^2=EG-F^2$$

111. We have seen that when a family of geodesics (whether concurrent or not) and their orthogonal geodesic parallels are taken as parametric curves, the arc-element is given by

$$ds^2=dp^2+D^2dq^2;$$

and so (§ 68) the Gauss measure of curvature is

$$\frac{1}{\alpha\beta}=-\frac{1}{D}\frac{\partial^2 D}{\partial p^2}.$$

Consider any closed area on the surface; when an infinitesimal element dS of this area is taken, its total curvature is

$$\frac{dS}{\alpha\beta};$$

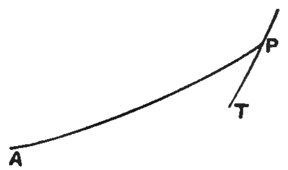
and so the total curvature of the closed area on the surface is

$$\iint\frac{dS}{\alpha\beta},$$

the double integral extending over the whole of that area. We proceed to obtain the expression, due to Gauss, for the total curvature of a geodesic triangle on the surface.

112. A preliminary property of geodesics must first be established. When the element of arc on the surface has the form

$$ds^2=dp^2+D^2dq^2,$$



we have (§ 68)

$$\Gamma=0,\quad \Gamma'=0,\qquad \Gamma''=-DD_1,$$
$$\Delta=0,\quad \Delta'=D_1/D,\quad \Delta''=D_2/D;$$

and so (§ 92) the general equations of geodesics are

$$\frac{d^2p}{ds^2}-DD_1\left(\frac{dq}{ds}\right)^2=0,\qquad \frac{d}{ds}\left(D^2\frac{dq}{ds}\right)-DD_2\left(\frac{dq}{ds}\right)^2=0.$$

Let A be an angular point of our geodesic triangle, and TP the opposite geodesic side; and let AP be a geodesic ($q=$ constant) from A to a current point P on the opposite side, so that $AP=p$. Then if ψ be the angle APT, we have

$$\cos\psi=\frac{dp}{ds},\qquad \sin\psi=D\frac{dq}{ds};$$

and therefore the first of the two equations becomes

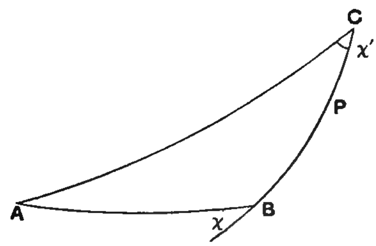
$$\frac{d}{ds}(\cos\psi) - D_1\frac{dq}{ds}\sin\psi = 0,$$

that is,

$$d\psi = -D_1 dq = -\frac{\partial D}{\partial p}dq,$$

which is the property in question. The second equation (as may easily be verified) leads to the same result.

113. Now consider a geodesic triangle ABC; we shall use geodesic polars. An element of area is $dp\,.\,Ddq$; and so the total curvature of the geodesic triangle is



$$= \iint \frac{Ddpdq}{\alpha\beta}$$

$$= -\iint \frac{\partial^2 D}{\partial p^2}dpdq.$$

Integrating with respect to p, and remembering (§ 68) that, at A, $\dfrac{\partial D}{\partial p}$ is equal to unity, we have

$$\int\left(1 - \frac{\partial D}{\partial p}\right)dq$$

as the integral; that is, the integral is equal to

$$\int(dq + d\psi).$$

Now $\int dq$, for the triangle, is equal to the angle A; and $\int d\psi$, for the triangle, is $\chi' - \chi$, that is, $C - (\pi - B)$. Hence *the total curvature of the geodesic triangle is*

$$A + B + C - \pi,$$

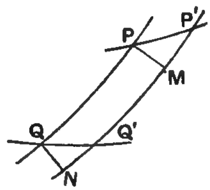
a result first established by Gauss.

When the surface is a sphere, the result is Girard's theorem on the area of a spherical triangle. When the surface is everywhere synclastic, the specific curvature is positive; when the surface is developable, the specific curvature is zero; and when the surface is everywhere anticlastic, the specific curvature is negative. Thus the quantity $A + B + C - \pi$ is positive, zero, or negative, according as the surface is synclastic, developable, or anticlastic. Two geodesics, diverging from a point on an anticlastic surface, cannot again intersect; the range (§ 89) of a geodesic on such a surface is unlimited.

If the surface is such that we can take a closed geodesic returning upon itself, and if we stop at the point of return, we have a special case. Then $\int dq = 2\pi$; and $\chi' = \chi$, because ψ returns to its initial value; hence the total curvature is 2π, that is, one-half the surface of the indicatrix sphere.

114. It is reasonable to expect that, through the notion of geodesic distance, it will be possible to construct for a surface relations which have some analogies with relations in a plane.

Thus, suppose a family of geodesics given, and let them be cut by any two curves. Let PQ and $P'Q'$ be two neighbouring geodesics of the family, cut by the curves PP' and QQ'; and let the geodesic parallels through P and Q be PM and QN. Then if p and $p+dp$ be the geodesic-distance coordinates for P and P', while p_0 and p_0+dp_0 are those for Q and Q', we have



$$PQ = p - p_0 = MN,$$
$$\begin{aligned} ds &= P'Q' - PQ \\ &= \{p + dp - (p_0 + dp_0)\} - (p - p_0) \\ &= dp - dp_0. \end{aligned}$$

But if $PP' = dt$, $QQ' = dt_0$, we have

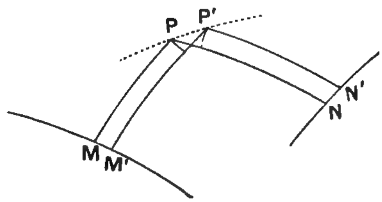
$$dp = P'M = dt \cos PP'M, \quad dp_0 = Q'N = dt_0 \cos QQ'N;$$

and therefore

$$ds = dt \cos PP'M - dt_0 \cos QQ'N,$$

being the expression for the increment of the geodesics cut by the curves.

Again, let P be any point on the surface; and from P drop two geodesic perpendiculars PM and PN upon a couple of selected curves, one belonging to one given family of geodesic parallels and the other belonging to another given family of geodesic parallels. When P describes a locus on the surface such that



$$PM \pm PN$$

is constant, then, if P' be a consecutive point on the locus, we have

$$dPM \pm dPN = 0,$$

so that

$$PP' \cos PP'M' \pm PP' \cos P'PN = 0,$$

and therefore

$$\cos PP'M' \pm \cos P'PN = 0.$$

Hence the tangent in the surface to the locus of P bisects the angle MPN, either externally or internally. Such loci are known as geodesic ellipses (for external bisection) and geodesic hyperbolas (for internal bisection).

To give analytical expression to this descriptive property, we choose initially, as coordinates of P, the geodesic distances u and v from the two curves; then

$$ds^2 = edu^2 + 2fdudv + gdv^2.$$

Let $u' =$ constant represent the geodesics which are orthogonal to the geodesic parallels $u =$ constant; then we have

$$ds^2 = du^2 + D^2 du'^2,$$

so that $ds^2 - du^2$ is a perfect square, when regarded as a function of the differential elements, that is,

$$(e-1)\, du^2 + 2f du dv + g dv^2$$

is a perfect square, when regarded as a function of du and dv. Hence

$$f^2 = g(e-1).$$

Similarly, because v is a geodesic distance,

$$f^2 = e(g-1).$$

Consequently

$$e = g, \quad f^2 = e(e-1);$$

and thus, taking $e = \operatorname{cosec}^2 \omega$, so that $f = \cos\omega \operatorname{cosec}^2 \omega$, the arc-element becomes

$$ds^2 = (du^2 + dv^2 + 2du dv \cos\omega) \operatorname{cosec}^2 \omega.$$

To indicate more explicitly the analogy between ellipses in a plane and geodesic ellipses, we take

$$u + v = 2U, \quad u - v = 2V;$$

then the arc-element takes the form

$$ds^2 = \frac{dU^2}{\sin^2 \frac{1}{2}\omega} + \frac{dV^2}{\cos^2 \frac{1}{2}\omega}.$$

The quantity ω depends upon the particular geodesic parallels chosen as the base of the geodesic perpendiculars, as well as upon the surface itself.

We have a special result (originally due to Liouville) to the following effect: *if a surface admits two families of geodesics which cut at a constant angle, the surface is developable.* For if ω is constant, e, f, g are constant; their derivatives are zero, and so $LN - M^2$ vanishes; that is, the specific curvature vanishes, which establishes the result.

The Equation $\Delta\phi = 1$.

115. We have seen that, in one method of determining the integral equation of geodesics upon a surface, it is necessary to integrate the general equation, which is an ordinary non-linear ordinary differential equation of the second order between p and q. But the fact, that the arc-element on the surface can be expressed differentially in a form which arises most simply when geodesic polar coordinates are used, can be employed to determine systems of geodesic parallels and the associated systems of orthogonal geodesics.

Let a family of geodesic parallels on a surface be represented by the equation

$$\phi(p, q) = \text{constant};$$

and let the equation

$$\psi(p, q) = \text{constant}$$

represent the family of orthogonal geodesics. Then, after preceding explanations, we know that the arc-element on the surface can be expressed in the form

$$d\phi^2 + D^2 d\psi^2,$$

where D is free from differential elements; hence

$$E dp^2 + 2F dp dq + G dq^2 = d\phi^2 + D^2 d\psi^2,$$

and therefore

$$E = \phi_1^2 \quad + D^2\psi_1^2,$$

$$F = \phi_1\phi_2 + D^2\psi_1\psi_2,$$

$$G = \phi_2^2 \quad + D^2\psi_2^2.$$

Consequently

$$(E - \phi_1^2)(G - \phi_2^2) - (F - \phi_1\phi_2)^2 = 0;$$

or, if we write

$$\Delta\phi = \frac{G\phi_1^2 - 2F\phi_1\phi_2 + E\phi_2^2}{V^2},$$

we have

$$\Delta\phi = 1,$$

as a necessary condition.

It is also a sufficient condition. For, when the relation

$$(E - \phi_1^2)(G - \phi_2^2) - (F - \phi_1\phi_2)^2 = 0$$

is satisfied, we have

$$ds^2 - d\phi^2 = (E - \phi_1^2) dp^2 + 2(F - \phi_1\phi_2) dp dq + (G - \phi_2^2) dq^2.$$

The right-hand side, regarded as a function of differential elements, is a perfect square because of the relation; and therefore

$$ds^2 - d\phi^2 = (A dp + B dq)^2 = D^2 d\psi^2.$$

The condition therefore is sufficient as well as necessary; and so we have the result:—

The general solution of the equation $\Delta\phi = 1$ determines a family of geodesic parallels cut orthogonally by a family of geodesics.

The function $\Delta\phi$ is called* the *first differential parameter* of the function ϕ.

Now this equation $\Delta\phi = 1$ is a partial differential equation of the first order in two independent variables. To integrate it, we can always use Charpit's method, though in special cases we may use simpler methods all

* After Beltrami who introduced it in his Bologna memoir of 1869, hereafter to be quoted.

of which can be exhibited as special standard forms of the general method*. The procedure in the general method is as follows. The subsidiary set of equations

$$\frac{dp}{\frac{2}{V^2}(G\phi_1 - F\phi_2)} = \frac{dq}{\frac{2}{V^2}(E\phi_2 - F\phi_1)} = \frac{d\phi_1}{-\frac{\partial}{\partial p}(\Delta\phi)} = \frac{d\phi_2}{-\frac{\partial}{\partial q}(\Delta\phi)}$$

is constructed. One integral of the set is required which, while distinct from $\Delta\phi = 1$, must contain ϕ_1 or ϕ_2 or both; let it be

$$f(p, q, \phi_1, \phi_2) = a.$$

This relation is combined with $\Delta\phi = 1$ to express ϕ_1 and ϕ_2 in terms of p and q; when their values are substituted in

$$d\phi = \phi_1 dp + \phi_2 dq,$$

the right-hand side becomes an exact differential; and the integral of this equation is obtained, by quadratures merely, in the form

$$\phi = \phi(p, q, a) = b.$$

Here a and b are arbitrary constants. Our desired geodesic parallels are given by

$$\phi(p, q, a) = b.$$

It will be noticed that, for their complete expression, we need one integration of a set of ordinary equations and one quadrature.

116. When a family of geodesic parallels, satisfying the equation $\Delta\phi = 1$, has been obtained, the family of orthogonal geodesics can be constructed in two ways.

It might happen that a somewhat special family of geodesic parallels is obtainable in a form

$$\phi(p, q) = b,$$

where ϕ contains no arbitrary constant a. Then

$$D\psi_1 = (E - \phi_1^2)^{\frac{1}{2}}, \quad D\psi_2 = (G - \phi_2^2)^{\frac{1}{2}};$$

and so the corresponding family of orthogonal geodesics, ψ = constant, is given by

$$(E - \phi_1^2)^{\frac{1}{2}} dp + (G - \phi_2^2)^{\frac{1}{2}} dq = 0.$$

The integral equation of the special family of geodesics is then obtained, not by a mere quadrature but by the integration of this equation.

When, however, it happens that the general families of geodesic parallels are obtained in the form

$$\phi(p, q, a) = b,$$

where ϕ now contains an arbitrary constant a, the orthogonal geodesics can be constructed by a direct process. We have

$$ds^2 = d\phi^2 + D^2 d\psi^2.$$

* See the author's *Treatise on Differential Equations*, 3rd ed., § 201.

On the right-hand side occur the quantities p, q, dp, dq, which are current variables, and also a, which is a parametric variable; while ds^2 does not itself explicitly involve a. Hence

$$0 = d\phi d \frac{\partial \phi}{\partial a} + \left(\frac{\partial D}{\partial a} d\psi + D d \frac{\partial \psi}{\partial a}\right) D d\psi,$$

all over the surface. Along each geodesic, we have

$$d\psi = 0;$$

and therefore, along each geodesic, we have

$$d\phi d \frac{\partial \phi}{\partial a} = 0,$$

so that

$$d\phi = 0, \text{ or } d \frac{\partial \phi}{\partial a} = 0.$$

Now we cannot have $d\psi = 0$ and $d\phi = 0$ together, for ψ and ϕ are functionally independent of one another. We therefore have

$$d\psi = 0, \quad d \frac{\partial \phi}{\partial a} = 0,$$

simultaneously; so that ψ is a function of $\frac{\partial \phi}{\partial a}$ alone. Merging the derivative function in the multiplier D, we can take

$$\psi = \frac{\partial \phi}{\partial a};$$

in other words, *the geodesics, which are orthogonal to the general families of geodesic parallels* $\phi(p, q, a) = b$, *are given by*

$$\frac{\partial \phi}{\partial a} = c,$$

where c *is an arbitrary constant.* Moreover, as a is not a purely additive constant in ϕ, this equation of the orthogonal geodesics contains two arbitrary constants a and c.

Further, the inference has already been drawn (§ 92) from the theory of ordinary equations of the second order that a *geodesic through an ordinary point on the surface is uniquely determined by its direction at the point.* The inference can be established as follows, without recourse to that theory. For the purpose, it will be sufficient to shew that the geodesic parallel $\phi(p, q, a) = b$ can be made, at the point, to adopt an assigned direction—of course, perpendicular to the assigned direction of the geodesic. The direction of the geodesic parallel is settled by the ratio ϕ_1/ϕ_2; if this ratio were independent of a, so that

$$\phi_1 = \phi_2 \lambda,$$

where λ is a function of p and q alone and is not a function of a, the two equations

$$\Delta\phi = 1, \quad \phi_1 = \phi_2 \lambda,$$

would determine ϕ_1 and ϕ_2 as quantities independent of a; and then $\phi(p, q, a)$ would not involve a, contrary to hypothesis. Thus ϕ_1/ϕ_2 involves the arbitrary constant a; this ratio can be made to assume any value, by taking all possible values for a; and so the geodesic parallel (and consequently the geodesic) through the point can be made to lie in any assigned direction at the point.

As regards the multiplier D, we have

$$0 = d\phi d\frac{\partial\phi}{\partial a} + \left(\frac{\partial D}{\partial a}d\psi + Dd\frac{\partial\psi}{\partial a}\right)Dd\psi,$$

everywhere on the surface. But

$$d\psi = d\frac{\partial\phi}{\partial a};$$

substituting, and dividing out by $d\psi$, we have

$$0 = d\phi + D\frac{\partial D}{\partial a}d\frac{\partial\phi}{\partial a} + D^2 d\frac{\partial^2\phi}{\partial a^2}$$

on the surface. Hence

$$0 = \phi_1 + D\frac{\partial D}{\partial a}\frac{\partial\phi_1}{\partial a} + D^2\frac{\partial^2\phi_1}{\partial a^2},$$

$$0 = \phi_2 + D\frac{\partial D}{\partial a}\frac{\partial\phi_2}{\partial a} + D^2\frac{\partial^2\phi_2}{\partial a^2};$$

and therefore, writing

$$\phi' = \frac{\partial\phi}{\partial a}, \quad \phi'' = \frac{\partial^2\phi}{\partial a^2}, \ldots,$$

and so on, we have

$$D^2 = \frac{J(\phi, \phi')}{J(\phi', \phi'')}, \quad D\frac{\partial D}{\partial a} = \frac{J(\phi'', \phi)}{J(\phi', \phi'')},$$

where $J(u, v)$ is the Jacobian of u and v with respect to p and q. Consequently

$$\frac{\partial}{\partial a}\left\{\frac{J(\phi, \phi')}{J(\phi', \phi'')}\right\} = 2\frac{J(\phi'', \phi)}{J(\phi', \phi'')},$$

and therefore

$$3J(\phi, \phi'')J(\phi', \phi'') = J(\phi, \phi')J(\phi', \phi''').$$

A first integral, proper to the surface, of this equation of the third order in a, is

$$\frac{J^3(\phi, \phi')}{J(\phi', \phi'')} = V^2,$$

which can be established independently.

It is easy to verify that the curves

$$\phi = \text{constant}, \quad \frac{\partial\phi}{\partial a} = \text{constant},$$

are orthogonal. The direction dp/dq of the former is such that

$$\phi_1 dp + \phi_2 dq = 0.$$

The direction $\delta p/\delta q$ of the latter is such that

$$\frac{\partial \phi_1}{\partial a}\delta p + \frac{\partial \phi_2}{\partial a}\delta q = 0.$$

These directions on the surface are perpendicular if

$$E\phi_2 \frac{\partial \phi_2}{\partial a} - F\left(\phi_1 \frac{\partial \phi_2}{\partial a} + \phi_2 \frac{\partial \phi_1}{\partial a}\right) + G\phi_1 \frac{\partial \phi_1}{\partial a} = 0.$$

But we have

$$E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2 = V^2,$$

and E, F, G, V do not contain a; hence the condition, necessary for orthogonality, is satisfied.

117. One other result may be noted. Suppose that the general equation of geodesics is given in the form

$$\psi(p,\ q,\ a) = c,$$

where a does not occur in a merely additive form in ψ; it is desirable to have the geodesic parallels. Now along any geodesic, we have

$$\psi_1\delta p + \psi_2\delta q = 0;$$

consequently the orthogonal direction dp/dq, being that of the geodesic parallel, is given by

$$(E\psi_2 - F\psi_1)\,dp + (F\psi_2 - G\psi_1)\,dq = 0.$$

Thus a quantity μ, independent of differential elements, must exist such that

$$\mu\,\{(E\psi_2 - F\psi_1)\,dp + (F\psi_2 - G\psi_1)\,dq\} = d\phi;$$

and ϕ is such that $\Delta\phi = 1$. Hence

$$\mu\,(E\psi_2 - F\psi_1) = \phi_1, \quad \mu\,(F\psi_2 - G\psi_1) = \phi_2,$$

$$\mu^2\,(E\psi_2^2 - 2F\psi_1\psi_2 + G\psi_1^2) = 1;$$

therefore

$$\phi_1 = (E\psi_2 - F\psi_1)\,(E\psi_2^2 - 2F\psi_1\psi_2 + G\psi_1^2)^{-\frac{1}{2}},$$

$$\phi_2 = (F\psi_2 - G\psi_1)\,(E\psi_2^2 - 2F\psi_1\psi_2 + G\psi_1^2)^{-\frac{1}{2}},$$

so that, with these values inserted, *the equation*

$$\phi_1 dp + \phi_2 dq = 0$$

represents the geodesic parallels, when $\psi = c$ represents the orthogonal geodesics.

A quadrature alone is necessary in order to have the integral equation $\phi(p,\ q,\ a) = b$.

A simple equivalent form can be given to the expressions for ϕ, ϕ_1 and ϕ_2. Along the geodesic we have

$$\psi_1\delta p + \psi_2\delta q = 0,$$

and therefore

$$\frac{\delta p}{\psi_2} = \frac{\delta q}{-\psi_1} = \frac{\delta s}{(E\psi_2^2 - 2F\psi_1\psi_2 + G\psi_1^2)^{\frac{1}{2}}}.$$

Hence

$$\phi_1 = E\frac{\delta p}{\delta s} + F\frac{\delta q}{\delta s},$$

$$\phi_2 = F\frac{\delta p}{\delta s} + G\frac{\delta q}{\delta s};$$

and therefore

$$\phi = \int \frac{(E + Fq')\,dp + (F + Gq')\,dq}{(E + 2Fq' + Gq'^2)^{\frac{1}{2}}},$$

where q', $= -\psi_1/\psi_2$, gives the direction of the geodesic. Now q' involves a and therefore the integral involves a; thus

$$\frac{\partial\phi}{\partial a} = \int \frac{V^2\dfrac{\partial q'}{\partial a}}{(E + 2Fq' + Gq'^2)^{\frac{3}{2}}}(dq - q'dp).$$

If then we regard

$$q' = -\frac{\psi_1}{\psi_2} = \theta(p,\ q,\ a)$$

as a first integral of the general equation of geodesics, a second integral (and therefore the primitive) is given by

$$c = \frac{\partial\phi}{\partial a} = \int \frac{V^2}{(E + 2F\theta + G\theta^2)^{\frac{3}{2}}}\frac{\partial\theta}{\partial a}(dq - \theta\,dp).$$

This is a known theorem of Jacobi's connected with the theory of the last multiplier *.

Ex. 1. Consider surfaces such that E, F, G are functions of only one of the parametric variables, say p. Let

$$\Theta = E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2 - (EG - F^2) = 0.$$

The subsidiary system for the integration of $\Theta = 0$ is

$$\frac{dp}{2(G\phi_1 - F\phi_2)} = \frac{dq}{2(E\phi_2 - F\phi_1)} = \frac{d\phi_1}{-\dfrac{\partial\Theta}{\partial p}} = \frac{d\phi_2}{-\dfrac{\partial\Theta}{\partial q}}.$$

In the present case, $\dfrac{\partial\Theta}{\partial q} = 0$; hence we must have $d\phi_2 = 0$ in the subsidiary system, that is, an integral is $\phi_2 = a$. (The integral can be obtained by expressing the equation $\Theta = 0$, for the present case, in one of the standard forms indicated in § 115.) This integral is to be combined with $\Theta = 0$; so we find

$$G\phi_1 = aF + V(G - a^2)^{\frac{1}{2}}.$$

Consequently,

$$\phi = \int \phi_1 dp + \phi_2 dq$$

$$= aq + \int \frac{1}{G}\{aF + V(G - a^2)^{\frac{1}{2}}\}\,dp;$$

and therefore the geodesic parallels are given by

$$aq + \int \frac{1}{G}\{aF + V(G - a^2)^{\frac{1}{2}}\}\,dp = b,$$

where a and b are arbitrary constants.

* See the author's *Treatise on Differential Equations*, 3rd ed., § 174, Ex. 3.

The orthogonal geodesics are

$$\frac{\partial\phi}{\partial a}=c,$$

that is,

$$q+\int\left\{\frac{F}{G}-\frac{aV}{G(G-a^2)^{\frac{1}{2}}}\right\}dp=c,$$

where a and c are arbitrary constants; and this equation accordingly is the general equation of geodesics upon the particular surface.

Let

$$VG^{-\frac{1}{2}}\,dp=du,\qquad dq+\frac{F}{G}\,dp=dv;$$

then the element of arc on the surface becomes

$$ds^2=du^2+G\,dv^2,$$

where G, a function of p alone, is a function therefore of u and not of v. Thus the surface is deformable into a surface of revolution.

The equation of the geodesics becomes

$$v-\int\frac{a\,du}{\{G(G-a^2)\}^{\frac{1}{2}}}=c.$$

Let ω be the angle at which the geodesic cuts the meridian; then

$$ds\sin\omega=G^{\frac{1}{2}}\,dv,\quad ds\cos\omega=du,$$

so that

$$\tan\omega=G^{\frac{1}{2}}\frac{dv}{du}=\frac{a}{(G-a^2)^{\frac{1}{2}}},$$

or

$$G^{\frac{1}{2}}\sin\omega=a,$$

which (§ 93) was the former first integral of geodesics on a surface of revolution.

Ex. 2. Consider the geodesics on the Liouville surfaces given by

$$ds^2=(P-Q)(R^2\,dp^2+S^2\,dq^2),$$

where P and R are functions of p only, while Q and S are functions of q only. (It is clear that the parametric curves yield an isometric system.)

The differential equation $\Delta\phi=1$ is now

$$\frac{1}{P-Q}\left(\frac{1}{R^2}\phi_1^{\,2}+\frac{1}{S^2}\phi_2^{\,2}\right)=1,$$

so that

$$P-\frac{1}{R^2}\phi_1^{\,2}=Q+\frac{1}{S^2}\phi_2^{\,2},$$

a standard form of equation. An integral is known to be

$$P-\frac{1}{R^2}\phi_1^{\,2}=a,\qquad Q+\frac{1}{S^2}\phi_2^{\,2}=a;$$

this result being derivable also from the subsidiary system in § 115. Hence

$$\phi_1=R(P-a)^{\frac{1}{2}},\quad \phi_2=S(a-Q)^{\frac{1}{2}};$$

therefore

$$\phi=\int R(P-a)^{\frac{1}{2}}\,dp+\int S(a-Q)^{\frac{1}{2}}\,dq.$$

The general equation of geodesics, being $\partial\phi/\partial a=$constant, now becomes

$$\psi=\int R\,(P-a)^{-\frac{1}{2}}\,dp-\int S\,(a-Q)^{-\frac{1}{2}}\,dq=c,$$

where a and c are arbitrary constants. This is the primitive of the general equation of the geodesics ; a first integral is

$$\frac{R}{(P-a)^{\frac{1}{2}}}\,dp-\frac{S}{(a-Q)^{\frac{1}{2}}}\,dq=0.$$

Let ω be the angle at which the geodesic cuts the parametric curve, $q=$constant ; then

$$ds\cos\omega=R\,(P-Q)^{\frac{1}{2}}\,dp,\quad ds\sin\omega=S\,(P-Q)^{\frac{1}{2}}\,dq,$$

and therefore, along the geodesic, we have

$$\frac{\cos\omega}{(P-a)^{\frac{1}{2}}}=\frac{\sin\omega}{(a-Q)^{\frac{1}{2}}},$$

that is,

$$P\sin^2\omega+Q\cos^2\omega=a,$$

which may be regarded as a first integral of the general differential equation of the geodesics.

Further, we have

$$d\psi=\frac{R}{(P-a)^{\frac{1}{2}}}\,dp-\frac{S}{(a-Q)^{\frac{1}{2}}}\,dq,$$

$$d\phi=R\,(P-a)^{\frac{1}{2}}\,dp+S\,(a-Q)^{\frac{1}{2}}\,dq,$$

so that

$$\begin{aligned}d\phi^2+(P-a)\,(a-Q)\,d\psi^2&=(P-Q)\,(R^2dp^2+S^2dq^2)\\&=ds^2,\end{aligned}$$

which puts in evidence the fact that the curves, $\psi=$constant, are a family of geodesics while the curves, $\phi=$constant, are their orthogonal geodesic parallels.

It is manifest that the geodesic curve touches a parametric curve given by $P=a$, if this equation has real roots, and a parametric curve given by $Q=a$, if this equation has real roots.

Note. The surfaces include, as special cases, planes, spheres, central quadrics. The applications to these surfaces are developed by Darboux*.

Ex. 3. Obtain, by Charpit's method, an integral equation of geodesic parallels on the surface

$$ds^2=\frac{1}{q}\,dp^2+\frac{1}{p}\,dq^2$$

in the form

$$q^{-\frac{3}{2}}\sin^3\omega-p^{-\frac{3}{2}}\cos^3\omega=a,$$

and an associated integral equation of geodesics in the form

$$q^{-\frac{3}{2}}\cos^3\omega+p^{-\frac{3}{2}}\sin^3\omega=c,$$

where a and c are arbitrary constants, and ω is the angle at which a geodesic cuts the parametric curve, $q=$constant.

* In his treatise, vol. iii, pp. 12—16.

118. Much simplification is introduced into the analysis connected with this branch of the theory of geodesics by referring the surface to its nul lines as parametric curves. The arc-element then (§ 56) has the form

$$ds^2 = 4\lambda\, dp dq.$$

The ordinary equations for geodesics become

$$\left.\begin{aligned} \lambda p'' + \lambda_1 p'^2 = 0, \quad \lambda q'' + \lambda_2 q'^2 = 0 \\ \frac{d^2 q}{dp^2} = -\frac{\lambda_2}{\lambda}\left(\frac{dq}{dp}\right)^2 + \frac{\lambda_1}{\lambda}\frac{dq}{dp} \end{aligned}\right\}.$$

The partial differential equation for geodesic parallels is

$$\phi_1 \phi_2 = \tfrac{1}{2} F = \lambda.$$

Consider, however, the ordinary equation for a geodesic. Along the curve, let t denote dq/dp; and suppose that a first integral has been obtained in a form

$$t = g\,(p, q, a),$$

where a is an arbitrary constant. Now

$$\frac{dt}{dp} = \frac{\partial g}{\partial p} + t\frac{\partial g}{\partial q} = \frac{\partial t}{\partial p} + t\frac{\partial t}{\partial q};$$

hence, as the ordinary equation of the second order has to be satisfied in connection with the supposed first integral $t = g$, we have

$$\frac{\partial t}{\partial p} + t\frac{\partial t}{\partial q} = -\frac{\lambda_2}{\lambda}t^2 + \frac{\lambda_1}{\lambda}t,$$

and therefore

$$\frac{\partial}{\partial q}\{(\lambda t)^{\frac{1}{2}}\} = \frac{\partial}{\partial p}\left\{\left(\frac{\lambda}{t}\right)^{\frac{1}{2}}\right\},$$

satisfied along the geodesic. Now, along any arc on the surface, we have

$$ds^2 = 4\lambda\, dp dq.$$

Along a geodesic, the element of arc is given by $d\phi$, so that (as $t\,dp = dq$)

$$\begin{aligned} d\phi &= 2\,(\lambda t)^{\frac{1}{2}}\, dp \\ &= (\lambda t)^{\frac{1}{2}}\, dp + \left(\frac{\lambda}{t}\right)^{\frac{1}{2}} dq. \end{aligned}$$

The last expression is a perfect differential because of the relation

$$\frac{\partial}{\partial q}\{(\lambda t)^{\frac{1}{2}}\} = \frac{\partial}{\partial p}\left\{\left(\frac{\lambda}{t}\right)^{\frac{1}{2}}\right\},$$

which has just been established; hence the element of arc is given by $d\phi$, so that (as $t\,dp = dq$)

$$\phi = \int\left\{(\lambda t)^{\frac{1}{2}}\, dp + \left(\frac{\lambda}{t}\right)^{\frac{1}{2}} dq\right\}.$$

Thus the value of ϕ can be obtained by quadratures; and we manifestly have

$$\phi_1 = (\lambda t)^{\frac{1}{2}}, \quad \phi_2 = \left(\frac{\lambda}{t}\right)^{\frac{1}{2}},$$

in accord with the partial differential equation of the geodesic parallels

$$\phi_1 \phi_2 = \lambda.$$

Hence we have the theorem*:—

When a first integral of the characteristic ordinary equation of a geodesic is known, the geodesic parallels can be obtained merely by quadratures.

Further, the theorem of § 116 can be deduced at once. Let

$$\psi(p, q, a) = c'$$

be the general equation of geodesics; then along any member of any of the families, we have

$$\psi_1 + t\psi_2 = \frac{\partial \psi}{\partial p} + t\frac{\partial \psi}{\partial q} = 0.$$

But

$$\frac{\partial \phi_1}{\partial a} = \frac{1}{2}\left(\frac{\lambda}{t}\right)^{\frac{1}{2}} \frac{\partial t}{\partial a}, \quad \frac{\partial \phi_2}{\partial a} = -\frac{1}{2}\left(\frac{\lambda}{t^3}\right)^{\frac{1}{2}} \frac{\partial t}{\partial a},$$

so that

$$\frac{\partial \phi_1}{\partial a} + t\frac{\partial \phi_2}{\partial a} = 0.$$

Consequently

$$\frac{\partial \phi_1}{\partial a}\psi_2 - \frac{\partial \phi_2}{\partial a}\psi_1 = 0,$$

so that

$$J\left(\frac{\partial \phi}{\partial a}, \psi\right) = 0.$$

As this Jacobian does not vanish in virtue of $\psi = c'$ because c' does not occur, it must be satisfied identically; there is therefore a functional relation between ψ and $\frac{\partial \phi}{\partial a}$, say

$$\frac{\partial \phi}{\partial a} = F(\psi).$$

Hence the geodesics are given by

$$\frac{\partial \phi}{\partial a} = F(c') = c,$$

where c is an arbitrary constant; and so we have again the known theorem for the derivation of the general integral equation of geodesics from the general integral equation of geodesic parallels.

* It harmonises with the theorem of Jacobi's on the last multiplier (§ 117) and was enunciated by Beltrami in this form, *Opere Mat.*, t. i, pp. 366—373.

Ex. 1. Consider the surface for which

$$\lambda = f(p-q).$$

The general ordinary equation for the geodesics becomes

$$\frac{dt}{dp} = (t^2+t)\frac{f'}{f},$$

while

$$\frac{d}{dp}(p-q) = 1-t;$$

hence

$$\frac{d}{dp}\{\log f(p-q)\} = (1-t)\frac{f'}{f} = \frac{1-t}{t^2+t}\frac{dt}{dp}$$

and therefore

$$\frac{(1+t)^2}{4t} = \frac{a}{f},$$

where a is an arbitrary constant. Hence

$$t = \frac{a^{\frac{1}{2}} - (a-f)^{\frac{1}{2}}}{a^{\frac{1}{2}} + (a-f)^{\frac{1}{2}}};$$

consequently

$$ft = \{a^{\frac{1}{2}} - (a-f)^{\frac{1}{2}}\}^2, \qquad \frac{f}{t} = \{a^{\frac{1}{2}} + (a-f)^{\frac{1}{2}}\}^2,$$

so that the arc along the geodesic is given by

$$\begin{aligned}\phi &= \int (ft)^{\frac{1}{2}}\,dp + \left(\frac{f}{t}\right)^{\frac{1}{2}} dq \\ &= \int \{a^{\frac{1}{2}}(dp+dq) - (a-f)^{\frac{1}{2}}(dp-dq)\} \\ &= a^{\frac{1}{2}}(p+q) - \int (a-\Theta)^{\frac{1}{2}}\,d\theta,\end{aligned}$$

where $\theta = p-q$, $\Theta = f(\theta)$. Thus the geodesic parallels are

$$a^{\frac{1}{2}}(p+q) - \int (a-\Theta)^{\frac{1}{2}}\,d\theta = b;$$

and the geodesics themselves are given by

$$p+q - \int \left(\frac{a}{a-\Theta}\right)^{\frac{1}{2}} d\theta = c.$$

Ex. 2. Obtain the integral equation of the geodesics on the surface

$$ds^2 = 4f(pq)\,dp\,dq$$

in the form

$$\log\frac{p}{q} = \int \left(\frac{a}{a+\theta\Theta}\right)^{\frac{1}{2}} \frac{d\theta}{\theta} + c,$$

where $\theta = pq$, $\Theta = f(\theta)$; and deduce the equation of the geodesic parallels.

Ex. 3. Shew that the geodesics on the surface

$$ds^2 = 4\{f(p-q) - g(p+q)\}\,dp\,dq$$

are given by the equation

$$\int \frac{d(p-q)}{\{f(p-q)-a\}^{\frac{1}{2}}} \pm \int \frac{d(p+q)}{\{g(p+q)-a\}^{\frac{1}{2}}} = c.$$

Note. This form, together with all derived from it by transformation of the variables, includes the cases, at present known, in which an integral equation of the geodesics can be expressed in finite terms.

Polynomial Integrals of $\Delta\phi = 1$.

119. We have seen that geodesics can always be deduced from a primitive integral equation of geodesic parallels. The latter can be obtained if we have an equation

$$f(\phi_1, \phi_2, p, q) = a,$$

where a is an arbitrary constant, together with the equation

$$\Delta = \Delta\phi - 1 = 0.$$

The condition of coexistence is the Jacobian relation

$$(f, \Delta) = 0,$$

which, in full, is

$$\frac{\partial f}{\partial p}\frac{\partial \Delta}{\partial \phi_1} - \frac{\partial f}{\partial \phi_1}\frac{\partial \Delta}{\partial p} + \frac{\partial f}{\partial q}\frac{\partial \Delta}{\partial \phi_2} - \frac{\partial f}{\partial \phi_2}\frac{\partial \Delta}{\partial q} = 0;$$

and any integral of the subsidiary system (being the subsidiary system in Charpit's method, § 115)

$$\frac{d\phi_1}{-\dfrac{\partial\Delta}{\partial p}} = \frac{d\phi_2}{-\dfrac{\partial\Delta}{\partial q}} = \frac{dp}{\dfrac{\partial\Delta}{\partial\phi_1}} = \frac{dq}{\dfrac{\partial\Delta}{\partial\phi_2}},$$

which involves ϕ_1 or ϕ_2 or both, can be used for the function $f(\phi_1, \phi_2, p, q)$. The two equations $\Delta = 0$, $f = a$ are to coexist; so the form of f is always modifiable by means of the equation $\Delta = 0$.

Now the number of cases in which an integral of the subsidiary system can be obtained (by which we usually mean that it can be obtained in finite terms) is comparatively small. Among these, some special attention has been devoted* to the cases when f is polynomial in ϕ_1 and ϕ_2; the conditions, necessary and sufficient for the existence of such a function f, can be obtained in the simplest instances.

Accordingly, suppose that f is polynomial of order n in ϕ_1 and ϕ_2; and let the terms in f of the same order m be gathered together and denoted by f_m, so that f is expressible in a form

$$f = f_n + f_{n-1} + f_{n-2} + \dots + f_2 + f_1 + f_0,$$

where the coefficients in $f_n, f_{n-1}, \dots$ are (or may be) functions of p and q. The actual expression of f can be modified by the use of the equation $\Delta\phi = 1$; as $\Delta\phi$ is quadratic in ϕ_1 and ϕ_2, a set of even terms in f will remain even, and a set of odd terms will remain odd, after such modification. The equation

$$(f, \Delta) = 0$$

* See Darboux's treatise, vol. iii, pp. 23—39, 66—85, where (p. 66) references are given; and a note by Kœnigs at the end (pp. 368—404) of the fourth volume.

is to be satisfied, always concurrently with the equation $\Delta\phi = 1$; that is, the equation

$$(f_n, \Delta) + (f_{n-1}, \Delta) + (f_{n-2}, \Delta) + \dots = 0$$

is to be satisfied concurrently with $\Delta\phi = 1$. Hence the even terms in f by themselves satisfy the equation, and the odd terms in f by themselves satisfy the equation, in the form

$$(f_n, \Delta) + (f_{n-2}, \Delta) + \dots\dots = 0,$$

$$(f_{n-1}, \Delta) + (f_{n-3}, \Delta) + \dots\dots = 0,$$

each concurrently with $\Delta\phi = 1$. Consequently, the odd powers of f taken together constitute an integral, and the even powers of f taken together constitute an integral.

Consider the aggregate of even powers

$$f_0 + f_2 + f_4 + \dots + f_{2\mu};$$

it can be transformed into

$$f_0 (\Delta\phi)^{\mu} + f_2 (\Delta\phi)^{\mu-1} + f_4 (\Delta\phi)^{\mu-2} + \dots + f_{2\mu},$$

that is, into a homogeneous polynomial of even order 2μ. Similarly the aggregate of odd powers

$$f_1 + f_3 + f_5 + \dots + f_{2\mu+1}$$

can be transformed into

$$f_1 (\Delta\phi)^{\mu} + f_3 (\Delta\phi)^{\mu-1} + f_5 (\Delta\phi)^{\mu-2} + \dots + f_{2\mu+1},$$

that is, into a homogeneous polynomial of odd order $2\mu + 1$. We are therefore led to inquire what are the integrals f, in the form of homogeneous polynomials in ϕ_1 and ϕ_2, of the lowest orders in succession.

We do not consider the case (if any) when f is of order zero, that is, when it does not involve ϕ_1 or ϕ_2. It cannot effectively be combined with $\Delta\phi = 1$ to determine ϕ_1 and ϕ_2, so as to lead to the quadrature necessary for the determination of ϕ.

As in § 118, we refer the surface to its nul lines, so that the arc-element is

$$ds^2 = 4\lambda\, dp\, dq;$$

and then

$$\Delta\phi = \frac{\phi_1\phi_2}{\lambda},$$

so that the differential equation for geodesic parallels becomes

$$\Delta = \phi_1\phi_2 - \lambda = 0.$$

When

$$f(\phi_1, \phi_2, p, q) = a$$

is an equation to be associated with $\Delta = 0$, the condition of coexistence $(f, \Delta) = 0$ becomes

$$\lambda_1 \frac{\partial f}{\partial \phi_1} + \lambda_2 \frac{\partial f}{\partial \phi_2} + \phi_2 \frac{\partial f}{\partial p} + \phi_1 \frac{\partial f}{\partial q} = 0.$$

We are concerned with integrals of this equation that are polynomial and homogeneous in ϕ_1 and ϕ_2.

120. When there is a linear integral, homogeneous in ϕ_1 and ϕ_2, it must be of the form

$$f = \alpha\phi_1 + \beta\phi_2 = 2a,$$

where α and β are functions of p and q. In order that it may be an integral, the equation

$$\alpha\lambda_1 + \beta\lambda_2 + \phi_2(\alpha_1\phi_1 + \beta_1\phi_2) + \phi_1(\alpha_2\phi_1 + \beta_2\phi_2) = 0$$

must be satisfied concurrently with

$$\phi_1\phi_2 = \lambda.$$

Hence

$$\alpha_2 = 0, \quad \beta_1 = 0, \quad \alpha\lambda_1 + \lambda\alpha_1 + \beta\lambda_2 + \lambda\beta_2 = 0.$$

From the first two, we have

$$\alpha = P, \quad \beta = Q,$$

where P is a function of p only and Q is a function of q only. Now α occurs in the combination $\alpha\phi_1$, that is, $\alpha \dfrac{\partial\phi}{\partial p}$; hence, if P is not zero, by taking a new variable $dp' = dp/P$, we do not alter the character of the arc-element and we make the new α equal to unity. Similarly, if Q is not zero, we can change the variable so as to make β equal to unity without altering the character of the arc-element. Also, P and Q do not vanish together, for otherwise f would be evanescent. Hence there are two cases effectively, viz.,

$$\text{(i)}, \quad \alpha = P = 1, \quad \beta = Q = 1,$$
$$\text{(ii)}, \quad \alpha = P = 0, \quad \beta = Q = 1.$$

For (i), the third condition becomes

$$\lambda_1 + \lambda_2 = 0,$$

so that

$$\lambda = k(p - q);$$

hence

$$ds^2 = 4k(p - q)\,dp\,dq.$$

We have

$$\phi_1 + \phi_2 = 2a, \quad \phi_1\phi_2 = \lambda = k;$$

hence

$$\phi_1 - \phi_2 = \pm 2(a^2 - k)^{\frac{1}{2}},$$

and so

$$d\phi = \{a \pm (a^2 - k)^{\frac{1}{2}}\}\,dp + \{a \mp (a^2 - k)^{\frac{1}{2}}\}\,dq.$$

Consequently the geodesic parallels are given by

$$a(p+q) \pm \int (a^2 - k)^{\frac{1}{2}} (dp - dq) = b;$$

and the geodesics are given by

$$p + q \pm a \int (a^2 - k)^{-\frac{1}{2}} (dp - dq) = c.$$

Moreover, writing

$$p = \tfrac{1}{2}(v + i\eta), \quad q = \tfrac{1}{2}(v - i\eta), \quad \{k(i\eta)\}^{\frac{1}{2}} d\eta = du, \quad k(i\eta) = U,$$

the arc-element is

$$ds^2 = du^2 + Udv^2,$$

so that the surface can be deformed into a surface of revolution; and with these variables, the geodesic parallels and the geodesics are given by the respective equations

$$av \pm \int (U - a^2)^{\frac{1}{2}} \frac{du}{U} = b, \quad v \mp a \int (U - a^2)^{-\frac{1}{2}} \frac{du}{U} = c.$$

For (ii), the third condition is $\lambda_2 = 0$, so that $\lambda = F(p)$, a function of p only. Writing

$$F(p)\, dp = dp',$$

(for modification of the variable p still is possible), we have the arc-element in the form

$$ds^2 = 4dp'dq.$$

This is a special form of the preceding case. Thus *the surfaces which provide a linear integral of the equation* $(f, \Delta) = 0$ *are deformable into surfaces of revolution.*

121. When the equation $(f, \Delta) = 0$ has a quadratic integral other than $\Delta\phi$, let it be

$$f = \alpha\phi_1^2 + 2\beta\phi_1\phi_2 + \gamma\phi_2^2 = 4a.$$

In order that it may be an integral at all, the equation

$$2(\alpha\phi_1 + \beta\phi_2)\lambda_1 + 2(\beta\phi_1 + \gamma\phi_2)\lambda_2$$
$$+ \phi_2(\alpha_1\phi_1^2 + 2\beta_1\phi_1\phi_2 + \gamma_1\phi_2^2) + \phi_1(\alpha_2\phi_1^2 + 2\beta_2\phi_1\phi_2 + \gamma_2\phi_2^2) = 0$$

must be satisfied concurrently with $\phi_1\phi_2 = \lambda$; the necessary conditions are

$$\alpha_2 = 0, \quad \gamma_1 = 0,$$
$$2\alpha\lambda_1 + 2\beta\lambda_2 + \lambda\alpha_1 + 2\lambda\beta_2 = 0,$$
$$2\beta\lambda_1 + 2\gamma\lambda_2 + 2\lambda\beta_1 + \lambda\gamma_2 = 0.$$

Hence

$$\alpha = P, \quad \gamma = Q,$$

where P is a function of p only and may be zero, and Q is a function of q only and may be zero; but α and γ do not vanish together, for then f would be a multiple of $\Delta\phi$—a possibility which is to be excluded. Thus there are two cases:—

(i), $\alpha = P$, not zero; $\gamma = Q$, not zero;

(ii), $\alpha = P$, not zero; $\gamma = 0$.

Case (i). The other two conditions are

$$\frac{\partial}{\partial q}(\lambda\beta) + \alpha^{\frac{1}{2}}\frac{\partial}{\partial p}(\lambda\alpha^{\frac{1}{2}}) = 0, \quad \frac{\partial}{\partial p}(\lambda\beta) + \gamma^{\frac{1}{2}}\frac{\partial}{\partial q}(\lambda\gamma^{\frac{1}{2}}) = 0;$$

hence

$$\frac{\partial}{\partial p}\left\{\alpha^{\frac{1}{2}}\frac{\partial}{\partial p}(\lambda\alpha^{\frac{1}{2}})\right\} = \frac{\partial}{\partial q}\left\{\gamma^{\frac{1}{2}}\frac{\partial}{\partial q}(\lambda\gamma^{\frac{1}{2}})\right\}.$$

Let

$$P^{-\frac{1}{2}}dp = dp', \quad Q^{-\frac{1}{2}}dq = dq', \quad \lambda P^{\frac{1}{2}}Q^{\frac{1}{2}} = \mu;$$

then the equation is

$$\frac{\partial^2\mu}{\partial p'^2} = \frac{\partial^2\mu}{\partial q'^2},$$

so that

$$\mu = g(p'+q') + h(p'-q'),$$

where g and h are any functions whatever. Also

$$-\frac{\partial}{\partial p'}(\lambda\beta) = \frac{\partial\mu}{\partial q'}, \quad -\frac{\partial}{\partial q'}(\lambda\beta) = \frac{\partial\mu}{\partial p'};$$

consequently

$$\lambda\beta = -g(p'+q') + h(p'-q').$$

The element of arc is

$$\begin{aligned} ds^2 &= 4\lambda\, dp\, dq \\ &= 4\mu\, dp'\, dq' \\ &= 4\{g(p'+q') + h(p'-q')\}\, dp'\, dq', \end{aligned}$$

and

$$\phi_1'\phi_2' = \mu = g(p'+q') + h(p'-q').$$

Also, as

$$\begin{aligned} 4a = f &= P\phi_1^2 + Q\phi_2^2 + 2\lambda\beta \\ &= \phi_1'^2 + \phi_2'^2 - 2g(p'+q') + 2h(p'-q'), \end{aligned}$$

we obtain

$$\begin{aligned} \phi_1' - \phi_2' &= 2\{a - h(p'-q')\}^{\frac{1}{2}}, \\ \phi_1' + \phi_2' &= 2\{a + g(p'+q')\}^{\frac{1}{2}}; \end{aligned}$$

and therefore

$$\begin{aligned} \phi &= \int (\phi_1' dp' + \phi_2' dq') \\ &= \int \{a + g(p'+q')\}^{\frac{1}{2}}(dp'+dq') + \int \{a - h(p'-q')\}^{\frac{1}{2}}(dp'-dq'). \end{aligned}$$

Consequently the geodesics are

$$\int \frac{dp'+dq'}{\{a+g(p'+q')\}^{\frac{1}{2}}} + \int \frac{dp'-dq'}{\{a-h(p'-q')\}^{\frac{1}{2}}} = c,$$

where a and c are arbitrary constants, and the radicals clearly can have either sign.

Surfaces, which have their arc-element of the foregoing form, are often called Liouville surfaces (Ex. 2, § 117).

Case (ii). The other two conditions now are

$$\frac{\partial}{\partial q}(\lambda\beta) + \alpha^{\frac{1}{2}}\frac{\partial}{\partial p}(\lambda\alpha^{\frac{1}{2}}) = 0, \quad \frac{\partial}{\partial p}(\lambda\beta) = 0.$$

Changing the variable p so that

$$P^{-\frac{1}{2}}dp = \alpha^{-\frac{1}{2}}dp = dp',$$

we have

$$\frac{\partial}{\partial q}(\lambda\beta) + \frac{\partial}{\partial p'}(\lambda\alpha^{\frac{1}{2}}) = 0, \quad \frac{\partial}{\partial p'}(\lambda\beta) = 0.$$

Hence

$$\lambda\beta = -Q,$$
$$\lambda\alpha^{\frac{1}{2}} = p'Q' + \bar{Q} = \mu,$$

where Q and $\bar{Q}$ are any functions of q. Thus

$$ds^2 = 4(p'Q' + \bar{Q})\,dp'dq,$$

a surface first given by Lie*; and

$$\phi_1'^2 = a - 2\lambda\beta = a + 2Q,$$
$$\phi_1'\phi_2 = p'Q' + \bar{Q}.$$

Consequently

$$\phi = \int(\phi_1'dp' + \phi_2 dq)$$
$$= (a + 2Q)^{\frac{1}{2}}p' + \int\frac{\bar{Q}}{(a+2Q)^{\frac{1}{2}}}dq;$$

and therefore the geodesics are

$$\frac{p'}{(a+2Q)^{\frac{1}{2}}} - \int\frac{\bar{Q}\,dq}{(a+2Q)^{\frac{3}{2}}} = c,$$

where a and c are arbitrary constants.

Note. If the surface is real, then ds^2 must be real and positive; hence p' and q are conjugate variables, and

$$p'Q' + \bar{Q} = qQ_0' + \bar{Q}_0$$

where $\bar{Q}_0$ is the conjugate of $\bar{Q}$ and Q_0' is the conjugate of Q'. Each side of the equation manifestly must be bilinear in p' and q; hence

$$\mu = ap'q + bp' + cq + e,$$

where a and e must be real, while b and c are conjugate. When a is not zero, linear transformations lead to

$$\mu = p'q + 1;$$

writing

$$p' = ue^{iv}, \quad q = ue^{-iv},$$

we have

$$ds^2 = 4(u^2+1)(du^2 + u^2dv^2).$$

* In his investigations on geodesics that admit infinitesimal transformations, *Math. Ann.*, t. xx (1882), pp. 357—454.

When a is zero, linear transformations lead to

$$\mu = p' + q;$$

writing

$$p' = u + iv, \quad q = u - iv,$$

we have

$$ds^2 = 8u(du^2 + dv^2).$$

Both surfaces are deformable into surfaces of revolution.

122. Returning to the Liouville surfaces, which constitute the more general case, we have

$$\alpha = P, \quad \gamma = Q, \quad P^{-\frac{1}{2}}dp = dp', \quad Q^{-\frac{1}{2}}dq = dq',$$

$$\lambda P^{\frac{1}{2}}Q^{\frac{1}{2}} = \quad g(p' + q') + h(p' - q'),$$

$$\lambda\beta = -g(p' + q') + h(p' - q').$$

The simplest instance of all arises when $\alpha = 1$, $\gamma = 1$, so that $p' = p$, $q' = q$; and then

$$\lambda = g(p + q) + h(p - q).$$

The geodesics are now given by

$$\int \frac{dp + dq}{\{a + g(p + q)\}^{\frac{1}{2}}} + \int \frac{dp - dq}{\{a - h(p - q)\}^{\frac{1}{2}}} = c.$$

Conversely, when λ is given in the form

$$\lambda = g(p + q) + h(p - q),$$

we manifestly can have

$$\alpha = 1, \quad \gamma = 1, \quad \lambda\beta = -g(p + q) + h(p - q),$$

as a set of coefficients satisfying all the conditions for the existence of a quadratic integral of the equation $(f, \Delta) = 0$, and so leading to the determination of general families of geodesics. But the question arises:—can there be more than one set of coefficients α, β, γ satisfying all the conditions for the existence of a quadratic integral, so that there would be more than one set of general families of geodesics upon the surface? To answer the question, we return to the conditions

$$2\alpha\lambda_1 + \alpha_1\lambda + 2\beta\lambda_2 + 2\lambda\beta_2 = 0, \quad 2\gamma\lambda_2 + \lambda\gamma_2 + 2\beta\lambda_1 + 2\lambda\beta_1 = 0;$$

eliminating β, we have

$$2\alpha\lambda_{11} + 3\alpha_1\lambda_1 + \alpha_{11}\lambda = 2\gamma\lambda_{22} + 3\gamma_2\lambda_2 + \gamma_{22}\lambda.$$

This equation, in which

$$\lambda = g(p + q) + h(p - q),$$

is to be satisfied by $\alpha = P$, $\gamma = Q$, where P is a function of p alone, and Q a function of q alone.

For the full discussion of this relation, reference should be made to the investigations by Kœnigs already (p. 123) quoted. Some simple examples may be adduced.

I. Let

$$\lambda = (p+q)^m,$$

where m is a constant. Then the equation becomes

$$2m(m-1)(\alpha-\gamma) + 3m(\alpha'-\gamma')(p+q) + (\alpha''-\gamma'')(p+q)^2 = 0,$$

where α' is written for α_1 and γ' for γ_2. Operating twice in succession with $\dfrac{\partial^2}{\partial p\,\partial q}$, we have

$$(3m+2)(\alpha''-\gamma'') + 2(\alpha'''-\gamma''')(p+q) = 0,$$

$$\alpha'''' - \gamma'''' = 0.$$

As $\alpha = \gamma$ when $p+q=0$, the last relation gives

$$\alpha = c_0p^4 + 4c_1p^3 + 6c_2p^2 + 4c_3p + c_4,$$

$$\gamma = c_0q^4 - 4c_1q^3 + 6c_2q^2 - 4c_3q + c_4.$$

When these are substituted in the last relation but one, we find

$$m = -2;$$

and then the critical equation is satisfied without any further condition. Hence on a surface for which

$$ds^2 = \frac{4}{(p+q)^2}\,dp\,dq,$$

we have

$$\left.\begin{aligned} P &= c_0p^4 + 4c_1p^3 + 6c_2p^2 + 4c_3p + c_4 \\ Q &= c_0q^4 - 4c_1q^3 + 6c_2q^2 - 4c_3q + c_4 \end{aligned}\right\};$$

in other words, there are five distinct sets of coefficients for quadratic integrals of the equation $(f, \Delta) = 0$ for this surface, and there are five distinct general families of geodesics. Also

$$\beta = -P^{\frac{1}{2}}Q^{\frac{1}{2}}$$

in this case; so that the quadratic integral is

$$P\phi_1^2 - 2P^{\frac{1}{2}}Q^{\frac{1}{2}}\phi_1\phi_2 + Q\phi_2^2 = a,$$

that is, it is the square of a linear integral; nevertheless, the five constants c_0, c_1, c_2, c_3, c_4 remain unconditioned.

II. Let

$$\lambda = \wp(p+q) - \wp(p-q),$$

where $\wp$ denotes the Weierstrass elliptic function. The critical equation is

$$2\alpha\lambda_{11} + 3\alpha'\lambda_1 + \alpha''\lambda = 2\gamma\lambda_{22} + 3\gamma'\lambda_2 + \gamma''\lambda;$$

it is satisfied by $\alpha = 1$, $\gamma = 1$. It is not difficult to verify that the equation is also satisfied by

$$\alpha = \wp(p), \quad \gamma = \wp(q);$$

and therefore, when regard is paid to the periodicity of λ, the equation is also satisfied by

$$\alpha = \wp(p + \omega_1), \quad \gamma = \wp(q + \omega_1),$$

$$\alpha = \wp(p + \omega_2), \quad \gamma = \wp(q + \omega_2),$$

$$\alpha = \wp(p + \omega_3), \quad \gamma = \wp(q + \omega_3);$$

in other words, we can take

$$\alpha = c_0 + c_1 \wp(p) + c_2 \wp(p + \omega_1) + c_3 \wp(p + \omega_2) + c_4 \wp(p + \omega_3),$$

$$\gamma = c_0 + c_1 \wp(q) + c_2 \wp(q + \omega_1) + c_3 \wp(q + \omega_2) + c_4 \wp(q + \omega_3).$$

The five constants c_0, c_1, c_2, c_3, c_4 are unconditioned; they occur in the general integral equation, which therefore includes five distinct general families of geodesics.

Ex. 1. Shew that, on a surface for which

$$ds^2 = \{(p-q)^{-2} + b\}\, dp\, dq,$$

where b is a constant,

$$\alpha = c_0 + c_1 p + c_2 p^2, \quad \gamma = c_0 + c_1 q + c_2 q^2.$$

Ex. 2. Shew that, on a surface for which

$$ds^2 = \{(p+q)^{-2} - (p-q)^{-2}\}\, dp\, dq,$$

we have

$$\left.\begin{aligned} \alpha &= c_0 p^{-2} + c_1 + c_2 p^2 + c_3 p^4 + c_4 p^6 \\ \gamma &= c_0 q^{-2} + c_1 + c_2 q^2 + c_3 q^4 + c_4 q^6 \end{aligned}\right\}.$$

Note. These examples are due to Kœnigs, who gives tables of the various cases. All these surfaces, which admit five distinct families of geodesics, have their specific curvature constant (or zero).

123. The preceding investigation has related to the use which can be made of a single integral of the subsidiary system of $(f, \Delta) = 0$ in the construction of the function ϕ which determines the geodesic parallels and the geodesics. It is conceivable that two integrals of that subsidiary system, f and k, both involving ϕ_1 and ϕ_2, should be known and that they could coexist. In that case, they must satisfy not merely the relations

$$(f, \Delta) = 0, \quad (k, \Delta) = 0,$$

as they will unconditionally because they are integrals of the subsidiary system, but the further relation

$$(f, k) = 0,$$

which is the condition of coexistence* of f and k. We then should have three equations

$$\Delta\phi = 1, \quad f = a, \quad k = b,$$

which coexist; eliminating ϕ_1 and ϕ_2, we obtain a relation involving two arbitrary constants which would be an equation of geodesic parallels.

But can the combination occur? We have seen that distinct quadratic integrals can exist for an appropriate surface; they will coexist if the Jacobian condition $(f, k) = 0$ is satisfied.

Accordingly, consider the surface

$$ds^2 = 4\lambda\, dp\, dq = 4\{g(p+q) + h(p-q)\}\, dp\, dq.$$

We know that there is a quadratic integral

$$f = \phi_1^2 + 2\beta\phi_1\phi_2 + \phi_2^2,$$

where

$$\lambda\beta = -g(p+q) + h(p-q).$$

Let another quadratic integral be

$$k = \alpha\phi_1^2 + 2\rho\phi_1\phi_2 + \gamma\phi_2^2,$$

where $\alpha = P$, $\gamma = Q$, and

$$2\alpha\lambda_1 + \alpha'\lambda + 2(\lambda\rho_2 + \lambda_2\rho) = 0,$$

$$2\gamma\lambda_2 + \gamma'\lambda + 2(\lambda\rho_1 + \lambda_1\rho) = 0.$$

If f and k coexist, then $(f, k) = 0$; that is,

$$(\phi_1 + \beta\phi_2)(\phi_1^2\alpha' + 2\rho_1\phi_1\phi_2) - (\alpha\phi_1 + \rho\phi_2)\, 2\beta_1\phi_1\phi_2$$
$$+ (\beta\phi_1 + \phi_2)(2\rho_2\phi_1\phi_2 + \phi_2^2\gamma') - (\rho\phi_1 + \gamma\phi_2)\, 2\beta_2\phi_1\phi_2 = 0,$$

which must be satisfied concurrently with $\phi_1\phi_2 = \lambda$. Hence

$$\alpha' = 0, \quad \gamma' = 0,$$

$$\rho_1 - \alpha\beta_1 + \beta\rho_2 - \rho\beta_2 = 0,$$

$$\rho_2 - \gamma\beta_2 + \beta\rho_1 - \rho\beta_1 = 0.$$

From the first two of these we have

$$\alpha = \text{constant} = a', \quad \gamma = \text{constant} = b'.$$

The relations, which allow k to be an integral, are now

$$a'\lambda_1 + \frac{\partial}{\partial q}(\lambda\rho) = 0, \quad b'\lambda_2 + \frac{\partial}{\partial p}(\lambda\rho) = 0,$$

and therefore

$$a'\lambda_{11} - b'\lambda_{22} = 0.$$

* We are not here dealing with the question of merely distinct integrals of $(f, \Delta) = 0$, but of coexistent integrals. When the integrals $f = a$, $k = b$ are distinct but not coexistent, the relations

$$\Delta\phi = 1, \quad f = a$$

lead to one family of geodesic parallels, while the relations

$$\Delta\phi = 1, \quad k = b$$

lead to another family.

But $\lambda_{11} = \lambda_{22}$; hence

$$a' = b', \ = c,$$

say. Thus we have

$$c\lambda_1 + \frac{\partial}{\partial q}(\lambda\rho) = 0, \quad c\lambda_2 + \frac{\partial}{\partial p}(\lambda\rho) = 0,$$

and therefore

$$\lambda\rho = -cg(p+q) + ch(p-q) = c\lambda\beta,$$

so that

$$\rho = c\beta.$$

The remaining conditions for the coexistence of f and g are satisfied. But

$$k = \alpha\phi_1^2 + 2\rho\phi_1\phi_2 + \gamma\phi_2^2 = cf;$$

therefore the integrals, when they coexist, are not independent.

It therefore follows that, if there are two independent quadratic integrals, they cannot be combined to give the equation of geodesics. Each of them, by itself, leads to a family of geodesics; the two integrals determine two distinct families of geodesics.

124. As another example, leading to a similar conclusion, consider the surface

$$ds^2 = 4\lambda\, dp\, dq = 4f(p-q)\, dp\, dq.$$

It possesses a linear integral

$$g = \phi_1 + \phi_2.$$

Can it possess a quadratic integral, independent of $\phi_1\phi_2/\lambda$, and of g^2? If so, let it be

$$h = \alpha\phi_1^2 + 2\beta\phi_1\phi_2 + \gamma\phi_2^2,$$

where $\alpha = P$, $\gamma = Q$,

$$2\alpha\lambda_1 + \alpha_1\lambda + 2(\beta\lambda_2 + \lambda\beta_2) = 0,$$

$$2\gamma\lambda_2 + \gamma_2\lambda + 2(\beta\lambda_1 + \lambda\beta_1) = 0.$$

The condition of coexistence is $(g, h) = 0$; that is, the equation

$$\alpha_1\phi_1^2 + 2\beta_1\phi_1\phi_2 + \gamma_2\phi_2^2 + 2\beta_2\phi_1\phi_2 = 0,$$

must be satisfied concurrently with $\phi_1\phi_2 = \lambda$. Hence

$$\alpha_1 = 0, \quad \gamma_2 = 0, \quad \beta_1 + \beta_2 = 0.$$

From the first two, we have

$$\alpha = \text{constant} = a', \quad \gamma = \text{constant} = b'.$$

Now $\lambda_1 + \lambda_2 = 0$; hence the earlier conditions give

$$a' = b', \ = c,$$

say. Then

$$c\lambda_1 + \beta\lambda_2 + \lambda\beta_2 = 0,$$

that is,

$$c\lambda_1 = \beta\lambda_1 + \lambda\beta_1;$$

and

$$c\lambda_2 + \beta\lambda_1 + \lambda\beta_1 = 0,$$

that is,

$$c\lambda_2 = \beta\lambda_2 + \lambda\beta_2;$$

thus

$$\beta\lambda = c\lambda + k,$$

where k is a constant. Hence

$$h = c\phi_1{}^2 + c\phi_2{}^2 + 2c\lambda + 2k$$

$$= cg^2 + 2k\frac{\phi_1\phi_2}{\lambda};$$

in other words, h is not independent of g and $\Delta\phi$.

It follows that the coexistent integrals are not independent. The independent integrals determine distinct families of geodesics; but they cannot be combined to determine one and the same family.

EXAMPLES.

1. Representing the surface of an anchor-ring by the equations

$$x = r\cos\phi,\quad y = r\sin\phi,\quad z = a\sin\theta,\quad r = c + a\cos\theta,$$

obtain the equation of geodesics in the form

$$d\phi = \frac{ah\,dr}{r(r^2 - h^2)^{\frac{1}{2}}\{a^2 - (r-c)^2\}^{\frac{1}{2}}},$$

where h is an arbitrary constant; and find an equation for determining the conjugate of any point on the geodesic.

2. Discuss the geodesics on a hyperboloid of one sheet, not of revolution; and describe their course on the surface. In particular shew that, when the parameter θ of integration (§ 101) vanishes, the geodesics are the generators of the surface.

3. On an ellipsoid taken as in § 101, a geodesic through an umbilicus cuts the section of the surface through the real umbilici at an angle v; shew that the arc-element at any point on the surface can be expressed in the form

$$ds^2 = du^2 + \frac{y^2}{\sin^2 v}dv^2,$$

where y is the distance of the point from the umbilical section, and

$$2du = \left\{\frac{p}{(a+p)(c+p)}\right\}^{\frac{1}{2}}dp \pm \left\{\frac{q}{(a+q)(c+q)}\right\}^{\frac{1}{2}}dq.$$

Shew that an umbilical geodesic does not return upon itself; and obtain equations for the lines of curvature through a point in the form

$$\tan\tfrac{1}{2}v\tan\tfrac{1}{2}v' = \text{constant},\quad \tan\tfrac{1}{2}v\cot\tfrac{1}{2}v' = \text{constant},$$

where v and v' are the angles at which the geodesics from the point to two umbilici, that are not diametrically opposite, cut the umbilical section.

4. Obtain a first integral of the equation of geodesics on the quadric

$$y^2/a+z^2/c=4x$$

in the form

$$P^{-\frac{1}{2}}pdp+Q^{-\frac{1}{2}}qdq=0,\quad P^{-\frac{1}{2}}p^2dp+Q^{-\frac{1}{2}}q^2dq=ds,$$

where

$$(c-a)\,y^2=4a\,(a+p)\,(a+q),\quad (a-c)\,z^2=4c\,(c+p)\,(c+q),$$
$$P=p\,(a+p)\,(c+p)\,(\theta+p),\quad Q=q\,(a+q)\,(c+q)\,(\theta+q),$$

θ being a constant of integration.

Trace the geodesics.

5. Shew that, through any point on a surface, there passes at least one geodesic such that four consecutive points of the curve lie on its circle of curvature; and obtain an equation for the direction in the form

$$Pdp^3+3Qdp^2dq+3Rdp\,dq^2+Sdq^3=0,$$

where P, Q, R, S are the derived magnitudes of the third order.

6. Shew that an equation of geodesics on the surface

$$ds^2=\frac{dp^2}{ap+bq}+\frac{dq^2}{a'p+b'q}$$

is given by

$$\prod_{r=1}^{3}\{(a'p+b'q)^{-\frac{1}{2}}\cos\omega-t_r\,(ap+bq)^{-\frac{1}{2}}\sin\omega\}^{a_r}=\text{constant},$$

where ω is the angle at which the geodesic cuts the curve $p=$constant, and t_1, t_2, t_3, a_1, a_2, a_3 are constants such that

$$\frac{a+a'u^2}{b-au+b'u^2-a'u^3}=\frac{a_1}{u-t_1}+\frac{a_2}{u-t_2}+\frac{a_3}{u-t_3}.$$

7. Let the equation $(f,\Delta)=0$ have a cubic homogeneous integral of the subsidiary system in the form

$$A\phi_1{}^3+B\phi_2{}^3+3\alpha\phi_1{}^2\phi_2+3\beta\phi_1\phi_2{}^2=a,$$

the equation $\Delta=0$ being $\phi_1\phi_2-\lambda=0$. Shew that

$$A=P,\quad B=Q,$$

where P is a function of p only and Q is a function of q only, which may not vanish together for a proper cubic integral.

When neither P nor Q vanishes, so that new variables p' and q' can be taken in the form

$$P^{-\frac{1}{3}}dp=dp',\quad Q^{-\frac{1}{3}}dq=dq',$$

shew that

$$\lambda\alpha P^{-\frac{1}{3}}=-\frac{\partial^2u}{\partial p'^2},\quad \lambda P^{\frac{1}{3}}Q^{\frac{1}{3}}=\frac{\partial^2u}{\partial p'\partial q'},\quad \lambda\beta Q^{-\frac{1}{3}}=\frac{\partial^2u}{\partial q'^2},$$

where u satisfies the equation

$$\frac{\partial}{\partial p'}\left(\frac{\partial^2u}{\partial p'^2}\frac{\partial^2u}{\partial p'\partial q'}\right)+\frac{\partial}{\partial q'}\left(\frac{\partial^2u}{\partial p'\partial q'}\frac{\partial^2u}{\partial q'^2}\right)=0.$$

When Q vanishes but not P, so that a new variable p' can be taken in the form $P^{-\frac{1}{3}}dp=dp'$, shew that

$$\lambda\beta=\overline{Q},$$

where $\overline{Q}$ is a function of q only. Shew also that

$$\lambda P^{\frac{1}{3}}=-\frac{\partial v}{\partial q},\quad \lambda\alpha P^{-\frac{1}{3}}=\frac{\partial v}{\partial p'},$$

where v satisfies the equation

$$\frac{\partial}{\partial p'}\left(\frac{\partial v}{\partial p'}\frac{\partial v}{\partial q}\right)+\frac{\partial}{\partial q}\left(\overline{Q}\,\frac{\partial v}{\partial q}\right)=0.$$

8. The equation $(f, \Delta)=0$ has a quartic homogeneous integral of its subsidiary system in the form

$$\phi_1^4+4\beta\phi_1^3\phi_2+6\gamma\phi_1^2\phi_2^2+4\beta'\phi_1\phi_2^3+\phi_2^4=a,$$

the equation $\Delta=0$ being $\phi_1\phi_2-\lambda=0$. Shew that, if

$$\lambda^2\gamma=\Gamma,$$

then

$$\lambda\beta=-\frac{\partial^2 u}{\partial p^2},\quad \lambda=\frac{\partial^2 u}{\partial p\partial q},\quad \lambda\beta'=-\frac{\partial^2 u}{\partial q^2},$$

$$\frac{3}{2}\frac{\partial\Gamma}{\partial p}=2\frac{\partial^2 u}{\partial q^2}\frac{\partial^3 u}{\partial p\partial q^2}+\frac{\partial^2 u}{\partial p\partial q}\frac{\partial^3 u}{\partial q^3},$$

$$\frac{3}{2}\frac{\partial\Gamma}{\partial q}=2\frac{\partial^2 u}{\partial p^2}\frac{\partial^3 u}{\partial p^2\partial q}+\frac{\partial^2 u}{\partial p\partial q}\frac{\partial^3 u}{\partial p^3},$$

so that u satisfies an equation of the fourth order.

If the equation has a quartic integral of its subsidiary system in the form

$$\phi_1^4+4\beta\phi_1^3\phi_2+6\gamma\phi_1^2\phi_2^2+4\beta'\phi_1\phi_2^3=a,$$

then, denoting $\lambda^2\gamma$ by Γ, shew that

$$\lambda\beta=\frac{\partial u}{\partial p},\quad \lambda=-\frac{\partial u}{\partial q},\quad \lambda\beta'=Q,$$

$$\frac{3}{2}\frac{\partial\Gamma}{\partial p}=Q'\frac{\partial u}{\partial q}+2Q\frac{\partial^2 u}{\partial q^2},$$

$$\frac{3}{2}\frac{\partial\Gamma}{\partial q}=2\frac{\partial u}{\partial p}\frac{\partial^2 u}{\partial p\partial q}+\frac{\partial u}{\partial q}\frac{\partial^2 u}{\partial p^2},$$

where Q is a function of q only, u satisfying an equation of the third order.

9. The equation $(f, \Delta)=0$ has a polynomial integral of its subsidiary system expressible in the form

$$a_n\phi_1^n+c_n\phi_2^n+a_{n-2}\phi_1^{n-2}+c_{n-2}\phi_2^{n-2}+\ldots=\text{constant},$$

the equation $\Delta=0$ being $\phi_1\phi_2=\lambda$. Shew that a_n cannot involve q, and that c_n cannot involve p; and obtain the relations satisfied by the remaining coefficients in the integral.

10. When the equation $(f, \Delta)=0$, where $\Delta=\phi_1\phi_2-\lambda=0$, has an integral of its subsidiary system in the form

$$\frac{\alpha\phi_1+\beta\phi_2}{\gamma\phi_1+\phi_2}=a,$$

where no one of the quantities α, β, γ vanishes, shew that

$$\alpha=P\gamma,\quad \beta=Q,$$

P being a function of p only, and Q a function of q only. Shew also that λ and γ satisfy the equations

$$\left.\begin{aligned}\frac{\partial}{\partial p}\{\lambda\gamma(P-Q)\}+\lambda Q'=0\\ \frac{\partial}{\partial q}\left\{\frac{\gamma}{\lambda(P-Q)}\right\}+\lambda P'=0\end{aligned}\right\};$$

and obtain the geodesic parallels in the form

$$\int\left\{\left(\frac{\lambda}{\gamma}\frac{a-Q}{P-a}\right)^{\frac{1}{2}}dp+\left(\gamma\lambda\frac{P-a}{a-Q}\right)^{\frac{1}{2}}dq\right\}=b.$$

CHAPTER VI.

General Curves on a Surface: Differential Invariants.

The present chapter consists of two connected parts, and relates to curves that have no particular organic relation to the surface but are specified by some assigned analytical definition.

In the first part, the expressions for the various geometric magnitudes are obtained in connection with simultaneous binary forms, associated at once with the curve and the surface; and it proves possible to obtain some relations among the magnitudes.

In the second part, there is a discussion of certain functions called differential invariants (sometimes differential parameters). They maintain their values unaltered through all changes in the superficial variables of reference, and so they represent geometrical magnitudes of the curve and the surface. Their expressions are constructed, and their geometrical significance is established.

Various methods have been devised for these differential invariants; and references to some of the authorities are given in § 133. The method here adopted is based upon Lie's theory of continuous groups and, in the form adopted, was the subject of a memoir by the author which is quoted in § 133. The reason for the adoption of this method, in spite of its laborious detail, which however becomes mechanically easier as soon as its algorithm is recognised, and in spite of its initial non-geometrical aspect, lies in its compelling quality. Besides giving the expressions of the covariants, it indicates how many of them are independent, and indicates also a merely algebraical method of expressing all the covariants in terms of an algebraically independent set; consequently, when once the geometrical significance of all the covariants is established, we know how many of the geometrical magnitudes are independent and we have all the relations (up to any order of derivation) that exist among the magnitudes.

General Curves on a Surface.

125. We now proceed to consider general curves on the surface, rather than special curves as in preceding chapters, especially for the purpose of obtaining the analytical expressions for the more important geometrical magnitudes. As the actual values of these magnitudes for a given curve must be the same whatever system of superficial coordinates be adopted, it follows that the various expressions must have an invariantive character under all changes of the coordinates. Hence connected with the surface,

and with a curve or curves on the surface, there will exist covariants and invariants persisting through all transformations of the parameters; so it becomes necessary to construct all such invariantive functions and to establish their geometric significance.

As before, we use p and q to denote the current parameters on the surface; the parametric curves are not assumed to be an orthogonal system. A curve on the surface can be selected either by some relation between p and q of the form

$$\phi(p, q) = 0,$$

or by having p and q given, explicitly or implicitly, as functions of some parameter, say s, the arc of the curve measured from a fixed point. We shall use the latter method first, and shall denote derivatives of p and q with regard to s by $p', p'', \ldots, q', q'', \ldots$.

It is convenient to recall some earlier results. Let

$$\begin{aligned}
I &= Ep'^2 + 2Fp'q' + Gq'^2,\\
A &= Lp'^2 + 2Mp'q' + Nq'^2,\\
W &= \frac{1}{V}\begin{vmatrix} Ep' + Fq', & Fp' + Gq' \\ Lp' + Mq', & Mp' + Nq' \end{vmatrix},\\
D_1 &= \Gamma p'^2 + 2\Gamma' p'q' + \Gamma'' q'^2 + p'',\\
D_2 &= \Delta p'^2 + 2\Delta' p'q' + \Delta'' q'^2 + q'',\\
D &= V(p'D_2 - q'D_1)\\
&= V\{p'q'' - q'p'' + \Delta p'^3 + (2\Delta' - \Gamma)p'^2q' + (\Delta'' - 2\Gamma')p'q'^2 - \Gamma'' q'^3\},\\
V^2 &= EG - F^2,\\
T^2 &= LN - M^2,\\
U &= EN - 2FM + GL;
\end{aligned}$$

and write, temporarily,

$$VW = C.$$

For all curves that are not nul lines, $I = 1$; for nul lines, $I = 0$. The asymptotic lines are given by $A = 0$; the lines of curvature by $C = 0$ or $W = 0$; the geodesic lines by $D_1 = 0$ or $D_2 = 0$ or $D = 0$.

Also we have

$$\frac{dI}{ds} = 0,$$

so that

$$(Ep' + Fq')D_1 + (Fp' + Gq')D_2 = 0;$$

hence

$$\frac{D_2}{Ep' + Fq'} = \frac{D_1}{-(Fp' + Gq')} = \frac{D}{V},$$

so that

$$D_2 = (Ep' + Fq')\frac{D}{V}, \quad D_1 = -(Fp' + Gq')\frac{D}{V};$$

and therefore

$$D^2 = ED_1^2 + 2FD_1D_2 + GD_2^2.$$

Now I and A are a couple of simultaneous quadratic forms; and their asyzygetically complete concomitant system (that is to say, the aggregate of linearly independent quantities that are invariantive for linear transformations of p' and q') is constituted by the set I, A, C, V^2, T^2, U. By a known result—which also can easily be verified directly—in the theory of binary forms, we have

$$C^2 = IAU - T^2I^2 - V^2A^2,$$

so that (introducing the mean measure of curvature and the Gaussian measure)

$$W^2 = IAH - KI^2 - A^2.$$

Thus, in the case of nul lines (the importance of which is analytical), we have

$$W = \pm iA.$$

In the case of asymptotic lines, we have

$$W = \pm iK^{\frac{1}{2}} = \pm iT/V.$$

In the case of all lines, other than nul lines, we have

$$\begin{aligned} W^2 &= -A^2 + AH - K \\ &= \left(\frac{1}{\alpha} - A\right)\left(A - \frac{1}{\beta}\right). \end{aligned}$$

Again, we have

$$\begin{aligned} WD &= \begin{vmatrix} Ep' + Fq', & Fp' + Gq' \\ Lp' + Mq', & Mp' + Nq' \end{vmatrix} \begin{vmatrix} p', & q' \\ D_1, & D_2 \end{vmatrix} \\ &= \begin{vmatrix} 1, & A \\ \Psi, & \Phi \end{vmatrix} = \Phi - A\Psi, \end{aligned}$$

where

$$\begin{aligned} \Psi &= (Ep' + Fq')D_1 + (Fp' + Gq')D_2 = 0, \\ \Phi &= (Lp' + Mq')D_1 + (Mp' + Nq')D_2 \\ &= p'^3(L\Gamma + M\Delta) + p'^2q'(2L\Gamma' + 2M\Delta' + M\Gamma + N\Delta) \\ &\quad + p'q'^2(L\Gamma'' + M\Delta'' + 2M\Gamma' + 2N\Delta') + q'^3(M\Gamma'' + N\Delta'') \\ &\quad + (Lp' + Mq')p'' + (Mp' + Nq')q''. \end{aligned}$$

Also

$$\begin{aligned} \frac{dA}{ds} &= 2(Lp' + Mq')p'' + 2(Mp' + Nq')q'' \\ &\quad + p'^3L_1 + p'^2q'(L_2 + 2M_1) + p'q'^2(2M_2 + N_1) + q'^3N_2; \end{aligned}$$

and therefore

$$\frac{dA}{ds} - 2\Phi = Pp'^3 + 3Qp'^2q' + 3Rp'q'^2 + Sq'^3$$
$$= \Omega,$$

say, on reduction and after introducing the derived magnitudes of the third order. We thus have

$$DW = \Phi,$$

$$\frac{dA}{ds} = 2DW + \Omega,$$

where Ω now is a cubic form associated with the former system.

126. The circular curvature and the torsion of the curve, the circular curvature and the torsion of the geodesic tangent to the curve, and the geodesic curvature of the curve, can be brought into analytical relation with the foregoing magnitudes. These geometrical quantities will be denoted by the following symbols, all the conventions (§§ 103, 104) as to signs of magnitudes and as to directions in which angles are measured remaining unaltered:—

$$\left.\begin{aligned}\rho &= \text{radius of circular curvature of the curve}\\ \sigma &= \text{radius of torsion of the curve}\end{aligned}\right\},$$

ρ' = radius of circular curvature of the geodesic tangent, being the radius of circular curvature of the normal section of the surface through the tangent,

σ' = radius of torsion of the geodesic tangent, (the last two grouped by a brace),

γ = radius of geodesic curvature of the curve.

Also we write

ϖ = inclination of the principal normal of the curve to the normal to the surface,

θ = inclination of the tangent of the curve to the line of curvature for which $1/\alpha$ is the principal curvature of the surface.

Now

$$x'' = x_{11}p'^2 + 2x_{12}p'q' + x_{22}q'^2 + x_1p'' + x_2q''$$
$$= AX + x_1D_1 + x_2D_2,$$

on substituting for x_{11}, x_{12}, x_{22}, their values (§ 34) in terms of X, x_1, x_2; and similarly

$$y'' = AY + y_1D_1 + y_2D_2,$$
$$z'' = AZ + z_1D_1 + z_2D_2.$$

By Meunier's theorem, we have

$$\frac{\cos\varpi}{\rho} = \frac{1}{\rho'} = A.$$

Let λ, μ, ν be the direction-cosines of the binormal of the curve; then

$$\begin{aligned}\lambda &= \rho\,(y'z'' - z'y'')\\ &= \rho A\,(Zy' - Yz') + \rho D_1\,(y'z_1 - z'y_1) + \rho D_2\,(y'z_2 - z'y_2)\\ &= \frac{\rho A}{V}\,\{x_1\,(Fp' + Gq') - x_2\,(Ep' + Fq')\} + \rho XD,\end{aligned}$$

on reduction; and similarly for μ and ν. Hence

$$\begin{aligned}\sin\varpi &= \lambda X + \mu Y + \nu Z\\ &= \rho D,\end{aligned}$$

so that

$$\frac{1}{\gamma} = \frac{\sin\varpi}{\rho} = D.$$

To obtain the torsion of the curve, we use the third Serret-Frenet formulæ

$$\frac{d\lambda}{ds} = -\frac{l}{\sigma}, \quad \frac{d\mu}{ds} = -\frac{m}{\sigma}, \quad \frac{d\nu}{ds} = -\frac{n}{\sigma},$$

where $l = \rho x''$, $m = \rho y''$, $n = \rho z''$. Now

$$\sin\varpi = \lambda X + \mu Y + \nu Z;$$

hence

$$\begin{aligned}\cos\varpi\,\frac{d\varpi}{ds} &= \Sigma X\,\frac{d\lambda}{ds} + \Sigma\lambda\,\frac{dX}{ds}\\ &= -\frac{\cos\varpi}{\sigma} + p'\Sigma\lambda X_1 + q'\Sigma\lambda X_2.\end{aligned}$$

But with the values of X_1 and X_2, as given in § 29, we have

$$\Sigma\lambda X_1 = \frac{\rho A}{V}\,\{M\,(Ep' + Fq') - L\,(Fp' + Gq')\},$$

$$\Sigma\lambda X_2 = \frac{\rho A}{V}\,\{N\,(Ep' + Fq') - M\,(Fp' + Gq')\};$$

and therefore

$$\cos\varpi\,\frac{d\varpi}{ds} = -\frac{\cos\varpi}{\sigma} + \frac{\rho A}{V}\,VW = -\frac{\cos\varpi}{\sigma} + W\cos\varpi.$$

Proceeding similarly from the equation

$$\cos\varpi = Xl + Ym + Zn,$$

and using the second Serret-Frenet formulæ, we find

$$-\sin\varpi\,\frac{d\varpi}{ds} = \frac{\sin\varpi}{\sigma} - \frac{\rho D}{V}\,VW = \frac{\sin\varpi}{\sigma} - W\sin\varpi.$$

Thus, from these relations, we have

$$\frac{d\varpi}{ds} + \frac{1}{\sigma} = W.$$

Further, this result shews that $\frac{1}{\sigma} + \frac{d\varpi}{ds}$ is the same for all curves through the point on the surface which have the same tangent as the given curve. Take the geodesic tangent; for that particular curve, ϖ is always zero, and $\frac{1}{\sigma'}$ is the torsion of the geodesic; hence

$$\left.\begin{aligned} \frac{1}{\sigma} + \frac{d\varpi}{ds} &= W \\ \frac{1}{\sigma'} &= W \\ \frac{1}{\sigma} + \frac{d\varpi}{ds} &= \frac{1}{\sigma'} \end{aligned}\right\},$$

in agreement with the result of § 106.

Also, we have

$$\left.\begin{aligned} \frac{1}{\rho'} &= \frac{\cos^2\theta}{\alpha} + \frac{\sin^2\theta}{\beta} \\ \frac{1}{\sigma'} &= \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\cos\theta\sin\theta \end{aligned}\right\}.$$

For the radius of spherical curvature, we have

$$\begin{aligned} R^2 &= \rho^2 + \sigma^2\left(\frac{d\rho}{ds}\right)^2 \\ &= \rho^2 + \rho^6\sigma^2(AA' + DD')^2. \end{aligned}$$

Again, from the equation

$$\frac{\cos\varpi}{\rho} = A,$$

we have

$$\begin{aligned} -\frac{\sin\varpi}{\rho}\frac{d\varpi}{ds} - \frac{\cos\varpi}{\rho^2}\frac{d\rho}{ds} &= \frac{dA}{ds} \\ &= 2DW + \Omega \\ &= 2\frac{\sin\varpi}{\rho}\left(\frac{d\varpi}{ds} + \frac{1}{\sigma}\right) + \Omega, \end{aligned}$$

and therefore

$$\left(\frac{2}{\sigma} + 3\frac{d\varpi}{ds}\right)\frac{\sin\varpi}{\rho} + \frac{\cos\varpi}{\rho^2}\frac{d\rho}{ds} = -\Omega.$$

127. We shall need the expressions for the various geometrical magnitudes belonging to the curve when it is given by an equation between p and q, say

$$\phi(p, q) = 0.$$

Writing

$$\Theta = (E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2)^{\frac{1}{2}},$$

we obtained (§ 105) an expression for the geodesic curvature in Bonnet's form

$$\frac{V}{\gamma} = \frac{\partial}{\partial p}\left(\frac{F\phi_2 - G\phi_1}{\Theta}\right) + \frac{\partial}{\partial q}\left(\frac{F\phi_1 - E\phi_2}{\Theta}\right).$$

Later, it will appear that this relation can usefully be taken in the modified form

$$-\frac{1}{\gamma} = \frac{V}{\Theta^3}(a\phi_2^2 - 2b\phi_1\phi_2 + c\phi_1^2),$$

where

$$a = \phi_{11} - \Gamma\phi_1 - \Delta\phi_2,$$
$$b = \phi_{12} - \Gamma'\phi_1 - \Delta'\phi_2,$$
$$c = \phi_{22} - \Gamma''\phi_1 - \Delta''\phi_2.$$

Thus the geodesic curvature is expressible, save for the factor $-V\Theta^{-3}$, as an algebraic quantity, homogeneous in ϕ_1 and ϕ_2, with coefficients that depend upon the curve and the surface.

Further, we have

$$\frac{1}{\rho'} = Lp'^2 + 2Mp'q' + Nq'^2$$
$$= \frac{1}{\Theta^2}(L\phi_2^2 - 2M\phi_1\phi_2 + N\phi_1^2).$$

Again

$$W = \frac{1}{V}\begin{vmatrix} Ep' + Fq', & Fp' + Gq' \\ Lp' + Mq', & Mp' + Nq' \end{vmatrix}$$
$$= \frac{1}{V\Theta^2}\begin{vmatrix} E\phi_2 - F\phi_1, & F\phi_2 - G\phi_1 \\ L\phi_2 - M\phi_1, & M\phi_2 - N\phi_1 \end{vmatrix};$$

and

$$\tan \varpi = \frac{\rho'}{\gamma}.$$

With these values of ϖ and W, the torsion of the curve is given by

$$\frac{1}{\sigma} + \frac{d\varpi}{ds} = W,$$

while the torsion of the geodesic tangent is given by

$$\frac{1}{\sigma'} = W.$$

Thus various magnitudes belonging to the curve and its geodesic tangent can be expressed, save for factors involving a power of V and a power of Θ, as algebraic quantities, homogeneous in ϕ_1 and ϕ_2, with coefficients that depend upon the curve and the surface.

Consider the special forms for the parametric curves; it is convenient to record the values of all the magnitudes.

For the curve $p=a$, we have

$$\phi_1=1, \quad \phi_2=0, \quad ds=G^{\frac{1}{2}}dq,$$

so that $\Theta=-G^{\frac{1}{2}}$; thus

$$A=\frac{N}{G}, \quad W=\frac{1}{GV}(FN-GM), \quad D=-\frac{V\Gamma''}{G^{\frac{3}{2}}},$$

$$\frac{\cos\varpi}{\rho}=\frac{N}{G}, \quad \frac{1}{\gamma}=\frac{\sin\varpi}{\rho}=-\frac{V\Gamma''}{G^{\frac{3}{2}}},$$

$$\frac{1}{\sigma}=\frac{N^2G^{\frac{1}{2}}}{N^2G+V^2\Gamma''^2}\frac{\partial}{\partial q}\left(\frac{V\Gamma''}{NG^{\frac{1}{2}}}\right)+\frac{1}{GV}(FN-GM),$$

$$\frac{1}{\sigma'}=\frac{1}{GV}(FN-GM).$$

For the curve $q=b$, we have

$$\phi_1=0, \quad \phi_2=1, \quad ds=E^{\frac{1}{2}}dp,$$

so that $\Theta=E^{\frac{1}{2}}$; thus

$$A=\frac{L}{E}, \quad W=\frac{1}{EV}(EM-FL), \quad D=\frac{V\Delta}{E^{\frac{3}{2}}},$$

$$\frac{\cos\varpi}{\rho}=\frac{L}{E}, \quad \frac{1}{\gamma}=\frac{\sin\varpi}{\rho}=\frac{V\Delta}{E^{\frac{3}{2}}},$$

$$\frac{1}{\sigma}=-\frac{L^2E^{\frac{1}{2}}}{L^2E+V^2\Delta^2}\frac{\partial}{\partial p}\left(\frac{V\Delta}{LE^{\frac{1}{2}}}\right)+\frac{1}{EV}(EM-FL),$$

$$\frac{1}{\sigma'}=\frac{1}{EV}(EM-FL).$$

Some Properties of Organic Curves.

128. One of the advantages of the preceding general forms is that, for particular curves such as those organically related to the surface, one or other of the covariants vanishes; the resulting relation frequently leads to geometric properties. We shall consider the lines in turn.

Consider, first, the lines of curvature on the surface. We have

$$W=0;$$

thus the torsion of the geodesic tangent is zero.

Again, the torsion of the line of curvature is given by

$$-\frac{1}{\sigma}=\frac{d\varpi}{ds};$$

hence, if a line of curvature is plane, its plane cuts the surface at a constant angle—a theorem due to Joachimsthal.

Again, if the line of curvature be also a line of curvature on another surface so that it is the intersection of two surfaces, we have

$$-\frac{1}{\sigma}=\frac{d\varpi}{ds},\quad -\frac{1}{\sigma}=\frac{d\varpi'}{ds},$$

for the two surfaces. Then

$$\varpi'-\varpi=\text{constant};$$

hence the two surfaces cut at a constant angle—a theorem also due to Joachimsthal.

Similarly, if two surfaces cut at a constant angle, and if the curve of intersection be a line of curvature on one surface, it is a line of curvature on the other also. For, as

$$\varpi'-\varpi=\text{constant},$$

we have

$$\frac{d\varpi'}{ds}=\frac{d\varpi}{ds},$$

and therefore

$$-\frac{1}{\sigma}+W'=-\frac{1}{\sigma}+W,$$

so that

$$W'=W.$$

If then either W or W' vanishes, both vanish—which establishes the proposition.

Further, if a plane cut a surface at a constant angle, the curve of intersection is a line of curvature on the surface. For

$$\frac{d\varpi}{ds}=0,$$

owing to the constancy of the angle, and

$$\frac{1}{\sigma}=0,$$

because the curve is plane; hence

$$W=0,$$

shewing that the curve is a line of curvature.

We have seen that inversion with respect to any centre conserves lines of curvature (§ 79); and we know that inversion changes a plane into a sphere. Hence we may expect some properties of spherical lines of curvature similar to the preceding properties of plane lines of curvature.

Suppose, then, that a line of curvature lies on a sphere. Its geodesic tangent is a great circle, that is, a plane curve; and therefore the torsion of the geodesic tangent is zero, so that, on the sphere, we have

$$-\frac{1}{\sigma}=\frac{d\varpi'}{ds}.$$

As the curve is a line of curvature on the surface,

$$-\frac{1}{\sigma}=\frac{d\varpi}{ds};$$

hence

$$\varpi'-\varpi=\text{constant},$$

or the sphere and the surface cut at a constant angle.

Similarly, if a sphere cuts a surface at a constant angle, the curve of intersection is a line of curvature on the surface. For

$$\frac{d\varpi'}{ds}=\frac{d\varpi}{ds}.$$

that is,

$$\frac{1}{\sigma}=\frac{1}{\sigma}-W,$$

and therefore

$$W=0,$$

proving the proposition.

129. Consider, next, the asymptotic lines on the surface. For them,

$$A=0,$$

that is, their directions are given by

$$Lp'^2+2Mp'q'+Nq'^2=0.$$

For one of the asymptotic lines, we have

$$\frac{p'}{M+iT}=\frac{q'}{-L}=\mu,$$

where

$$\mu^2\{2EM^2-2FLM-ELN+GL^2+2iT(EM-FL)\}=1.$$

Now

$$\begin{aligned}W&=\frac{1}{V}\begin{vmatrix} Ep'+Fq', & Fp'+Gq' \\ Lp'+Mq', & Mp'+Nq'\end{vmatrix}\\ &=\frac{iT\mu^2}{V}\begin{vmatrix} EM-FL+iET, & FM-GL+iFT \\ L \quad , & M+iT\end{vmatrix}\\ &=\frac{iT}{V}=iK^{\frac{1}{2}}.\end{aligned}$$

Similarly, for the other asymptotic line

$$\frac{p'}{M-iT}=\frac{q'}{-L},$$

we have

$$W=-\frac{iT}{V}=-iK^{\frac{1}{2}}.$$

Both results are in accord with the relation

$$W^2 = AI\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - A^2 - I^2\frac{1}{\alpha\beta},$$

when $A = 0$, $I = 1$.

Now, because $A = 0$ for the asymptotic lines, we have

$$\frac{\cos\varpi}{\rho} = 0.$$

Hence, when the lines are not linear generators, we have

$$\varpi = \tfrac{1}{2}\pi.$$

In that case, we infer the following properties:—

(i) the geodesic curvature of an asymptotic line is equal to the circular curvature;

(ii) the torsion of an asymptotic line is equal to the torsion of its geodesic tangent;

(iii) the square of the torsion of an asymptotic line is equal to the specific curvature of the surface with its sign changed, so that the asymptotic lines on a surface of constant specific curvature have constant torsion;

(iv) the torsions of the two asymptotic lines at any point are equal and opposite.

Next, consider a section of the surface made by any plane drawn through the tangent to an asymptotic line. At the point on this plane section, p' and q' are the same as for the asymptotic line; and so, there, we have

$$A = 0.$$

Hence, for the plane section, we have

$$\frac{\cos\varpi_1}{\rho_1} = 0.$$

This condition can be satisfied in two ways.

We may have

$$\varpi_1 \gtrless \tfrac{1}{2}\pi, \quad \rho_1 = \infty;$$

so that then, for any plane section of the surface other than its section by its tangent plane, the point of contact is a point of inflexion.

Or we may have

$$\varpi_1 = \tfrac{1}{2}\pi,$$

and then ρ_1 can be merely finite; but ϖ_1 is equal to $\frac{1}{2}\pi$ only at the point and not everywhere along the plane curve, so that the quantity $d\varpi_1/ds$ does not vanish. Now, in general, we have (§ 126)

$$\left(\frac{2}{\sigma} + 3\frac{d\varpi}{ds}\right)\frac{\sin\varpi}{\rho} + \frac{\cos\varpi}{\rho^2}\frac{d\rho}{ds} = -\Omega,$$

so that, for the asymptotic line, we have

$$\frac{2}{\rho\sigma} = -\Omega.$$

At the point, Ω is the same for the asymptotic line and the plane section, being $Pp'^3 + 3Qp'^2q' + 3Rp'q'^2 + Sq'^3$; and for the plane section, we have $1/\sigma_1 = 0$. Hence, for the plane section,

$$3\frac{d\varpi_1}{ds}\frac{1}{\rho_1} = -\Omega,$$

so that

$$\frac{2}{\rho\sigma} = \frac{3}{\rho_1}\frac{d\varpi_1}{ds}.$$

Again, at the point, W is the same for the asymptotic line and the plane section. Hence for the asymptotic line

$$\frac{1}{\sigma} = W,$$

and for the plane section

$$0 = \frac{d\varpi_1}{ds} - W,$$

so that

$$\frac{d\varpi_1}{ds} = \frac{1}{\sigma}.$$

Consequently

$$2\rho_1 = 3\rho,$$

a result due to Beltrami.

130. Consider, next, geodesics on the surface. We then have

$$D = 0, \quad D_1 = 0, \quad D_2 = 0.$$

No value of the ratio p'/q' is determined by these equations; but we know (§ 92) that any value of the ratio at the point determines uniquely a geodesic on a part of the surface enclosing no singularity.

The direction of the geodesic through the point having maximum or minimum curvature is obtained by making

$$\frac{L\mu^2 + 2M\mu + N}{E\mu^2 + 2F\mu + G},$$

where $\mu = p'/q'$, a maximum or minimum. The necessary condition is

$$\begin{vmatrix} E\mu + F, & F\mu + G \\ L\mu + M, & M\mu + N \end{vmatrix} = 0,$$

that is,

$$W = 0;$$

so that the directions of the particular geodesics are those of the lines of curvature—a known result. Further, the torsion of a geodesic is always given by

$$\frac{1}{\sigma'} = W;$$

hence

(i) at a point of contact of a geodesic with a line of curvature, the torsion is zero;

(ii) if a geodesic be either a plane curve or a line of curvature, it is both;

both being known propositions (§§ 129, 66). Also, as $A = 1/\rho$ for geodesics, we have

$$\begin{aligned}\frac{1}{\sigma'^2} = W^2 &= \frac{1}{\rho}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - \frac{1}{\rho^2} - \frac{1}{\alpha\beta} \\ &= \left(\frac{1}{\alpha} - \frac{1}{\rho}\right)\left(\frac{1}{\rho} - \frac{1}{\beta}\right) \\ &= \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)^2 \sin^2\theta\cos^2\theta,\end{aligned}$$

as before. Manifestly the geodesic of maximum torsion bisects the angle between the lines of curvature.

As regards nul lines on a surface, being conjugate imaginaries on a real surface, their properties are entirely analytical. As $I = 0$ for nul lines, the relation

$$W^2 = AI\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - A^2 - \frac{1}{\alpha\beta} I^2$$

gives

$$W^2 = -A^2.$$

Let r and r' be the analytical quantities corresponding to the radii of curvature of normal sections of the surface through the tangents to the nul lines, and let s and s' be the analytical quantities corresponding to the radii of torsion of the nul lines; then

$$\frac{1}{s} = W_1 = iA_1 = \frac{i}{r},$$

$$\frac{1}{s'} = W_2 = iA_2 = \frac{i}{r'},$$

and therefore

$$\frac{1}{ss'} + \frac{1}{rr'} = 0.$$

131. We have seen that a geodesic can be a line of curvature, and then the curve is plane. It is natural to inquire what analytical combinations of characteristics among lines of curvature, asymptotic lines, geodesics, and nul lines are possible.

(i) When a curve is a line of curvature and asymptotic, then

$$A = 0, \quad W = 0,$$

so that either $I = 0$ or $K = 0$. When we restrict ourselves to real curves on real surfaces, we must have $K = 0$; the surface is developable, and the curve is a generator. Without that restriction, we could have $I = 0$, so that the curve is a nul line.

(ii) When a curve is a line of curvature and geodesic, then

$$W = 0, \quad D = 0, \quad D_1 = 0, \quad D_2 = 0.$$

Thus $\varpi = 0$, so that

$$\frac{1}{\sigma} = 0;$$

the curve is plane, and it arises as a normal plane section of the surface.

(iii) When a curve is a line of curvature and a nul line, then

$$W = 0, \quad I = 0;$$

and therefore $A = 0$. We again have part of case (i).

(iv) When a curve is asymptotic and geodesic, then

$$A = 0, \quad D = 0.$$

Thus

$$\frac{1}{\gamma} = 0, \quad \frac{1}{\rho} = 0, \quad \frac{1}{\rho'} = 0;$$

the curve is a straight line, being a generator of a ruled surface.

(v) When a curve is asymptotic and nul, then

$$A = 0, \quad I = 0;$$

hence $W = 0$. We have case (iii).

(vi) When a curve is geodesic and nul, then

$$I = 0, \qquad D = 0.$$

Hence

$$W = iA, \quad \varpi = 0.$$

The analytical quantities corresponding to the circular curvature and the torsion of the (imaginary) curve are connected by the relation

$$\rho^2 + \sigma^2 = 0.$$

(vii) When a curve is a line of curvature, a geodesic, and is asymptotic, then

$$W = 0, \quad D = 0, \quad A = 0,$$

so that either $I = 0$ or $K = 0$. We again have case (i).

Differential Invariants.

132. The preceding results shew that many of the quantities connected with a surface and with curves on the surface are expressed as functions connected with binary forms. But a set of parameters on a surface can be changed at will, while the quantities themselves are unaltered in value; hence the expression in terms of the new parameters must be equal to the earlier expression. In other words, we have an invariant or a covariant of the forms.

Now it is important to know all the invariants and covariants which can occur, as well as their geometrical significance. It is equally important to know what relations may exist among them, for these will be relations among the geometrical quantities themselves. Moreover, it is desirable to know the tale of invariants of different kinds, such as those involving the fundamental magnitudes of the surface without reference to any assigned curve, and those involving the fundamental magnitudes of the first kind (but not the magnitudes of the second kind) and any assigned curve or curves on the surface.

Differential invariants (or differential parameters) were introduced by Lamé for relations of space. The association with the theory of surfaces was first made* by Beltrami, to whom many of the early results are due. Another method, based more definitely on the pure algebra of the theory of forms, was initiated by Christoffel†, and has led the way to many investigations‡.

Differential invariants of the type considered must belong to the general class of differential invariants which constitute Lie's generalisation of the theory of the concomitants of homogeneous forms. It proves possible to adapt Lie's methods, used in the theory of continuous groups, for the construction of the functions required§; and so, as the process also indicates the amount of independence among the magnitudes constructed, we shall use it for the immediate purpose.

* In his memoir, *Mem. Acc. Bologna*, 2[da] ser., t. viii (1869), pp. 549—590. Beltrami there gives also a sketch of the early history of the subject. An account of the theory, developed on the basis of Beltrami's researches, is given by Darboux in his third volume, pp. 193—217.

† *Crelle*, t. lxx (1869), pp. 46—70.

‡ Special mention should be made of a memoir by Ricci and Levi-Civita, *Math. Ann.*, t. liv (1901), pp. 125—201.

§ It was first effected by Žorawski for one class of the invariants; see his memoir, *Acta Math.*, t. xvi (1893), pp. 1—64. The method was modified to some extent, and the construction of all classes of the invariants up to a certain order, was effected in a memoir by the author, *Phil. Trans.*, (1903), pp. 329—402; and certain new relations among the geometrical magnitudes of a surface are there given.

133. Some simple examples will indicate the kind of invariance which is to be characteristic.

Let the parametric variables p and q be changed to other independent parametric variables p' and q'; and write

$$\mathbf{J} = \frac{\partial p'}{\partial p}\frac{\partial q'}{\partial q} - \frac{\partial p'}{\partial q}\frac{\partial q'}{\partial p} = p_1'q_2' - p_2'q_1',$$

so that $\mathbf{J}$ does not vanish. Let E', F', G' denote the fundamental magnitudes of the first kind with the new variables; and similarly for the other magnitudes. Then

$$\begin{aligned} Edp^2 + 2Fdpdq + Gdq^2 &= ds^2 \\ &= E'dp'^2 + 2F'dp'dq' + G'dq'^2, \end{aligned}$$

and therefore

$$\begin{aligned} E &= E'p_1'^2 + 2F'p_1'q_1' + G'q_1'^2, \\ F &= E'p_1'p_2' + F'(p_1'q_2' + q_1'p_2') + G'q_1'q_2', \\ G &= E'p_2'^2 + 2F'p_2'q_2' + G'q_2'^2. \end{aligned}$$

Hence

$$V^2 = V'^2\mathbf{J}^2;$$

that is, the function V'^2 is equal to the function V^2 save for multiplication by a power of $\mathbf{J}$. We call V^2 a *relative* invariant.

Again, we have

$$X = \frac{1}{V}(y_1z_2 - y_2z_1);$$

hence

$$\begin{aligned} X' &= \frac{1}{V'}\left(\frac{\partial y}{\partial p'}\frac{\partial z}{\partial q'} - \frac{\partial y}{\partial q'}\frac{\partial z}{\partial p'}\right) \\ &= \frac{1}{V'\mathbf{J}}\left(\frac{\partial y}{\partial p}\frac{\partial z}{\partial q} - \frac{\partial y}{\partial q}\frac{\partial z}{\partial p}\right) = X; \end{aligned}$$

and similarly $Y' = Y$, $Z' = Z$. Quantities like X, Y, Z are *absolute* invariants, or (more simply) invariants.

Further,

$$\begin{aligned} Ldp^2 + 2Mdpdq + Ndq^2 &= \frac{ds^2}{\rho} \\ &= L'dp'^2 + 2M'dp'dq' + N'dq'^2, \end{aligned}$$

so that the relations between L', M', N' and L, M, N are the same as those between E', F', G' and E, F, G. Similarly for the derived quantities of the third order; we have

$$\begin{aligned} Pdp^3 &+ 3Qdp^2dq + 3Rdpdq^2 + Sdq^3 \\ &= \frac{d}{ds}\left(\frac{1}{\rho}\right)ds^3 \\ &= P'dp'^3 + 3Q'dp'^2dq' + 3R'dp'dq'^2 + S'dq'^3, \end{aligned}$$

so that

$$P = P'p_1'^3 + 3Q'p_1'^2 q_1' + 3R'p_1'q_1'^2 + S'q_1'^3,$$

$$Q = P'p_1'^2 p_2' + Q'(2p_1'p_2'q_1' + p_1'^2 q_2') + R'(p_2'q_1'^2 + 2p_1'q_1'q_2') + S'q_1'^2 q_2',$$

$$R = P'p_1'p_2'^2 + Q'(2p_1'p_2'q_2' + p_2'^2 q_1') + R'(p_1'q_2'^2 + 2p_2'q_1'q_2') + S'q_1'q_2'^2,$$

$$S = P'p_2'^3 + 3Q'p_2'^2 q_2' + 3R'p_2'q_2'^2 + S'q_2'^3.$$

Then

$$LN - M^2 = (L'N' - M'^2)\,\mathbf{J}^2,$$

$$EN - 2FM + GL = (E'N' - 2F'M' + G'L')\,\mathbf{J}^2,$$

$$\begin{vmatrix} P, & 2Q, & R, & 0 \\ 0, & P, & 2Q, & R \\ Q, & 2R, & S, & 0 \\ 0, & Q, & 2R, & S \end{vmatrix} = \begin{vmatrix} P', & 2Q', & R', & 0 \\ 0, & P', & 2Q', & R' \\ Q', & 2R', & S', & 0 \\ 0, & Q', & 2R', & S' \end{vmatrix} \mathbf{J}^4,$$

the last being the discriminant of the cubic form; thus

$$\frac{1}{V^2}(LN - M^2) = \frac{1}{V'^2}(L'N' - M'^2),$$

$$\frac{1}{V^2}(EN - 2FM + GL) = \frac{1}{V'^2}(E'N' - 2F'M' + G'L'),$$

are invariants, being the two measures of curvature of the surface; and

$$\frac{1}{V^4}\begin{vmatrix} P, & 2Q, & R, & 0 \\ 0, & P, & 2Q, & R \\ Q, & 2R, & S, & 0 \\ 0, & Q, & 2R, & S \end{vmatrix}$$

also is an invariant of the surface.

But we also have covariants, as well as invariants. Let

$$W = \frac{1}{V}\begin{vmatrix} Edp + Fdq, & Fdp + Gdq \\ Ldp + Mdq, & Mdp + Ndq \end{vmatrix},$$

where $W = 0$ is the equation for the lines of curvature. From the foregoing relations, we have

$$\begin{aligned} Edp + Fdq &= E'p_1'dp' + F'(p_1'dq' + q_1'dp') + G'q_1'dq' \\ &= (E'dp' + F'dq')\,p_1' + (F'dp' + G'dq')\,q_1', \end{aligned}$$

and so for the other constituents in W; hence

$$\begin{aligned} W &= \frac{1}{V}\begin{vmatrix} E'dp' + F'dq', & F'dp' + G'dq' \\ L'dp' + M'dq', & M'dp' + N'dq' \end{vmatrix}\begin{vmatrix} p_1', & q_1' \\ p_2', & q_2' \end{vmatrix} \\ &= \frac{1}{V'}\begin{vmatrix} E'dp' + F'dq', & F'dp' + G'dq' \\ L'dp' + M'dq', & M'dp' + N'dq' \end{vmatrix} \\ &= W'. \end{aligned}$$

Thus W is an absolute covariant or (more simply) a covariant. The invariantive character of W is to be expected; for the lines of curvature must be the same, whatever parameters be used.

134. But there are other types of covariants. Take any curve

$$\phi(p, q) = \text{constant};$$

in the new variables, let it be

$$\phi'(p', q') = \text{same constant},$$

so that

$$\phi(p, q) = \phi'(p', q').$$

Then

$$\phi_1 = \phi_1' p_1' + \phi_2' q_1',$$
$$\phi_2 = \phi_1' p_2' + \phi_2' q_2',$$

and therefore

$$\begin{aligned} E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2 &= \phi_1'^2 (Ep_2'^2 - 2Fp_1'p_2' + Gp_1'^2) \\ &\quad + 2\phi_1'\phi_2' \{Ep_2'q_2' - F(p_1'q_2' + p_2'q_1') + Gp_1'q_1'\} \\ &\quad + \phi_2'^2 (Eq_2'^2 - 2Fq_1'q_2' + Gq_1'^2) \\ &= \mathbf{J}^2 (E'\phi_2'^2 - 2F'\phi_1'\phi_2' + G'\phi_1'^2). \end{aligned}$$

Consequently

$$\frac{1}{V^2}(E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2) = \frac{1}{V'^2}(E'\phi_2'^2 - 2F'\phi_1'\phi_2' + G'\phi_1'^2);$$

and therefore, if

$$\Delta(\phi) = \frac{1}{V^2}(E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2),$$

then $\Delta(\phi)$ is an absolute covariant, connected with the curve upon the surface. It is Beltrami's *first differential parameter*.

As $\Delta(\phi)$ is an absolute covariant, so also is $\Delta(\phi + \lambda\psi)$ for all arbitrary constant values of λ. Now

$$\Delta(\phi + \lambda\psi) = \Delta(\phi) + 2\lambda\,\Delta(\phi, \psi) + \lambda^2\Delta(\psi),$$

where

$$\Delta(\phi, \psi) = \frac{1}{V^2}\{E\phi_2\psi_2 - F(\phi_1\psi_2 + \psi_1\phi_2) + G\phi_1\psi_1\}.$$

Hence $\Delta(\phi, \psi)$ is another absolute covariant, connected with two curves, $\phi = \text{constant}$ and $\psi = \text{constant}$, upon the surface. Sometimes it is called an intermediate covariant, sometimes a mixed covariant.

Further, let

$$\Phi(\phi, \psi) = \frac{1}{V} J\left(\frac{\phi, \psi}{p, q}\right);$$

then

$$\Phi'(\phi', \psi') = \frac{1}{V'} J\left(\frac{\phi', \psi'}{p', q'}\right)$$

$$= \frac{1}{V'} J\left(\frac{\phi', \psi'}{p, q}\right) J\left(\frac{p, q}{p', q'}\right)$$

$$= \frac{1}{V'} J\left(\frac{\phi, \psi}{p, q}\right) \frac{1}{\mathbf{J}} = \Phi(\phi, \psi),$$

so that $\Phi(\phi, \psi)$ is another absolute covariant, intermediate to the two curves. But it is to be noted that the covariants so far obtained are not algebraically independent of one another; they are connected by the relation

$$\Delta(\phi)\,\Delta(\psi) - \Delta^2(\phi, \psi) = \Phi^2(\phi, \psi).$$

Again, we have

$$G\phi_1 - F\phi_2 = \phi_1'(Gp_1' - Fp_2') + \phi_2'(Gq_1' - Fq_2')$$

$$= \mathbf{J}q_2'(G'\phi_1' - F'\phi_2') + \mathbf{J}p_2'(F'\phi_1' - E'\phi_2'),$$

so that

$$\frac{G\phi_1 - F\phi_2}{V} = q_2' \frac{G'\phi_1' - F'\phi_2'}{V'} + p_2' \frac{F'\phi_1' - E'\phi_2'}{V'}.$$

Similarly

$$\frac{F\phi_1 - E\phi_2}{V} = q_1' \frac{G'\phi_1' - F'\phi_2'}{V'} + p_1' \frac{F'\phi_1' - E'\phi_2'}{V'}.$$

Hence

$$\frac{\partial}{\partial p}\left(\frac{G\phi_1 - F\phi_2}{V}\right) - \frac{\partial}{\partial q}\left(\frac{F\phi_1 - E\phi_2}{V}\right)$$

$$= \left(p_1' \frac{\partial}{\partial p'} + q_1' \frac{\partial}{\partial q'}\right)\left(\frac{G\phi_1 - F\phi_2}{V}\right) - \left(p_2' \frac{\partial}{\partial p'} + q_2' \frac{\partial}{\partial q'}\right)\left(\frac{F\phi_1 - E\phi_2}{V}\right)$$

$$= \mathbf{J} \frac{\partial}{\partial p'}\left(\frac{G'\phi_1' - F'\phi_2'}{V'}\right) - \mathbf{J} \frac{\partial}{\partial q'}\left(\frac{F'\phi_1' - E'\phi_2'}{V'}\right),$$

on reduction and substitution. Let

$$\Delta_2(\phi) = \frac{1}{V} \frac{\partial}{\partial p}\left(\frac{G\phi_1 - F\phi_2}{V}\right) + \frac{1}{V} \frac{\partial}{\partial q}\left(\frac{-F\phi_1 + E\phi_2}{V}\right);$$

then

$$\Delta_2(\phi) = \Delta_2(\phi'),$$

that is, $\Delta_2(\phi)$ is an absolute covariant. It is Beltrami's *second differential parameter*.

We thus have the set of covariants

$$\Phi(\phi, \psi), \quad \Delta(\phi), \quad \Delta(\phi, \psi), \quad \Delta_2(\phi).$$

By repeating the operations, we have other covariants

$$\Delta(\Delta\phi), \quad \Delta(\phi, \Delta\phi), \quad \Phi(\phi, \Delta\phi),$$

and so on, to any extent. Darboux proves* that any covariant, which involves two or more functions ϕ, ψ, ... and their derivatives, with E, F, G and their derivatives, can be obtained through the adequate repetition of the symbolical operations Δ and Φ.

135. To illustrate the use of these differential parameters and other covariants, let $p' = \phi(p, q)$, $q' = \psi(p, q)$, be taken as new parametric variables; then any arc-element upon the surface can be expressed in the form

$$ds^2 = \bar{E}d\phi^2 + 2\bar{F}d\phi d\psi + \bar{G}d\psi^2.$$

Now

$$\Delta(\phi) = \bar{G}/\bar{V}^2, \quad \Delta(\psi) = \bar{E}/\bar{V}^2, \quad \Delta(\phi, \psi) = -\bar{F}/\bar{V}^2,$$

on substitution; then

$$1/\bar{V}^2 = \Delta(\phi)\Delta(\psi) - \Delta^2(\phi, \psi) = \Phi^2(\phi, \psi),$$

and so

$$\bar{E} = \frac{\Delta(\psi)}{\Phi^2(\phi, \psi)}, \quad \bar{F} = -\frac{\Delta(\phi, \psi)}{\Phi^2(\phi, \psi)}, \quad \bar{G} = \frac{\Delta(\phi)}{\Phi^2(\phi, \psi)}.$$

Consequently the arc-element upon the surface becomes

$$ds^2 = \frac{\Delta(\psi)\,d\phi^2 - 2\Delta(\phi, \psi)\,d\phi d\psi + \Delta(\phi)\,d\psi^2}{\Phi^2(\phi, \psi)}.$$

When the new parametric curves are nul lines for the surface, we must have

$$\Delta(\psi) = 0, \quad \Delta(\phi) = 0;$$

that is, the nul lines for a surface are obtainable by taking two functionally independent solutions of the equation

$$\Delta(\chi) = 0.$$

When the new parametric curves are an orthogonal isometric system for the surface, we must have

$$\Delta(\phi) = \Delta(\psi), \quad \Delta(\phi, \psi) = 0;$$

that is, an orthogonal isometric system for a surface is obtainable by taking two functionally independent solutions of the equations

$$\Delta(\theta) = \Delta(\vartheta), \quad \Delta(\theta, \vartheta) = 0;$$

and then the arc-element is given by

$$ds^2 = \frac{d\theta^2 + d\vartheta^2}{\Delta(\theta)}.$$

* *Treatise*, t. iii, pp. 203, 204.

The variables for the isometric system are connected with the variables for the nul lines by the customary relation (§ 60); for, writing $\chi = \theta \pm i\vartheta$, where θ and ϑ are real, we have

$$\Delta(\chi) = \Delta(\theta) - \Delta(\vartheta) \pm 2i\Delta(\theta, \vartheta),$$

so that the equation for the variables of the nul lines leads to the equations for the variables of the isometric lines.

Again, by direct substitution in the expression for the second differential parameter, we have

$$\Delta_2(\theta) = 0, \quad \Delta_2(\vartheta) = 0;$$

thus both the parametric variables for an orthogonal isometric system satisfy the equation $\Delta_2(\mu) = 0$.

Ex. Taking the arc-element on a surface in the form

$$ds^2 = (1+p^2)\,dx^2 + 2pq\,dx\,dy + (1+q^2)\,dy^2,$$

prove that parallel planes cut a minimal surface in isometric curves.

Lastly for the purpose of immediate illustration, we can prove, by the method adopted in § 134 for $\Delta_2(\phi)$, that

$$\frac{1}{V}\left\{\frac{\partial}{\partial p}\left(\frac{F\phi_2 - G\phi_1}{\Theta}\right) + \frac{\partial}{\partial q}\left(\frac{F\phi_1 - E\phi_2}{\Theta}\right)\right\},$$

where Θ denotes $(E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2)^{\frac{1}{2}}$, is an absolute covariant. In order to obtain its geometrical significance, we specialise one of the new parametric curves, and we take $\phi(p, q) = q$. Then $\phi_1 = 0$, $\phi_2 = 1$, $\Theta = E^{\frac{1}{2}}$; the absolute covariant becomes

$$\frac{1}{V}\frac{\partial}{\partial p}\left(\frac{F}{E^{\frac{1}{2}}}\right) - \frac{1}{V}\frac{\partial}{\partial q}(E^{\frac{1}{2}}),$$

that is, $V\Delta E^{-\frac{3}{2}}$. But (§ 127) this quantity is the geodesic curvature of the curve $q =$ constant; hence we again have Bonnet's result

$$\frac{1}{\gamma} = \frac{1}{V}\left\{\frac{\partial}{\partial p}\left(\frac{F\phi_2 - G\phi_1}{\Theta}\right) + \frac{\partial}{\partial q}\left(\frac{F\phi_1 - E\phi_2}{\Theta}\right)\right\}.$$

136. The results in § 134 stir a larger question. We are challenged with the problem of finding and interpreting all the invariants upon a surface, and all the covariantive functions, which are connected with curves upon the surface and involve E, F, G, L, M, N as well as their derivatives. Merely for purposes of finite enumeration, we shall take derivatives only up to a finite order; and for purposes of precise illustration, we shall take only low orders of the derivatives of the various quantities. Moreover, we only want those covariants which are algebraically independent of one another; our quest is not for an asyzygetically complete system.

As regards the quantities to be included, we shall take E, F, G and their derivatives of the first order; in place of these derivatives, we shall take the six equivalent quantities Γ, Δ, Γ', Δ', Γ'', Δ'' of § 34, as convenient for our purpose, though not convenient if higher derivatives were required. The transformations of these quantities under the transformations of the variables involve the first and the second derivatives of p' and q' with respect to p and q. The laws of change for L, M, N are the same as those for E, F, G; and therefore to the retained order of derivatives of p' and q', we take L, M, N and their derivatives of the first order. But the last are not independent of one another, owing to the Mainardi-Codazzi relations; in place of them, we shall take the four derived magnitudes of the third order P, Q, R, S. We can take one curve, $\phi =$ constant, on the surface; or we can take two independent curves, $\phi =$ constant, and $\psi =$ constant, on the surface; it is no use taking three curves ϕ, ψ, $\chi =$ constants, for there is a functional relation between three functions of two variables. In the first instance, we shall take one curve, $\phi =$ constant, for the present purpose. The quantity ϕ itself will not occur; the relation $\phi = \phi'$ contains no derivatives of p' and q', but it provides the means of obtaining relations between the derivatives of ϕ and ϕ'. Clearly we shall have derivatives of ϕ of the first and the second orders, as these involve the retained order of derivatives of p' and q'. Thus our invariantive functions involve $E, F, G, \Gamma, \Delta, \Gamma', \Delta', \Gamma'', \Delta'', L, M, N, P, Q, R, S,$ ϕ_{10}, ϕ_{01}, ϕ_{20}, ϕ_{11}, ϕ_{02}, where*

$$u_{mn} = \frac{\partial^{m+n} u}{\partial p^m \partial q^n},$$

for all quantities u, and for all values of m and n.

Ultimately we require absolute invariants. These can be obtained as ratios of relative invariants; as the analysis for relative invariants is simpler than for absolute invariants, we construct the relative invariants. If f be any such function, and if f' be the same function under the new variables, our definition is that the relation

$$f = \mathrm{J}^{\mu} f'$$

must be satisfied for some integer value of the index μ.

137. To utilise this equation we have recourse to Lie's theory of continuous groups, particularly to the fundamental proposition† that a continuous group is determined by the aggregate of infinitesimal transformations which it contains. Accordingly, we shall deal only with infinitesimal transformations of p and q which (in Lie's notation) are

$$p' = p + \xi(p, q)\, dt, \quad q' = q + \eta(p, q)\, dt,$$

* This double-suffix notation is convenient for the expression of derivatives of all orders, though it is less convenient than the earlier notation for derivatives of the first order alone.

† *Theorie der Berührungstransformationen*, vol. iii, p. 597; see also Campbell, *Continuous groups*, p. 80.

where powers of dt above the first are neglected; and then, to secure all kinds of relations between p', q', p, q, we take ξ and η to be completely arbitrary functions of p and q. As the quantities retained for our invariants involve derivatives of p' and q' up to the second order, we shall have derivatives of ξ and η of the first and second orders.

As regards **J**, we have

$$\begin{aligned}\mathbf{J} &= (1+\xi_{10}dt)(1+\eta_{01}dt) - \xi_{01}dt\,\eta_{10}dt \\ &= 1 + (\xi_{10}+\eta_{01})\,dt,\end{aligned}$$

on neglecting dt^2. Also

$$f' = f + \frac{df}{dt}dt;$$

hence our defining relation becomes

$$\frac{df}{dt} + \mu(\xi_{10}+\eta_{01})f = 0,$$

which is to be satisfied for all functions ξ and η. We thus need the increment of f; and this arises through the increments of the various quantities it contains.

We have $\phi = \phi'$, and therefore

$$\begin{aligned}\phi_{10} &= p_1'\phi_{10}' + q_1'\phi_{01}' = (1+\xi_{10}dt)\,\phi_{10}' + \eta_{10}dt\,\phi_{01}', \\ \phi_{20} &= \left\{(1+\xi_{10}dt)\frac{\partial}{\partial p'} + \eta_{10}dt\frac{\partial}{\partial q'}\right\}\{(1+\xi_{10}dt)\,\phi_{10}' + \eta_{10}dt\,\phi_{01}'\} \\ &= (1+2\xi_{10}dt)\,\phi_{20}' + 2\eta_{10}dt\,\phi_{11}' + (\xi_{20}\phi_{10}' + \eta_{20}\phi_{01}')\,dt,\end{aligned}$$

on neglecting squares of dt, and so for other derivatives of ϕ. Hence

$$\phi_{10}' - \phi_{10} = -(\xi_{10}\phi_{10}' + \eta_{10}\phi_{01}')\,dt.$$

On the right-hand side we can replace ϕ_{10}' and ϕ_{01}' by ϕ_{10} and ϕ_{01} respectively, as we neglect squares of dt; hence

$$-\frac{d\phi_{10}}{dt} = \xi_{10}\phi_{10} + \eta_{10}\phi_{01}.$$

Similarly for the other derivatives; the required tale of results is:—

$$\left.\begin{aligned}-\frac{d\phi_{10}}{dt} &= \xi_{10}\phi_{10} + \eta_{10}\phi_{01} \\ -\frac{d\phi_{01}}{dt} &= \xi_{01}\phi_{10} + \eta_{01}\phi_{01} \\ -\frac{d\phi_{20}}{dt} &= 2\xi_{10}\phi_{20} + 2\eta_{10}\phi_{11} + \xi_{20}\phi_{10} + \eta_{20}\phi_{01} \\ -\frac{d\phi_{11}}{dt} &= \xi_{10}\phi_{11} + \xi_{01}\phi_{20} + \eta_{10}\phi_{02} + \eta_{01}\phi_{11} + \xi_{11}\phi_{10} + \eta_{11}\phi_{01} \\ -\frac{d\phi_{02}}{dt} &= 2\xi_{01}\phi_{11} + 2\eta_{01}\phi_{02} + \xi_{02}\phi_{10} + \eta_{02}\phi_{01}\end{aligned}\right\}.$$

To obtain the increments of the magnitudes of the surface, we proceed in the same way. We have

$$\begin{aligned} E &= E'p_1'^2 + 2F'p_1'q_1' + G'q_1'^2 \\ &= (1 + 2\xi_{10}dt)\,E' + 2\eta_{10}dt\,F', \end{aligned}$$

which at once gives dE/dt. Similarly for all the magnitudes; the required tale of results is:—

$$\left.\begin{aligned} -\frac{dE}{dt} &= 2\xi_{10}E + 2\eta_{10}F \\ -\frac{dF}{dt} &= \xi_{10}F + \xi_{01}E + \eta_{10}G + \eta_{01}F \\ -\frac{dG}{dt} &= 2\xi_{01}F + 2\eta_{01}G \end{aligned}\right\},$$

$$\left.\begin{aligned} -\frac{dL}{dt} &= 2\xi_{10}L + 2\eta_{10}M \\ -\frac{dM}{dt} &= \xi_{10}M + \xi_{01}L + \eta_{10}N + \eta_{01}M \\ -\frac{dN}{dt} &= 2\xi_{01}M + 2\eta_{01}N \end{aligned}\right\},$$

$$\left.\begin{aligned} -\frac{dP}{dt} &= 3\xi_{10}P + 3\eta_{10}Q \\ -\frac{dQ}{dt} &= 2\xi_{10}Q + \xi_{01}P + 2\eta_{10}R + \eta_{01}Q \\ -\frac{dR}{dt} &= \xi_{10}R + 2\xi_{01}Q + \eta_{10}S + 2\eta_{01}R \\ -\frac{dS}{dt} &= 3\xi_{01}R + 3\eta_{01}S \end{aligned}\right\}.$$

To obtain the increments of Γ, Δ, Γ', Δ', Γ'', Δ'', we can proceed from the equations of the type

$$x_{20} = LX + x_{10}\Gamma + x_{01}\Delta$$

in § 34, noting that X, Y, Z are invariants, and using the preceding results. We find

$$\begin{aligned} -\frac{d\Gamma}{dt} &= \xi_{10}\Gamma - \xi_{01}\Delta + 2\eta_{10}\Gamma' + \xi_{20} \\ -\frac{d\Delta}{dt} &= 2\xi_{10}\Delta + \eta_{10}(2\Delta' - \Gamma) - \eta_{01}\Delta + \eta_{20} \end{aligned}$$

$$\left.\begin{aligned}
-\frac{d\Gamma'}{dt} &= \xi_{01}(\Gamma-\Delta')+\eta_{10}\Gamma''+\eta_{01}\Gamma'+\xi_{11}\\
-\frac{d\Delta'}{dt} &= \xi_{10}\Delta'+\xi_{01}\Delta+\eta_{10}(\Delta''-\Gamma')+\eta_{11}\\
-\frac{d\Gamma''}{dt} &= -\xi_{10}\Gamma''+\xi_{01}(2\Gamma'-\Delta'')+2\eta_{01}\Gamma''+\xi_{02}\\
-\frac{d\Delta''}{dt} &= 2\xi_{01}\Delta'-\eta_{10}\Gamma''+\eta_{01}\Delta''+\eta_{02}
\end{aligned}\right\}.$$

138. The arguments that can occur in a covariantive function f are twenty-one in number, viz. ϕ_{10}, ϕ_{01}, ϕ_{20}, ϕ_{11}, ϕ_{02}; $E, F, G, \Gamma, \Delta, \Gamma', \Delta', \Gamma'', \Delta''$; L, M, N; P, Q, R, S. Denoting any one of them by u, we have

$$\frac{df}{dt}=\sum_u \frac{\partial f}{\partial u}\frac{du}{dt}.$$

The value of $\frac{du}{dt}$ has been obtained for each argument; hence the critical equation becomes

$$-\sum_u \frac{\partial f}{\partial u}\frac{du}{dt}=\mu(\xi_{10}+\eta_{01})f.$$

The equation must be satisfied for all arbitrary functions ξ and η whatever; and therefore the coefficients of all the derivatives of ξ and of η on the two sides must be respectively equal to one another. We thus obtain ten equations in all, arising through the coefficients of ξ_{10}, ξ_{01}; η_{10}, η_{01}; ξ_{20}, ξ_{11}, ξ_{02}; η_{20}, η_{11}, η_{02}. They are:—

$$\begin{aligned}
\mu f = 2E\frac{\partial f}{\partial E}+F\frac{\partial f}{\partial F}+2L\frac{\partial f}{\partial L}+M\frac{\partial f}{\partial M}+3P\frac{\partial f}{\partial P}+2Q\frac{\partial f}{\partial Q}+R\frac{\partial f}{\partial R}&\\
+\Gamma\frac{\partial f}{\partial \Gamma}+2\Delta\frac{\partial f}{\partial \Delta}+\Delta'\frac{\partial f}{\partial \Delta'}-\Gamma''\frac{\partial f}{\partial \Gamma''}&\\
+\phi_{10}\frac{\partial f}{\partial \phi_{10}}+2\phi_{20}\frac{\partial f}{\partial \phi_{20}}+\phi_{11}\frac{\partial f}{\partial \phi_{11}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i}),&
\end{aligned}$$

$$\begin{aligned}
\mu f = F\frac{\partial f}{\partial F}+2G\frac{\partial f}{\partial G}+M\frac{\partial f}{\partial M}+2N\frac{\partial f}{\partial N}+Q\frac{\partial f}{\partial Q}+2R\frac{\partial f}{\partial R}+3S\frac{\partial f}{\partial S}&\\
-\Delta\frac{\partial f}{\partial \Delta}+\Gamma'\frac{\partial f}{\partial \Gamma'}+2\Gamma''\frac{\partial f}{\partial \Gamma''}+\Delta''\frac{\partial f}{\partial \Delta''}&\\
+\phi_{01}\frac{\partial f}{\partial \phi_{01}}+\phi_{11}\frac{\partial f}{\partial \phi_{11}}+2\phi_{02}\frac{\partial f}{\partial \phi_{02}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii}),&
\end{aligned}$$

coming from the coefficients of ξ_{10} and η_{01} respectively; and

$$0 = E\frac{\partial f}{\partial F} + 2F\frac{\partial f}{\partial G} + L\frac{\partial f}{\partial M} + 2M\frac{\partial f}{\partial N} + P\frac{\partial f}{\partial Q} + 2Q\frac{\partial f}{\partial R} + 3R\frac{\partial f}{\partial S}$$

$$- \Delta\frac{\partial f}{\partial \Gamma} + (\Gamma - \Delta')\frac{\partial f}{\partial \Gamma'} + \Delta\frac{\partial f}{\partial \Delta'} + (2\Gamma' - \Delta'')\frac{\partial f}{\partial \Gamma''} + 2\Delta'\frac{\partial f}{\partial \Delta''}$$

$$+ \phi_{10}\frac{\partial f}{\partial \phi_{01}} + \phi_{20}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{02}} \quad \text{.........(I)},$$

$$0 = 2F\frac{\partial f}{\partial E} + G\frac{\partial f}{\partial F} + 2M\frac{\partial f}{\partial L} + N\frac{\partial f}{\partial M} + 3Q\frac{\partial f}{\partial P} + 2R\frac{\partial f}{\partial Q} + S\frac{\partial f}{\partial R}$$

$$+ 2\Gamma'\frac{\partial f}{\partial \Gamma} + (2\Delta' - \Gamma)\frac{\partial f}{\partial \Delta} + \Gamma''\frac{\partial f}{\partial \Gamma'} + (\Delta'' - \Gamma')\frac{\partial f}{\partial \Delta'} - \Gamma''\frac{\partial f}{\partial \Delta''}$$

$$+ \phi_{01}\frac{\partial f}{\partial \phi_{10}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{20}} + \phi_{02}\frac{\partial f}{\partial \phi_{11}} \quad \text{.........(II)},$$

$$0 = \frac{\partial f}{\partial \Gamma} + \phi_{10}\frac{\partial f}{\partial \phi_{20}} \quad \text{.........(III)},$$

$$0 = \frac{\partial f}{\partial \Gamma'} + \phi_{10}\frac{\partial f}{\partial \phi_{11}} \quad \text{.........(IV)},$$

$$0 = \frac{\partial f}{\partial \Gamma''} + \phi_{10}\frac{\partial f}{\partial \phi_{02}} \quad \text{.........(V)},$$

$$0 = \frac{\partial f}{\partial \Delta} + \phi_{01}\frac{\partial f}{\partial \phi_{20}} \quad \text{.........(VI)},$$

$$0 = \frac{\partial f}{\partial \Delta'} + \phi_{01}\frac{\partial f}{\partial \phi_{11}} \quad \text{.........(VII)},$$

$$0 = \frac{\partial f}{\partial \Delta''} + \phi_{01}\frac{\partial f}{\partial \phi_{02}} \quad \text{.........(VIII)},$$

coming from the remaining coefficients. This is the aggregate of equations arising out of the critical equations.

Conversely, a function f, that satisfies these equations in connection with a suitable integer value of μ, possesses the property

$$f = \mathbf{J}^{\mu} f',$$

that is, it is a covariant. Hence what is required for our purpose is the aggregate of algebraically independent functions satisfying these ten equations, the last eight of which are homogeneous and linear in the derivatives of f.

The theory of such equations, as well as the method of integration, is known*; so we proceed first to integrate them, and then apply the theory to indicate an independent aggregate.

139. Consider the last six of the equations, viz. (III)—(VIII), by themselves. All the Poisson-Jacobi conditions of coexistence are satisfied identically. Hence they form a complete Jacobian system. The total number of variables, with respect to which derivatives of f occur, is nine—viz., the three second derivatives of ϕ, and the six quantities Γ, Δ, Γ', Δ', Γ'', Δ''. Thus the total number of algebraically independent integrals, involving some or other of these nine variables, is three; for the number of such integrals is the excess of the number of such variables over the number of equations in a complete Jacobian system. Now it is easy to verify that the quantities

$$\phi_{20}-\Gamma\phi_{10}-\Delta\phi_{01},\quad \phi_{11}-\Gamma'\phi_{10}-\Delta'\phi_{01},\quad \phi_{02}-\Gamma''\phi_{10}-\Delta''\phi_{01},$$

satisfy the equations; and it is manifest that they are algebraically independent of one another. Hence, writing

$$\left.\begin{aligned} a &= \phi_{20}-\Gamma\ \phi_{10}-\Delta\ \phi_{01}\\ b &= \phi_{11}-\Gamma'\phi_{10}-\Delta'\phi_{01}\\ c &= \phi_{02}-\Gamma''\phi_{10}-\Delta''\phi_{01}\end{aligned}\right\},$$

we have a, b, c as the three integrals above indicated.

If, then, we take f to be any function of a, b, c, and of ϕ_{10}, ϕ_{01}, E, F, G, L, M, N, P, Q, R, S, then the six equations are satisfied; and the most general function of those arguments is the most general integral of those six equations. We therefore now limit f to be a function of those arguments; and we need take no further notice of the six equations. To avoid confusion, we denote the function f, in its new form, by g.

Let the equations (i), (ii), (I), (II) be written

$$\mu f=\nabla_1 f,\quad \mu f=\nabla_2 f,\quad 0=\Delta_1 f,\quad 0=\Delta_2 f.$$

We easily find

$$\begin{aligned} &\nabla_1 a = 2a, && \nabla_1 b = b, && \nabla_1 c = 0,\\ &\nabla_2 a = 0, && \nabla_2 b = b, && \nabla_2 c = 2c,\\ &\Delta_1 a = 0, && \Delta_1 b = a, && \Delta_1 c = 2b,\\ &\Delta_2 a = 2b, && \Delta_2 b = c, && \Delta_2 c = 0.\end{aligned}$$

* See the author's *Theory of Differential Equations*, vol. v, chap. iii.

Then as f, now denoted by g, has become a function of a, b, c, ϕ_{10}, ϕ_{01}, E, F, G, L, M, N, P, Q, R, S only, the four equations take the form

$$\mu g = 2E\frac{\partial g}{\partial E} + F\frac{\partial g}{\partial F} + 2L\frac{\partial g}{\partial L} + M\frac{\partial g}{\partial M} + 3P\frac{\partial g}{\partial P} + 2Q\frac{\partial g}{\partial Q} + R\frac{\partial g}{\partial R}$$
$$+ 2a\frac{\partial g}{\partial a} + b\frac{\partial g}{\partial b} + \phi_{10}\frac{\partial g}{\partial \phi_{10}} \quad \text{.....................(i)},$$

$$\mu g = F\frac{\partial g}{\partial F} + 2G\frac{\partial g}{\partial G} + M\frac{\partial g}{\partial M} + 2N\frac{\partial g}{\partial N} + Q\frac{\partial g}{\partial Q} + 2R\frac{\partial g}{\partial R} + 3S\frac{\partial g}{\partial S}$$
$$+ b\frac{\partial g}{\partial b} + 2c\frac{\partial g}{\partial c} + \phi_{01}\frac{\partial g}{\partial \phi_{01}} \quad \text{.....................(ii)},$$

$$0 = E\frac{\partial g}{\partial F} + 2F\frac{\partial g}{\partial G} + L\frac{\partial g}{\partial M} + 2M\frac{\partial g}{\partial N} + P\frac{\partial g}{\partial Q} + 2Q\frac{\partial g}{\partial R} + 3R\frac{\partial g}{\partial S}$$
$$+ a\frac{\partial g}{\partial b} + 2b\frac{\partial g}{\partial c} + \phi_{10}\frac{\partial g}{\partial \phi_{01}} \quad \text{.....................(I)},$$

$$0 = 2F\frac{\partial g}{\partial E} + G\frac{\partial g}{\partial F} + 2M\frac{\partial g}{\partial L} + N\frac{\partial g}{\partial M} + 3Q\frac{\partial g}{\partial P} + 2R\frac{\partial g}{\partial Q} + S\frac{\partial g}{\partial R}$$
$$+ 2b\frac{\partial g}{\partial a} + c\frac{\partial g}{\partial b} + \phi_{01}\frac{\partial g}{\partial \phi_{10}} \quad \text{.....................(II)}.$$

These four equations satisfy the Poisson-Jacobi conditions of coexistence, and so they are a complete system. When we take

$$\text{equation (i)} - \text{equation (ii)} = 0$$

with (I) and (II), we have a complete Jacobian system, each being linear and homogeneous in the derivatives of f. The arguments, with respect to which derivatives are taken, are fifteen in number; this complete Jacobian system contains three equations; and therefore, by the customary theorem, there are twelve algebraically independent solutions.

Further, we take a new equation, given by

$$\text{equation (i)} + \text{equation (ii)} = 0,$$

so that we have substituted two equivalent equations for (i) and (ii). The solutions to be obtained will be homogeneous in certain groups of the quantities; let any one of them be

of degree n_1 in E, F, G,
.... n_2 ... L, M, N,
.... n_3 ... P, Q, R, S,
.... n_4 ... a, b, c,
.... n_5 ... ϕ_{10}, ϕ_{01};

then this new equation is satisfied if

$$2\mu = 2n_1 + 2n_2 + 3n_3 + 2n_4 + n_5,$$

so that $n_3 + n_5$ must be an even integer.

140. Now the three equations

$$\text{(I)}, \quad \text{(II)}, \quad \text{equation (i)} - \text{equation (ii)} = 0,$$

are the complete Jacobian system of the differential equations for the invariants and covariants of the simultaneous system of binary forms

$$\begin{aligned} w_2 &= (E, F, G \between \phi_{01}, -\phi_{10})^2, \\ w_2' &= (L, M, N \between \phi_{01}, -\phi_{10})^2, \\ w_2'' &= (a, b, c \between \phi_{01}, -\phi_{10})^2, \\ w_3 &= (P, Q, R, S \between \phi_{01}, -\phi_{10})^3; \end{aligned}$$

and we therefore require an algebraically complete (not an asyzygetically complete) set of concomitants of these binary forms, the set to contain twelve members. An algebraically complete set is not unique; it can be modified by exclusion and inclusion, provided it remains an algebraically complete set of twelve members.

Such a set can be taken initially as follows:—

$$w_2 = (E, F, G \between \phi_{01}, -\phi_{10})^2,$$

$$V^2 = EG - F^2,$$

$$w_2' = (L, M, N \between \phi_{01}, -\phi_{10})^2,$$

$$T^2 = LN - M^2,$$

$$\begin{aligned} J &= \tfrac{1}{4} J\left(\frac{w_2, w_2'}{\phi_{01}, -\phi_{10}}\right) \\ &= (EM - FL)\,\phi_{01}^2 - (EN - GL)\,\phi_{01}\phi_{10} + (FN - GM)\,\phi_{10}^2, \end{aligned}$$

$$w_2'' = (a, b, c \between \phi_{01}, -\phi_{10})^2,$$

$$I = Ec - 2Fb + Ga,$$

$$\begin{aligned} J' &= \tfrac{1}{4} J\left(\frac{w_2, w_2''}{\phi_{01}, -\phi_{10}}\right) \\ &= (Eb - Fa)\,\phi_{01}^2 - (Ec - Ga)\,\phi_{01}\phi_{10} + (Fc - Gb)\,\phi_{10}^2, \end{aligned}$$

$$w_3 = (P, Q, R, S \between \phi_{01}, -\phi_{10})^3,$$

$$\begin{aligned} \delta = \{E^2S - 3EFR + (EG + 2F^2)\,Q - FGP\}\,\phi_{01} \\ - \{EFS - (EG + 2F^2)\,R + 3FGQ - G^2P\}\,\phi_{10}, \end{aligned}$$

$$\delta' = (ER - 2FQ + GP)\,\phi_{01} - (ES - 2FR + GQ)\,\phi_{10},$$

$$J'' = \tfrac{1}{6} J \left(\frac{w_2, w_3}{\phi_{01}, -\phi_{10}} \right)$$
$$= (EQ - FP)\,\phi_{01}^3 - (2ER - FQ - GP)\,\phi_{01}^2\phi_{10}$$
$$+ (ES + FR - 2GQ)\,\phi_{01}\phi_{10}^2 - (FS - GR)\,\phi_{10}^3.$$

In terms of the members of this algebraically complete set, every other concomitant of the system can be expressed; and each member of the set is a relative invariant or covariant.

To obtain the absolute invariants and covariants, we require the index μ of each of the foregoing quantities, as given by

$$\mu = n_1 + n_2 + n_4 + \tfrac{1}{2}(3n_3 + n_5).$$

We easily find

$$\mu = 2, \text{ for } w_2, V^2, w_2', T^2, w_2'', I;$$
$$\mu = 3, \text{ for } J, J', w_3, \delta';$$
$$\mu = 4, \text{ for } \delta, J'';$$

and therefore *an algebraically complete set of absolute invariants and covariants, eleven in number, is given by*

$$\frac{w_2}{V^2}, \frac{w_2'}{V^2}, \frac{T^2}{V^2}, \frac{w_2''}{V^2}, \frac{I}{V^2}, \frac{J}{V^3}, \frac{J'}{V^3}, \frac{w_3}{V^3}, \frac{\delta'}{V^3}, \frac{\delta}{V^4}, \frac{J''}{V^4}.$$

As already indicated, the system can be modified by the exclusion of some of the retained concomitants and the subsequent inclusion of some of the omitted concomitants, the same in number, and independent of one another when the set is restored to completeness.

Some instances of concomitants, omitted from the system and expressible in terms of its members, can easily be given; they will be deferred until the geometrical significance of the retained concomitants has been established, so that their geometrical significance can be given simultaneously.

141. Two directions at any point of a curve on a surface are specially determined by the curve, viz. the tangent to the curve, and the direction which lies in the tangent plane to the surface and is normal to the curve. These two directions may be denoted by $\frac{dp}{ds}, \frac{dq}{ds}$; and $\frac{dp}{dn}, \frac{dq}{dn}$; respectively.

Now we have, for the tangent,

$$\phi_{10}\frac{dp}{ds} + \phi_{01}\frac{dq}{ds} = 0,$$

together with the universal equation

$$E\left(\frac{dp}{ds}\right)^2 + 2F\frac{dp}{ds}\frac{dq}{ds} + G\left(\frac{dq}{ds}\right)^2 = 1;$$

as in §§ 26, 105, we take

$$\frac{dp}{ds} = w_2^{-\frac{1}{2}}\phi_{01}, \quad \frac{dq}{ds} = -w_2^{-\frac{1}{2}}\phi_{10}.$$

Again, for the direction of the normal in the tangent plane, we have

$$E\frac{dp}{ds}\frac{dp}{dn} + F\left(\frac{dp}{ds}\frac{dq}{dn} + \frac{dp}{dn}\frac{dq}{ds}\right) + G\frac{dq}{ds}\frac{dq}{dn} = 0,$$

and, as dn is an element of arc on the surface, we have

$$E\left(\frac{dp}{dn}\right)^2 + 2F\frac{dp}{dn}\frac{dq}{dn} + G\left(\frac{dq}{dn}\right)^2 = 1;$$

therefore

$$\frac{dp}{dn} = -\frac{1}{V}\left(F\frac{dp}{ds} + G\frac{dq}{ds}\right) = \frac{1}{Vw_2^{\frac{1}{2}}}(-F\phi_{01} + G\phi_{10}),$$

$$\frac{dq}{dn} = \frac{1}{V}\left(E\frac{dp}{ds} + F\frac{dq}{ds}\right) = \frac{1}{Vw_2^{\frac{1}{2}}}(E\phi_{01} - F\phi_{10}).$$

Before proceeding to the identification of the invariants, we obtain the simple interpretation of the Beltrami operators Δ and Φ. We have

$$\frac{d\phi}{ds} = 0.$$

Next, we have

$$\frac{d\phi}{dn} = \phi_{10}\frac{dp}{dn} + \phi_{01}\frac{dq}{dn}$$

$$= \frac{w_2^{\frac{1}{2}}}{V},$$

so that, writing

$$\frac{d\phi}{dn} = B,$$

we have

$$\frac{w_2}{V^2} = B^2,$$

where w_2/V^2 is Beltrami's first differential parameter $\Delta(\phi)$. Also, for any other quantity ψ—such as, for instance, occurs in the equation of a curve, $\psi = \text{constant}$—we have

$$\frac{d\psi}{ds} = \psi_{10}\frac{dp}{ds} + \psi_{01}\frac{dq}{ds}$$

$$= w_2^{-\frac{1}{2}} J\left(\frac{\psi, \phi}{p, q}\right)$$

$$= -\frac{1}{B}\Phi(\phi, \psi),$$

$$\frac{d\psi}{dn} = \psi_{10}\frac{dp}{dn} + \psi_{01}\frac{dq}{dn}$$

$$= \frac{1}{Vw_2^{\frac{1}{2}}}\{E\phi_{01}\psi_{01} - F(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + G\phi_{10}\psi_{10}\}$$

$$= \frac{1}{B}\Delta(\phi, \psi),$$

ds and dn in the differentiation being determined by the curve $\phi = \text{constant}$. Hence (except as to the invariant factor $-B$) Beltrami's invariant $\Phi(\phi, \psi)$ is $\frac{d\psi}{ds}$, and (except as to the factor B) his invariant $\Delta(\phi, \psi)$ is $\frac{d\psi}{dn}$; and repetitions of the Beltrami differential operators Φ and Δ are, effectively, repetitions of the operators $\frac{d}{ds}$ and $\frac{d}{dn}$. Moreover, we have

$$\frac{dp}{ds_q} = E^{-\frac{1}{2}}, \quad \frac{dq}{ds_p} = G^{-\frac{1}{2}},$$

$$\frac{dp}{dn_p} = -\frac{G^{\frac{1}{2}}}{V}, \quad \frac{dp}{dn_q} = -\frac{F}{VE^{\frac{1}{2}}}, \quad \frac{dq}{dn_p} = \frac{F}{VG^{\frac{1}{2}}}, \quad \frac{dq}{dn_q} = \frac{E^{\frac{1}{2}}}{V}.$$

Hence we infer at once the theorem (§ 134) of Darboux, that any covariant, which involves two or more functions ϕ, ψ, ... and their derivatives, with E, F, G and their derivatives, can be obtained through the adequate repetition of the symbolical operations Δ and Φ.

142. Coming more directly to the significance of the invariants in the retained complete aggregate, we shall denote the various geometrical magnitudes by the same symbols as in § 126. We already have

$$\frac{w_2}{V^2} = B^2,$$

where

$$B = \frac{d\phi}{dn}.$$

Next,

$$w_2' = (L, M, N \between \phi_{01}, -\phi_{10})^2$$

$$= w_2\left\{L\left(\frac{dp}{ds}\right)^2 + 2M\frac{dp}{ds}\frac{dq}{ds} + N\left(\frac{dq}{ds}\right)^2\right\}$$

$$= \frac{w_2}{\rho'};$$

and therefore

$$\frac{w_2'}{V^2} = \frac{B^2}{\rho'}.$$

Next,

$$\frac{T^2}{V^2} = \frac{LN - M^2}{EG - F^2}$$

$$= K = \frac{1}{\alpha\beta},$$

the Gaussian measure of curvature. Further,

$$J = (EM - FL)\,\phi_{01}^2 - (EN - GL)\,\phi_{01}\phi_{10} + (FN - GM)\,\phi_{10}^2$$

$$= w_2\left\{(EM - FL)\left(\frac{dp}{ds}\right)^2 + (EN - GL)\frac{dp}{ds}\frac{dq}{ds} + (FN - GM)\left(\frac{dq}{ds}\right)^2\right\}$$

$$= w_2 VW$$

$$= w_2 V\frac{1}{\sigma'};$$

and therefore

$$\frac{J}{V^3} = \frac{B^2}{\sigma'}.$$

Next, we have

$$\phi_{10}p' + \phi_{01}q' = 0,$$

$$\phi_{10}p'' + \phi_{01}q'' + \phi_{20}p'^2 + 2\phi_{11}p'q' + \phi_{02}q'^2 = 0;$$

so that

$$w_2^{\frac{1}{2}}(p'q'' - q'p'') + \frac{1}{w_2}(\phi_{20}\phi_{01}^2 - 2\phi_{11}\phi_{01}\phi_{10} + \phi_{02}\phi_{10}^2) = 0.$$

Hence, using the symbol D of § 125, we have

$$w_2^{\frac{3}{2}}\left[\frac{D}{V} - \Delta p'^3 - (2\Delta' - \Gamma)\,p'^2q' - (\Delta'' - 2\Gamma')\,p'q'^2 - \Gamma''q'^3\right]$$
$$+ \phi_{20}\phi_{01}^2 - 2\phi_{11}\phi_{01}\phi_{10} + \phi_{02}\phi_{10}^2 = 0.$$

and therefore

$$w_2^{\frac{3}{2}}\frac{D}{V} + w_2'' = 0.$$

Consequently,

$$\frac{w_2''}{V^2} = -D\frac{w_2^{\frac{3}{2}}}{V^3}$$

$$= -\frac{B^3}{\gamma},$$

where $\frac{1}{\gamma}$ is the geodesic curvature*.

Next, we have

$$\frac{d}{ds} = \frac{dp}{ds}\frac{\partial}{\partial p} + \frac{dq}{ds}\frac{\partial}{\partial q}$$

$$= w_2^{-\frac{1}{2}}\left(\phi_{01}\frac{\partial}{\partial p} - \phi_{10}\frac{\partial}{\partial q}\right);$$

* See § 105.

so, operating on the equation

$$B^2 = \frac{w_2}{V^2},$$

we have

$$w_2^{\frac{1}{2}}\, 2B\frac{dB}{ds} = \phi_{01}\frac{\partial}{\partial p}\left(\frac{w_2}{V^2}\right) - \phi_{10}\frac{\partial}{\partial q}\left(\frac{w_2}{V^2}\right)$$

$$= \frac{2}{V^2}J',$$

on reduction. Consequently

$$\frac{J'}{V^3} = B^2\frac{dB}{ds}.$$

Similarly, we have

$$\frac{d}{dn} = \frac{dp}{dn}\frac{\partial}{\partial p} + \frac{dq}{dn}\frac{\partial}{\partial q},$$

so that

$$Vw_2^{\frac{1}{2}}\frac{d}{dn} = (G\phi_{10} - F\phi_{01})\frac{\partial}{\partial p} + (E\phi_{01} - F\phi_{10})\frac{\partial}{\partial q};$$

so, operating on the equation

$$B^2 = \frac{w_2}{V^2},$$

we find

$$Vw_2^{\frac{1}{2}}\, 2B\frac{dB}{dn} = \left\{(G\phi_{10} - F\phi_{01})\frac{\partial}{\partial p} + (E\phi_{01} - F\phi_{10})\frac{\partial}{\partial q}\right\}\frac{w_2}{V^2}$$

$$= \frac{2}{V^2}w_2 I - 2w_2'',$$

on reduction*. Hence, substituting for w_2 and w_2'', we have

$$\frac{I}{V^2} = \frac{dB}{dn} - \frac{B}{\gamma}.$$

It is easy also to verify, as regards Beltrami's second differential parameter, that

$$\Delta_2(\phi) = \frac{I}{V^2}.$$

In order to interpret w_3, we must return to the initial definition (§ 40) of the derived magnitudes of the third order whereby they were connected with the variation of the curvature of the normal section of the surface through the tangent to the curve, that is, with the variation of the circular

* In making the reductions, here and elsewhere, the algebra can be greatly abbreviated by using the known property of covariants that they are uniquely determined by their "leading terms." Thus in the foregoing reduction, it is sufficient to take account of the highest power of ϕ_{01}, when once the quantity I (which is the intermediate invariant of w_2 and w_2'') has been segregated.

curvature of the tangent geodesic. Accordingly, let *arc-variation along the geodesic* be denoted by $\frac{d}{dt}$; if a quantity u does not relate to contact, then

$$\frac{du}{ds}=\frac{du}{dt},$$

because the curve and the geodesic touch; if u relates to contact of any order, then

$$\frac{du}{ds}-\frac{du}{dt}$$

is usually not zero because usually there is a non-vanishing geodesic contingence. Examples will occur later. Meanwhile, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{\rho'}\right)&=(P, Q, R, S\between\frac{dp}{ds}, \frac{dq}{ds})^3\\&=w_2^{-\frac{3}{2}}(P, Q, R, S\between\phi_{01}, -\phi_{10})^3\\&=w_2^{-\frac{3}{2}}w_3;\end{aligned}$$

and therefore

$$\frac{w_3}{V^3}=B^3\frac{d}{dt}\left(\frac{1}{\rho'}\right).$$

Next, for H, which denotes the measure of mean curvature of the surface, we have (§ 42)

$$V^2\frac{\partial H}{\partial p}=GP-2FQ+ER,\qquad V^2\frac{\partial H}{\partial q}=GQ-2FR+ES.$$

Hence

$$\begin{aligned}w_2^{\frac{1}{2}}\frac{dH}{ds}&=\phi_{01}\frac{\partial H}{\partial p}-\phi_{10}\frac{\partial H}{\partial q}\\&=\frac{1}{V^2}\delta';\end{aligned}$$

and therefore

$$\frac{\delta'}{V^3}=B\frac{dH}{ds}.$$

Similarly

$$\begin{aligned}Vw_2^{\frac{1}{2}}\frac{dH}{dn}&=(G\phi_{10}-F\phi_{01})\frac{\partial H}{\partial p}+(E\phi_{01}-F\phi_{10})\frac{\partial H}{\partial q}\\&=\frac{1}{V^2}\delta;\end{aligned}$$

and therefore

$$\frac{\delta}{V^4}=B\frac{dH}{dn}.$$

Proceeding as before, we have

$$\begin{aligned}w_2^{\frac{1}{2}}\frac{d}{ds}\left(\frac{B^2}{\rho'}\right)&=\left(\phi_{01}\frac{\partial}{\partial p}-\phi_{10}\frac{\partial}{\partial q}\right)\frac{w_2'}{V^2}\\&=\frac{w_3}{V^2}+\frac{2}{V^2}\frac{w_2'J'-w_2''J}{w_2},\end{aligned}$$

on reduction, so that we have no unidentified covariant on the right-hand side. But, substituting the values of all the covariants as known, we have

$$\frac{d}{ds}\left(\frac{1}{\rho'}\right) = \frac{d}{dt}\left(\frac{1}{\rho'}\right) + \frac{2}{\gamma\sigma'},$$

an illustration of the foregoing statement that arc-variation of a quantity, connected with the contact between a curve and its geodesic tangent, is not the same along the curve as along the geodesic tangent.

The result could also have been derived from the relation

$$\frac{dA}{ds} = \Omega + 2DW,$$

given in § 125; for when the values of A, Ω, D, W are inserted, we have

$$\frac{d}{ds}\left(\frac{1}{\rho'}\right) = \frac{d}{dt}\left(\frac{1}{\rho'}\right) + \frac{2}{\gamma\sigma'}.$$

Similarly, we have

$$\begin{aligned} V w_2^{\frac{1}{2}} \frac{d}{dn}\left(\frac{B^2}{\rho'}\right) &= \left\{(G\phi_{10} - F\phi_{01})\frac{\partial}{\partial p} + (E\phi_{01} - F\phi_{10})\frac{\partial}{\partial q}\right\}\frac{w_2'}{V^2} \\ &= \frac{J''}{V^2} + \frac{2}{V^2}\left(Iw_2' - \frac{JJ'}{w_2} - \frac{w_2' w_2''}{w_2} V^2\right). \end{aligned}$$

Substituting the values of the covariants already known, we find

$$\frac{J''}{V^4} = B^3 \frac{d}{dn}\left(\frac{1}{\rho'}\right) + 2\frac{B^2}{\sigma'}\frac{dB}{ds}.$$

This result completes the establishment of the significance of the covariants in the algebraically complete set as retained.

The following is the aggregate of the results which have been obtained:—

$$\frac{w_2}{V^2} = B^2 = \left(\frac{d\phi}{dn}\right)^2 = \Delta(\phi),$$

$$\frac{w_2'}{V^2} = \frac{B^2}{\rho'},$$

$$\frac{T^2}{V^2} = K = \frac{1}{\alpha\beta},$$

$$\frac{J}{V^3} = \frac{B^2}{\sigma'},$$

$$\frac{w_2''}{V^2} = -\frac{B^3}{\gamma},$$

$$\frac{I}{V^2} = \frac{dB}{dn} - \frac{B}{\gamma};$$

$$\frac{J'}{V^3} = B^2\frac{dB}{ds}.$$

$$\frac{w_3}{V^3} = B^3 \frac{d}{dt}\left(\frac{1}{\rho'}\right) = B^3 \left\{\frac{d}{ds}\left(\frac{1}{\rho'}\right) - \frac{2}{\gamma\sigma'}\right\},$$

$$\frac{\delta}{V^4} = B\frac{dH}{dn},$$

$$\frac{\delta'}{V^3} = B\frac{dH}{ds},$$

$$\frac{J''}{V^4} = B^3 \frac{d}{dn}\left(\frac{1}{\rho'}\right) + 2\frac{B^2}{\sigma'}\frac{dB}{ds}.$$

143. To illustrate the theorem that, to the order of derivation included, this system is algebraically complete, some examples will be taken.

The quantity $EN - 2FM + GL$ is the intermediate invariant of w_2 and w_2', and its index is 2; so we write

$$H = \frac{1}{V^2}(EN - 2FM + GL),$$

H being the mean curvature as usual; thus H is an absolute invariant. Now

$$J^2 = (EN - 2FM + GL)\, w_2 w_2' - T^2 w_2^2 - V^2 w_2'^2;$$

and therefore, when the values of the quantities are substituted from the above set, and a factor $B^4 V^6$ is removed,

$$\frac{1}{\sigma'^2} = \frac{H}{\rho'} - K - \frac{1}{\rho'^2},$$

shewing that H is expressible in terms of the retained quantities. A more familiar form of the relation is

$$\frac{1}{\sigma'^2} = \left(\frac{1}{\alpha} - \frac{1}{\rho'}\right)\left(\frac{1}{\rho'} - \frac{1}{\beta}\right).$$

The quantity $Lc - 2Mb + Na$ is the intermediate invariant of w_2' and w_2'', with index 2; thus $(Lc - 2Mb + Na)V^{-2}$ is an absolute invariant. Now

$$w_2(Lc - 2Mb + Na) = Iw_2' + HV^2 w_2'' - \frac{2}{w_2} JJ' - \frac{2}{w_2} V^2 w_2' w_2'',$$

and H is expressible as above; consequently the new absolute invariant is expressible in terms of members of the system. When the values of the invariants are inserted, we find

$$\begin{aligned}\frac{1}{V^2}(Lc - 2Mb + Na) &= \frac{1}{\rho'}\left(\frac{dB}{dn} + \frac{B}{\gamma}\right) - \frac{B}{\gamma}H - \frac{2}{\sigma'}\frac{dB}{ds} \\ &= \frac{1}{\rho'}\frac{dB}{dn} - \frac{2}{\sigma'}\frac{dB}{ds} - B\frac{\rho'}{\gamma}\left(\frac{1}{\sigma'^2} + \frac{1}{\alpha\beta}\right).\end{aligned}$$

The quantity $ac - b^2$ is the one invariant of w_2'', regarded as a binary form, and its index is 2; hence $(ac - b^2)V^{-2}$ is an absolute invariant of the system. Now

$$J'^2 = Iw_2 w_2'' - (ac - b^2)\, w_2^2 - V^2 w_2''^2,$$

so that $(ac-b^2)V^{-2}$ is expressible in terms of members of the system. When their values are inserted, we find

$$\frac{1}{V^2}(ac-b^2) = -\left(\frac{dB}{ds}\right)^2 - \frac{B}{\gamma}\frac{dB}{dn}.$$

There is an invariant intermediate to w_2, w_2', w_2'', viz.,

$$\begin{vmatrix} E, & F, & G \\ L, & M, & N \\ a, & b, & c \end{vmatrix},$$

and its index is 3. It must be expressible in terms of members of the system; in fact,

$$\begin{vmatrix} E, & F, & G \\ L, & M, & N \\ a, & b, & c \end{vmatrix} = \frac{IJ}{w_2} + \frac{2}{w_2^2}V^2w_2'J' - \frac{1}{w_2}V^2HJ' - \frac{2}{w_2^2}V^2Jw_2'',$$

and H is expressible in terms of members of the system. When the values of the invariants are inserted, we find

$$\begin{vmatrix} E, & F, & G \\ L, & M, & N \\ a, & b, & c \end{vmatrix} V^{-3} = \left(\frac{2}{\rho'} - \frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{dB}{ds} + \left(\frac{dB}{dn} + \frac{B}{\gamma}\right)\frac{1}{\sigma'}.$$

The cubic w_3 has the quadricovariant H_3, where

$$H_3 = (PR-Q^2)\,\phi_{01}^2 - (PS-QR)\,\phi_{01}\phi_{10} + (QS-R^2)\,\phi_{10}^2,$$

and its index is 4. Now

$$J''^2 = w_2w_3\delta' - V^2w_3^2 - w_2^2H_3,$$

and therefore

$$\frac{H_3}{V^4} = B^2\frac{dH}{ds}\frac{d}{dt}\left(\frac{1}{\rho}\right) - B^2\left\{\frac{d}{dt}\left(\frac{1}{\rho}\right)\right\}^2 - \left\{B\frac{d}{dn}\left(\frac{1}{\rho'}\right) + \frac{2}{\sigma'}\frac{dB}{ds}\right\}^2.$$

Similarly, it has a cubicovariant Φ_3, where

$$\Phi_3 = (P^2S - 3PQR + 2Q^3)\,\phi_{01}^3 - (3PQS - 6PR^2 + 3Q^2R)\,\phi_{01}^2\phi_{10}$$
$$+ (-3PRS + 6Q^2S - 3QR^2)\,\phi_{01}\phi_{10}^2 - (PS^2 - 3QRS + 2R^3)\,\phi_{10}^3,$$

and its index is 6. Now

$$w_2^3\Phi_3 = w_2w_3^2\delta - 3w_2w_3\delta'J'' + 2V^2w_3^2J'' - 2J''^3;$$

insertion of the values of the known covariants leads to the value of Φ_3.

144. These examples indicate a way of obtaining the value of a covariant of the system. It is sufficient to express the covariant in terms of the fundamental members of the algebraically complete system and then to substitute, in the expression, the values of those members which occur.

But the process can be used, in the same way, for another purpose. It may happen that geometrical magnitudes exist, which lie within the order

of derivation retained and which do not occur in the set of those which have occurred. They necessarily are covariants of the system, and consequently are expressible in terms of members of the system; thus they can be evaluated in terms of the set of magnitudes retained. Hence we are led to relations among the geometrical magnitudes.

As an example, consider the Gaussian measure of curvature K. Both the quantities dK/ds and dK/dn lie within the order of derivation retained. Now

$$\frac{dK}{ds} = w_2^{-\frac{1}{2}}\left(\phi_{01}\frac{\partial}{\partial p} - \phi_{10}\frac{\partial}{\partial q}\right)K,$$

so that (§ 42)

$$w_2^{\frac{1}{2}}V^2\frac{dK}{ds} = (NP - 2MQ + LR)\phi_{01} - (NQ - 2MR + LS)\phi_{10}.$$

Also

$$w_2^{\frac{1}{2}}V\frac{dK}{dn} = \left\{(G\phi_{10} - F\phi_{01})\frac{\partial}{\partial p} + (E\phi_{01} - F\phi_{10})\frac{\partial}{\partial q}\right\}K,$$

so that

$$w_2^{\frac{1}{2}}V^3\frac{dK}{dn} = \{SEL - R(2EM + FL) + Q(EN + 2FM) - PFN\}\phi_{01}$$
$$- \{SFL - R(2FM + GL) + Q(FN + 2GM) - PGN\}\phi_{10}.$$

When we express these covariants in terms of the members of the complete system, we have

$$w_2^{\frac{1}{2}}V^2\frac{dK}{ds} = \frac{w_2'}{w_2^2 w_3}(w_2 w_3\delta' - V^2 w_3^2) - \frac{2}{w_2^2}JJ'' + \frac{w_3}{w_2^2 w_2'}(w_2 w_2' HV^2 - V^2 w_2'^2).$$

Substituting the values of the covariants which are known, this gives

$$\frac{dK}{ds} = \frac{1}{\rho'}\frac{dH}{ds} + \left(H - \frac{2}{\rho'}\right)\frac{d}{dt}\left(\frac{1}{\rho'}\right) + \frac{2}{\sigma'}\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{4}{B\sigma'^2}\frac{dB}{ds},$$

the relation required. Other forms can be given to it. We have

$$\frac{1}{\sigma'^2} = \frac{H}{\rho'} - K - \frac{1}{\rho'^2},$$

so that

$$\frac{2}{\sigma'^3}\frac{d\sigma'}{ds} = \frac{dK}{ds} - \frac{1}{\rho'}\frac{dH}{ds} - \left(H - \frac{2}{\rho'}\right)\frac{d}{ds}\left(\frac{1}{\rho'}\right)$$
$$= \left(H - \frac{2}{\rho'}\right)\left\{\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{d}{ds}\left(\frac{1}{\rho'}\right)\right\} - \frac{2}{\sigma'}\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{4}{B\sigma'^2}\frac{dB}{ds},$$

and therefore

$$\frac{1}{\sigma'^2}\frac{d\sigma'}{ds} = -\left(H - \frac{2}{\rho'}\right)\frac{1}{\gamma} - \frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{B\sigma'}\frac{dB}{ds},$$

that is,

$$\frac{d}{ds}\left(\frac{1}{\sigma'}\right) = \frac{2}{B\sigma'}\frac{dB}{ds} + \frac{d}{dn}\left(\frac{1}{\rho'}\right) + \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{2}{\rho'}\right)\frac{1}{\gamma}.$$

Proceeding similarly with the covariantive expression for $w_2^{\frac{1}{2}} V^3 \dfrac{dK}{dn}$, we find

$$\frac{dK}{dn} = \frac{1}{\rho'}\frac{dH}{dn} + \left(H - \frac{2}{\rho'}\right)\frac{d}{dn}\left(\frac{1}{\rho'}\right) + \frac{2}{\sigma'}\frac{d}{dt}\left(\frac{1}{\rho'}\right)$$
$$- \frac{4}{B\rho'\sigma'}\frac{dB}{ds} - \frac{2}{\sigma'}\frac{dH}{ds} + \frac{2H}{B\sigma'}\frac{dB}{ds};$$

and this can similarly be changed to a relation

$$\frac{d}{dn}\left(\frac{1}{\sigma'}\right) = -\frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2}{\gamma\sigma'} - \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{2}{\rho'}\right)\frac{1}{B}\frac{dB}{ds} + \frac{dH}{ds}.$$

And so for other instances*.

145. The preceding investigation is concerned with invariants which arise in connection with a single curve upon the surface. It was pointed out (§ 136) that we might consider two, but that we could not consider profitably more than two, independent curves upon the surface. The method adopted for invariants connected with a single curve is applicable to the construction of invariants connected with two curves

$$\phi = \text{constant}, \quad \psi = \text{constant}.$$

We shall develop the results only for the simplest case—when the order of differentiation among the equations of transformation is only the first, instead of the second as in the preceding analysis.

In that simple case, the arguments which can occur in an invariant f are $E, F, G, L, M, N, \phi_{10}, \phi_{01}, \psi_{10}, \psi_{01}$. Every such invariant f satisfies four partial equations constructed in the same way as the ten equations in § 137; the four equations are

$$\mu f = 2E\frac{\partial f}{\partial E} + F\frac{\partial f}{\partial F} + 2L\frac{\partial f}{\partial L} + M\frac{\partial f}{\partial M} + \phi_{10}\frac{\partial f}{\partial \phi_{10}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}},$$
$$\mu f = F\frac{\partial f}{\partial F} + 2G\frac{\partial f}{\partial G} + M\frac{\partial f}{\partial M} + 2N\frac{\partial f}{\partial N} + \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{01}\frac{\partial f}{\partial \psi_{01}}.$$
$$0 = E\frac{\partial f}{\partial F} + 2F\frac{\partial f}{\partial G} + L\frac{\partial f}{\partial M} + 2M\frac{\partial f}{\partial N} + \phi_{10}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{01}},$$
$$0 = 2F\frac{\partial f}{\partial E} + G\frac{\partial f}{\partial F} + 2M\frac{\partial f}{\partial L} + N\frac{\partial f}{\partial M} + \phi_{01}\frac{\partial f}{\partial \phi_{10}} + \psi_{01}\frac{\partial f}{\partial \psi_{10}}.$$

These equations satisfy the Poisson-Jacobi conditions of coexistence. Taking the equation, which arises from the difference of the first two equations, and associating it with the last two, the set of three equations thus constituted is a complete Jacobian system. The number of variables, with respect to which differentiation occurs, is ten, being the total of the arguments which can occur in f; hence the number of independent solutions is seven, being the excess of the number of variables over the number of equations in the complete Jacobian system. Every solution of the equations, that is, every

* Several are given in the memoir by the author already (§ 132) cited.

covariant within the order of differentiation of the relations of transformation, can be expressed in terms of those seven solutions. Moreover, if a solution is homogeneous and

$$\begin{array}{l} \text{of order } m_1 \text{ in } E,\ F,\ G, \\ \ldots\ldots\ldots\ m_2 \ldots\ L,\ M,\ N, \\ \ldots\ldots\ldots\ m_3 \ldots\ \phi_{10},\ \phi_{01}, \\ \ldots\ldots\ldots\ m_4 \ldots\ \psi_{10},\ \psi_{01}, \end{array}$$

the equation, which arises by taking the sum of the first two equations, is satisfied if

$$\mu = m_1 + m_2 + \tfrac{1}{2}(m_3 + m_4).$$

Now the four equations shew that every solution is a concomitant of the system of binary forms

$$(E,\ F,\ G \between \phi_{01},\ -\phi_{10})^2,$$
$$(L,\ M,\ N \between \phi_{01},\ -\phi_{10})^2,$$
$$(\psi_{10},\ \psi_{01} \between \phi_{01},\ -\phi_{10}),$$

or, what is the same thing in an algebraically complete system, is a concomitant of the system of binary forms

$$(E,\ F,\ G \between \psi_{01},\ -\psi_{10})^2,$$
$$(L,\ M,\ N \between \psi_{01},\ -\psi_{10})^2,$$
$$(\phi_{10},\ \phi_{01} \between \psi_{01},\ -\psi_{10}).$$

We shall take them as concomitants of the first of these two systems.

An algebraically complete set of solutions (each one of which is a relative covariant) is made up of the set:

$$u = (E,\ F,\ G \between \phi_{01},\ -\phi_{10})^2,$$
$$u' = (L, M, N \between \phi_{01},\ -\phi_{10})^2,$$
$$V^2 = EG - F^2,$$
$$T^2 = LN - M^2,$$
$$w = (\psi_{10},\ \psi_{01} \between \phi_{01},\ -\phi_{10}),$$
$$J = \begin{vmatrix} E\phi_{01} - F\phi_{10}, & F\phi_{01} - G\phi_{10} \\ L\phi_{01} - M\phi_{10}, & M\phi_{01} - N\phi_{10} \end{vmatrix},$$
$$V^2\Delta(\phi,\ \psi) = \nabla = \begin{vmatrix} E\phi_{01} - F\phi_{10}, & F\phi_{01} - G\phi_{10} \\ \psi_{10}, & \psi_{01} \end{vmatrix}.$$

For these relative covariants, we find

$$\mu = 1, \text{ for } w,$$
$$\mu = 2, \text{ for } u,\ u',\ V^2,\ T^2,\ \nabla,$$
$$\mu = 3, \text{ for } J:$$

and therefore an algebraically complete set of absolute covariants within the order retained is made up of the six functions

$$\frac{w}{V},\ \frac{u}{V^2},\ \frac{u'}{V^2},\ \frac{T^2}{V^2},\ \frac{\nabla}{V^2},\ \frac{J}{V^3}.$$

146. The symbols already used to denote the geometrical quantities related to the curve $\phi =$ constant, will be retained. An elementary arc along the curve $\psi =$ constant will be denoted by ds', and one in the surface normal to that curve will be denoted by dn'. We write

$$B' = \frac{d\psi}{dn'};$$

and take

$\rho'' =$ radius of circular curvature of the geodesic tangent to the curve $\psi =$ constant,

$\sigma'' =$ radius of torsion of that geodesic tangent,

$\lambda =$ angle between the two curves, ϕ, $\psi =$ constants.

Also other simple covariantive forms occur, within the order of variation retained; among them, we note the following:—

$$v = (E,\ F,\ G \between \psi_{01},\ -\psi_{10})^2,$$

$$v' = (L,\ M,\ N \between \psi_{01},\ -\psi_{10})^2,$$

$$\bar{J} = \begin{vmatrix} E\psi_{01} - F\psi_{10}, & F\psi_{01} - G\psi_{10} \\ L\psi_{01} - M\psi_{10}, & M\psi_{01} - N\psi_{10} \end{vmatrix},$$

$$h = L\phi_{01}\psi_{01} - M(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + N\phi_{10}\psi_{10},$$

$$\Lambda = (EM - FL)\,\phi_{01}\psi_{01} - \tfrac{1}{2}(EN - GL)(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + (FN - GM)\,\phi_{10}\psi_{10}.$$

Each of these must, of course, be expressible in terms of the members of the algebraically complete set already retained.

In proceeding to the geometrical interpretation of the foregoing covariants, we shall as far as possible use the earlier results applying in the case of a single curve.

As before, we have

$$\frac{dp}{ds} = u^{-\frac{1}{2}}\phi_{01}, \quad \frac{dq}{ds} = -u^{-\frac{1}{2}}\phi_{10},$$

$$\frac{dp}{ds'} = v^{-\frac{1}{2}}\psi_{01}, \quad \frac{dq}{ds'} = -v^{-\frac{1}{2}}\psi_{10},$$

$$\frac{dp}{dn} = \frac{1}{Vu^{\frac{1}{2}}}(-F\phi_{01} + G\phi_{10}), \qquad \frac{dq}{dn} = \frac{1}{Vu^{\frac{1}{2}}}(E\phi_{01} - F\phi_{10}),$$

$$\frac{dp}{dn'} = \frac{1}{Vv^{\frac{1}{2}}}(-F\psi_{01} + G\psi_{10}), \quad \frac{dq}{dn'} = \frac{1}{Vv^{\frac{1}{2}}}(E\psi_{01} - F\psi_{10}).$$

Then (§ 142) we have

$$\frac{u}{V^2} = B^2 = \left(\frac{d\phi}{dn}\right)^2 = \Delta(\phi), \qquad \frac{u'}{V^2} = \frac{B^2}{\rho'}, \qquad \frac{T^2}{V^2} = K = \frac{1}{\alpha\beta}, \qquad \frac{J}{V^3} = \frac{B^2}{\sigma'},$$

$$\frac{1}{\sigma'^2} = \left(\frac{1}{\alpha} - \frac{1}{\rho'}\right)\left(\frac{1}{\rho'} - \frac{1}{\beta}\right);$$

and similarly for the curve $\psi = $ constant, we have

$$\frac{v}{V^2} = B'^2 = \left(\frac{d\psi}{dn'}\right)^2 = \Delta(\psi), \quad \frac{v'}{V^2} = \frac{B'^2}{\rho''}, \quad \frac{\bar{J}}{V^3} = \frac{B'^3}{\sigma''},$$

$$\frac{1}{\sigma''^2} = \left(\frac{1}{\alpha} - \frac{1}{\rho''}\right)\left(\frac{1}{\rho''} - \frac{1}{\beta}\right).$$

Also

$$\cos\lambda = E\frac{dp}{ds}\frac{dp}{ds'} + F\left(\frac{dp}{ds}\frac{dq}{ds'} + \frac{dp}{ds'}\frac{dq}{ds}\right) + G\frac{dq}{ds}\frac{dq}{ds'}$$

$$= u^{-\frac{1}{2}}v^{-\frac{1}{2}}\nabla;$$

hence

$$\frac{\nabla}{V^2} = BB'\cos\lambda.$$

Similarly

$$\sin\lambda = V\left(\frac{dp}{ds}\frac{dq}{ds'} - \frac{dq}{ds}\frac{dp}{ds'}\right)$$

$$= -u^{-\frac{1}{2}}v^{-\frac{1}{2}}Vw;$$

hence

$$\frac{w}{V} = -BB'\sin\lambda.$$

It follows at once that

$$\frac{w^2}{V^2} + \frac{\nabla^2}{V^4} = B^2B'^2 = \frac{uv}{V^4},$$

giving the expression for v in terms of the members of the algebraically complete aggregate.

Again, we have

$$u^2v' = u'\nabla^2 - 2wJ\nabla - w^2u'V^2 + uw^2(EN - 2FM + GL);$$

when the values of the various quantities are inserted, the equation reduces to

$$\frac{1}{\rho''} = \frac{\cos 2\lambda}{\rho'} + \frac{\sin 2\lambda}{\sigma'} + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\sin^2\lambda.$$

Similarly, we have

$$u^2\bar{J} = J\nabla^2 + \nabla w\{2V^2u' - u(EN - 2FM + GL)\} - V^2Jw^2;$$

when the values of the various quantities are inserted, the equation reduces to

$$\frac{1}{\sigma''} = \frac{\cos 2\lambda}{\sigma'} - \left\{\frac{1}{\rho'} - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right\}\sin 2\lambda.$$

Both these results can be verified, by using Euler's theorem on the curvature of a normal section and the equation for the torsion of a geodesic given in § 107. Moreover, we at once have

$$\frac{\cos\lambda}{\rho''} - \frac{\sin\lambda}{\sigma''} = \frac{\cos\lambda}{\rho'} + \frac{\sin\lambda}{\sigma'},$$

$$\frac{\cos\lambda}{\sigma''} + \left\{\frac{1}{\rho''} - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right\}\sin\lambda = \frac{\cos\lambda}{\sigma'} - \left\{\frac{1}{\rho'} - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right\}\sin\lambda.$$

Next, we have

$$uh = u'\nabla - Jw,$$

and therefore

$$\frac{h}{V^2} = BB'\left(\frac{\cos\lambda}{\rho'} + \frac{\sin\lambda}{\sigma'}\right).$$

We also have

$$vh = v'\nabla + \bar{J}w,$$

leading to

$$\frac{h}{V^2} = BB'\left(\frac{\cos\lambda}{\rho''} - \frac{\sin\lambda}{\sigma''}\right),$$

the two expressions for h being equal.

Also

$$u\Lambda = J\nabla + V^2wu' - \tfrac{1}{2}wu\,(EN - 2FM + GL),$$

and therefore

$$\frac{\Lambda}{V^3} = \frac{BB'\cos\lambda}{\sigma'} - \left\{\frac{1}{\rho'} - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right\} BB'\sin\lambda.$$

We also have

$$v\Lambda = \bar{J}\nabla - V^2wv' + \tfrac{1}{2}wv\,(EN - 2FM + GL),$$

and therefore

$$\frac{\Lambda}{V^3} = \frac{BB'\cos\lambda}{\sigma''} + \left\{\frac{1}{\rho''} - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right\} BB'\sin\lambda,$$

the two expressions for Λ being equal.

Some of the corresponding results relating to differential invariants within the next order of derivation of the equations of transformation of the independent variables, which lead to relations between the geometrical magnitudes involved, are given in the author's memoir cited (§ 132). And further results are derivable, in this field of research, by the use of the same method.

EXAMPLES.

1. Shew that, if two systems of orthogonal curves have constant geodesic curvatures, they are isometric curves.

2. With the notation adopted in § 146, for the circular curvature and the torsion of two curves ϕ=constant and ψ=constant, and for other magnitudes connected with the curves, prove that

$$\frac{1}{\rho'\rho''} = \frac{\sin^2\lambda}{\alpha\beta} + \left(\frac{\cos\lambda}{\rho'} + \frac{\sin\lambda}{\sigma'}\right)^2,$$

$$\frac{1}{\sigma'\sigma''} = -\frac{1}{4}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 \sin^2\lambda + \left[\frac{\cos\lambda}{\sigma'} - \left\{\frac{1}{\rho'} - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right\}\sin\lambda\right]^2.$$

3. The orthogonal trajectories of the curves $\phi(p, q) = c$ are drawn; denoting by $1/\bar{\gamma}$ the geodesic curvature of one of these trajectories, prove that

$$\bar{\gamma}J' = -B^3V^3,$$

where J' is the covariant of § 142. Interpret this result for the case when the curves $\phi = c$ are geodesic parallels.

4. Prove that

$$\frac{d^2\Upsilon}{ds\,dn} - \frac{d^2\Upsilon}{dn\,ds} = -\frac{1}{\gamma}\frac{d\Upsilon}{ds} + \frac{1}{B}\frac{dB}{ds}\frac{d\Upsilon}{dn} = -\frac{1}{\gamma}\frac{d\Upsilon}{ds} - \frac{1}{\bar{\gamma}}\frac{d\Upsilon}{dn},$$

where Υ is any absolute invariant of the surface, $1/\bar{\gamma}$ is the geodesic curvature of the orthogonal trajectory of the curves ϕ = constant, and the other quantities in the equation have the customary significance. Prove the result also when $\Upsilon = B$.

5. Shew that, if Υ denotes any quantity (such as H or K), which is related to any position on the surface and the expression of which is independent of any direction through that position,

$$\frac{d^2\Upsilon}{ds^2} - \frac{d^2\Upsilon}{dt^2} = \frac{1}{\gamma}\frac{d\Upsilon}{dn},$$

where ds, dn, dt are elementary arcs, along any curve, normal to the curve, and along its geodesic tangent, respectively.

6. With the notation of § 142, whereby d/ds denotes arc-variation along any curve, ϕ = constant, while d/dt denotes arc-variation along its geodesic, shew that

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right) - \frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) = 2\,\frac{d}{ds}\left(\frac{1}{\gamma\sigma'}\right) + \frac{3}{\gamma}\frac{d}{dt}\left(\frac{1}{\sigma'}\right),$$

where the quantities ρ', σ', γ have the customary significance in connection with the curve and the surface. Shew also that

$$\frac{d}{ds}\left\{\frac{d}{dt}\left(\frac{1}{\rho'}\right)\right\} - \frac{d}{dt}\left\{\frac{d}{ds}\left(\frac{1}{\rho'}\right)\right\} = \frac{1}{\gamma^3}\frac{d}{dt}\left(\frac{\gamma^2}{\sigma'}\right).$$

7. Writing $u_2 = -F\phi_{01} + G\phi_{10}$, $v_2 = E\phi_{01} - F\phi_{10}$, $u_2' = -M\phi_{01} + N\phi_{10}$, $v_2' = L\phi_{01} - M\phi_{10}$, obtain the following results, as interpretations of the respective covariants :—

$$(L,\ M,\ N \between u_2,\ v_2)^2 = V^4 B^2 \left(H - \frac{1}{\rho'}\right);$$

$$(E,\ F,\ G \between u_2',\ v_2')^2 = V^4 B^2 \left(\frac{H}{\rho'} - K\right);$$

$$(a,\ b,\ c \between u_2,\ v_2)^2 = V^4 B^2 \frac{dB}{dn};$$

$$(EM - FL,\ EN - GL,\ FN - GM \between u_2,\ v_2)^2 = -V^5 \frac{B^2}{\sigma'};$$

$$(Lb - Ma,\ Lc - Na,\ Mc - Nb \between \phi_{01},\ -\phi_{10})^2 = V^3 \left(\frac{B^2}{\rho'}\frac{dB}{ds} + \frac{B^3}{\gamma\sigma'}\right);$$

$$(P,\ Q,\ R,\ S \between u_2,\ v_2)^3 = -V^6 \left\{B^3 \frac{d}{dn}\left(\frac{1}{\rho'}\right) + 2\,\frac{B^2}{\sigma'}\frac{dB}{ds} - B^3 \frac{dH}{dn}\right\};$$

$$(LQ - MP,\ 2LR - MQ - NP,\ LS + MR - 2NQ,\ MS - NR \between \phi_{01},\ -\phi_{10})^3$$
$$= V^4 \left[B^3 \left\{\frac{1}{\rho'}\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{1}{\sigma'}\frac{d}{dt}\left(\frac{1}{\rho'}\right)\right\} + \frac{2}{\rho'}\frac{B^2}{\sigma'}\frac{dB}{ds}\right];$$

$$(aQ - bP,\ 2aR - bQ - cP,\ aS + bR - 2cQ,\ bS - cR \between \phi_{01},\ -\phi_{10})^3$$
$$= -V^4 \left\{B^3 \frac{dB}{ds}\frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{B^4}{\gamma}\frac{d}{dn}\left(\frac{1}{\rho'}\right)\right\}.$$

Evaluate also the following covariantive magnitudes :—

$$\{L^2S - 3LMR + (LN + 2M^2)\,Q - MNP\}\,\phi_{01} - \{LMS - (LN + 2M^2)\,R + 3MNQ - N^2P\}\,\phi_{10};$$

$$(P,\ Q,\ R,\ S \between u_2',\ v_2')^3;$$

$$(aR - 2bQ + cP)\,\phi_{01} - (aS - 2bR + cQ)\,\phi_{10};$$

and the discriminant (§ 133) of the cubic form $(P,\ Q,\ R,\ S \between \phi_{01},\ -\phi_{10})^3$.

CHAPTER VII.

Comparison of Surfaces.

The present chapter deals with three methods used for the comparison of surfaces.

The process of conformal representation had its origin in Lagrange's theory of maps of surfaces of revolution. It was generalised by Gauss, and now has become an important feature in the theory of functions of a complex variable*.

The theory of geodesic representation of surfaces, whereby geodesics are to be conserved, proves to be of somewhat limited range. It was initiated by Beltrami, and has been developed† by Dini and Darboux.

For spherical representation of surfaces and for the use of tangential coordinates, special reference should be made to Darboux's Treatise (Book ii, ch. vii, in vol. i; and Book viii, ch. viii, in vol. iv) and to Bianchi (ch. v, in vol. i).

147. We now pass from the discussion of curves on surfaces to the more direct consideration of surfaces themselves; and we begin with the representation of surfaces on one another. This comparison of surfaces is important alike in practice and in theory. In practice, it includes the whole matter of maps, on whatever principle they are constructed. In theory, it admits the derivation of classes of properties of a surface, by assigning one or other of the simpler surfaces on which a surface can be represented under postulated laws and conditions.

In all the representations, that will be considered, a surface and its representation correspond point by point; and even of such representations, only some will be discussed. Such theories as those of inversion and polar reciprocation belong mainly to the domain of non-differential geometry, and will find no place here. We shall limit the immediate discussion to three kinds of representation, called conformal, spherical, and geodesic respectively.

In conformal representation, the aim is to secure the detailed arrangement of two surfaces so that, when they are compared, there shall be the greatest amount of similarity possible. Thus two spherical surfaces can be coordinated so that the only element lacking from complete similarity is the scale; and even the scale is uniform. But a spherical surface and a plane cannot be coordinated to that degree of similarity.

* For references, see the chapters on conformal representation in the author's *Theory of functions*.

† References are given in §§ 154, 156.

In the geodesic representation of one surface upon another, the purpose is to relate the surfaces in such a way that geodesics on one of them always correspond to geodesics upon the other. Thus if a surface can be geodesically represented upon a plane, its geodesics are derived from straight lines by the equation of relation. But, as will be seen, it is not always possible to make any two surfaces geodesic representatives of one another.

Of spherical representation, we have already had one particular instance in the spherical indicatrix connected with a skew curve. General attention is concentrated more upon the tangent plane at its point of contact than upon the point itself; for we draw, through the centre of a sphere of radius unity, a line parallel to the normal, and we take the point, where this line cuts the spherical surface, as the image of the point on the surface. Thus any configuration on the original surface will have its representation in a configuration on the spherical surface; and the question has to be solved as to how far a spherical image can be substituted for the original configuration or can determine it. It will appear (§ 159), however, that developable surfaces do not admit of spherical representation.

There is still one other correspondence of surfaces of the utmost importance; it occurs when a surface is deformed in any manner that excludes stretching or tearing. The theory of deformation is reserved for a separate chapter.

Conformal Representation.

148. In general, it is impossible to depict two surfaces so that all the arcs upon one of them correspond exactly to all the arcs upon the other; the relation would require that, subject to a uniform scale of change, the one surface would be deformable into the other. But it is possible to bring them into relation with one another so that all the infinitesimal arcs at a point on one shall correspond to all the infinitesimal arcs at a point on the other, the magnification between the arcs being the same for all of them at the two points. The magnification will, however, vary from point to point; so the similarity between the surfaces exists between infinitesimal areas, and is not uniform over the whole surface. It follows from elementary geometry that, because of the uniform magnification at a point, the angle between two corresponding infinitesimal arcs is unaltered; in other words, the indicated relation conserves angles. This relation is called a *conformal representation.* It is essential to the constitution of a geographical map, made as perfect as possible; it secures similarity in detail, even when similarity at large cannot be obtained.

When two surfaces are conformally related to one another, and one of them is conformally related to a third, the other also is conformally related to that third surface, because at corresponding points the three surfaces are

similar to one another. Hence, in order to bring any two surfaces into conformal relation with one another, it is sufficient to have each of them conformally related to some standard surface; and the simpler this standard surface is, the simpler the equations for conformation. We therefore take a plane as the standard surface; and so our problem is the construction of an equation (or of equations) which shall represent a surface conformally on a plane.

Accordingly, at a point P take any arc ds on the given surface; and in the plane at a point P', made arbitrarily to correspond with P, take any arc ds'. Then if

$$m\,ds = ds',$$

where the magnification m is some function of position, not involving the differential elements ds or ds' in any way, and if this relation holds for all arcs at P and P', we secure detailed similarity at the points between the surface and the plane. It is a differential equation; its integral gives a conforming relation; and we have to see what elements of generality it contains that may be at our disposal. Denoting the point in the plane by x and y, the equation is

$$E\,dp^2 + 2F\,dp\,dq + G\,dq^2 = m^{-2}(dx^2 + dy^2).$$

In the first place, it is clear that the lines $x = a'$, $y = b'$, give a double set of orthogonal isometric lines; and we therefore, in effect, have to determine the orthogonal isometric lines of the surface as a practically equivalent problem.

Let $u = a$, $v = b$ be nul lines through P, so that (§ 60)

$$du = \mu\,\{E\,dp + (F + iV)\,dq\}, \quad dv = \mu_0\,\{E\,dp + (F - iV)\,dq\},$$

where μ and μ_0 are free from differential elements; then u and v are conjugate complex variables; and μ and μ_0 are conjugate magnitudes, which may be entirely real in special cases. Also, let

$$z = x + iy, \quad z_0 = x - iy;$$

then our differential equation has the form

$$du\,dv = \frac{E}{m^2\mu\mu_0}\,dz\,dz_0 = \lambda\,dz\,dz_0,$$

where λ is a real positive quantity.

When du vanishes, $dz\,dz_0$ also vanishes, that is, either dz vanishes or dz_0 vanishes. Suppose that du and dz vanish together; then u and z are constant together. But u is a function of two variables p and q, and z is a function of two variables x and y; hence u and z can be constant together, only if some functional relation of the form

$$u = f(z)$$

exists, f denoting any unrestricted function of its argument. Let $g(z_0)$ be the conjugate of $f(z)$; then

$$v = g(z_0),$$

which now is a consequence of the relation between the complex variables u and z.

Had we taken dz_0 instead of dz as the magnitude vanishing with du, we should have had a relation

$$u = F(z_0),$$

with a conjugate consequence $v = G(z)$. The effect is only to change the sign of y, or to take a reflexion of the configuration in the axis of x—a process that does not affect conformation; and so we can regard the first relation as covering both cases.

Moreover, as regards the result, no limitation has been imposed on the function f. Hence we have the theorem:—

> *A surface is conformally represented on a plane by the relation*
>
> $$u = f(x + iy),$$
>
> *where f is any function whatever, and u is a parameter of nul lines of the surface.*

And we infer that every conformal representation of the surface must be derived through some appropriate form of the function f. Also, we have

$$v = g(x - iy),$$

where g is the conjugate of f; hence

$$du\,dv = f'(x + iy)\,g'(x - iy)\,(dx^2 + dy^2),$$

so that

$$\frac{1}{m^2} = \frac{\mu\mu_0}{E} f'(x + iy)\,g'(x - iy),$$

giving the magnitude m associated with the function f.

Different functions f give different conformal representations of the surface on the plane. All these representations on the plane are conformal with one another; that is, different functions f lead to conformal representations of the plane upon itself.

In the above result, the function f is general. It may be made special by the assignment of appropriate conditions or requirements, additional to the conformal quality of the representation.

Ex. Consider the surface

$$ds^2 = \mu^2 (dp^2 + e^{2p} dq^2),$$

manifestly deformable into a surface of revolution, μ being a constant. The Gaussian measure of curvature is $-\mu^{-2}$, so that the surface is a pseudo-sphere (§ 54).

The nul lines are given by

$$dp \pm ie^p dq = 0;$$

that is, we may take

$$e^{-p} + iq = f(x+iy),$$

for any form of the function f; and then we have a copy of the surface on the plane.

The general equation of geodesics on the surface is easily found, by the method of §§ 115, 116, to be

$$e^{-2p} + (q-a)^2 = b,$$

where a and b are arbitrary constants.

If we take $f(x+iy) = x+iy$, then the figures in the map corresponding to geodesics are

$$x^2 + (y-a)^2 = b,$$

that is, circles having their centres on the axis of y.

149. The simplest class of cases, in its analytical aspect, arises when the surface to be represented conformally on a plane is itself a plane. When w is the complex variable of any point in the plane to be represented, the conforming relation is

$$w = f(z);$$

or, as f is any function of its argument, the conforming relation can be taken to be

$$F(w, z) = 0,$$

where F is a quite general function of its two arguments.

Round this equation, and specially connected with the particularisation of the function F, so as to satisfy one or other of special conditions, there has grown a vast body of investigations belonging to the theory of functions of complex variables; and a multitude of functional properties find their elucidation through the conformal representation of the two planes of w and of z. As such investigations really belong to the theory of functions and only secondarily to differential geometry, an account of them must be sought elsewhere*.

150. An extensive class of important cases, which really were the base of Lagrange's investigations into maps and map-making, is provided by surfaces of revolution.

Let r denote the distance of a point on the surface from the axis of revolution, z its height above some plane perpendicular to the axis of revolution, ϕ the longitude (relative to some fixed meridian) of the meridian through the point, and $d\sigma$ the element of arc of the meridian at the point. We have

$$d\sigma^2 = dr^2 + dz^2,$$

* An account of the functional theory of conformal representation of planes will be found in the author's *Theory of functions of a complex variable*, (2nd ed.), chapters xix and xx.

so that r, z, $d\sigma$ are functions of the current parameter of the meridian; and

$$ds^2 = (d\sigma^2 + r^2 d\phi^2)$$

$$= r^2 (d\psi^2 + d\phi^2),$$

where

$$d\psi = \frac{d\sigma}{r}.$$

so that ψ is a function of the current parameter of the meridian only. Then the relation

$$\phi + i\psi = f(x + iy),$$

for any form of the function f, gives a conformal representation of the surface of revolution upon the plane; and the magnification m, being the ratio of an elementary arc on the surface to an elementary arc on the plane, is given by

$$1 = m^2 r^2 f'(x + iy)\, g'(x - iy),$$

where $g(x - iy)$ is the conjugate of $f(x + iy)$.

If we take the conforming relation to be

$$x + iy = F(\phi + i\psi),$$

then

$$m^2 r^2 = F'(\phi + i\psi)\, G'(\phi - i\psi).$$

Manifestly the lines in the map, that represent the meridians on the surface, are given by the equation

$$f(x + iy) + g(x - iy) = \text{constant};$$

and the lines in the map, that represent the parallels of latitude on the surface, are given by the equation

$$f(x + iy) - g(x - iy) = \text{constant}.$$

151. The surface of revolution which occurs most frequently in this connection, through geographical and astronomical problems, is the sphere. The natural current parameter to choose for the meridian is the latitude λ, so that

$$r = a \cos \lambda, \quad d\sigma = a\, d\lambda,$$

where a is the radius of the sphere; and then

$$d\psi = \frac{d\sigma}{r} = \frac{d\lambda}{\cos \lambda},$$

so that

$$\operatorname{sech} \psi = \cos \lambda,$$

the constant of integration being chosen so that λ and ψ vanish together. The conforming relation is

$$\phi + i\psi = f(x + iy);$$

the magnification m is given by

$$\frac{1}{m^2 a^2 \cos^2 \lambda} = f'(x+iy)\, g'(x-iy);$$

and various conformal representations are given by various forms of the function f.

There are three forms of f which are of special importance—two for geographical maps, and one for star-maps.

For the first form, we take $f(u) = u/k$, where k is a real constant; then

$$k(\phi + i\psi) = x + iy,$$

so that

$$x = k\phi, \quad y = k\psi.$$

Thus the meridians (ϕ = constant) and the parallels of latitude (ψ = constant) are two sets of straight lines in the map; they are perpendicular to one another, as is to be expected under the conservation of angles. Meridians, with a constant difference of longitude, become equidistant parallel straight lines. Parallels of latitude, with a constant difference of latitude and lying on the same side of the equator, become parallel straight lines whose distance from one another increases towards the pole. Also $g(x-iy) = \frac{1}{k}(x-iy)$; hence

$$m = \frac{k}{a} \sec \lambda.$$

Thus the magnification is uniform along a parallel of latitude; and it increases along a meridian away from the equator, the increase being very rapid towards the pole. This map is known as *Mercator's projection.*

But though the meridians become straight lines, no other great circles become straight lines.

For the second form, we take

$$\begin{aligned} x + iy &= ke^{i(\phi + i\psi)} \\ &= ke^{-\psi + i\phi}, \end{aligned}$$

so that

$$x = ke^{-\psi} \cos \phi, \quad y = ke^{-\psi} \sin \phi.$$

Also

$$f(x+iy) = \frac{1}{i} \log \frac{x+iy}{k},$$

so that

$$g(x-iy) = -\frac{1}{i} \log \frac{x-iy}{k};$$

hence

$$m^2 a^2 \cos^2 \lambda = x^2 + y^2,$$

and therefore

$$m = \frac{k}{a} e^{-\psi} \sec \lambda = \frac{k}{a} \frac{1}{1 + \sin \lambda}.$$

The meridians ($\phi = \text{constant}$) are represented by the concurrent straight lines

$$y = x \tan \phi.$$

The parallels of latitude ($\psi = \text{constant}$, $\lambda = \text{constant}$) are the concentric circles

$$x^2 + y^2 = k^2 e^{-2\psi} = k^2 \frac{1 - \sin \lambda}{1 + \sin \lambda},$$

of course orthogonal to the concurrent meridian lines. This map is known as the *stereographic projection*; the South pole is the origin of projection.

For the third form, we take

$$\begin{aligned} x + iy &= k e^{ic(\phi + i\psi)} \\ &= k e^{-c\psi + ic\phi}, \end{aligned}$$

where k and c are real constants; and c is different from unity, being a disposable constant used to secure some property or to satisfy some special condition. We have

$$x = k e^{-c\psi} \cos c\phi, \quad y = k e^{-c\psi} \sin c\phi.$$

Also,

$$\phi + i\psi = \frac{1}{ic} \log \frac{x + iy}{k} = f(x + iy),$$

$$\phi - i\psi = -\frac{1}{ic} \log \frac{x - iy}{k} = g(x - iy);$$

hence

$$m^2 a^2 \cos^2 \lambda = c^2 (x^2 + y^2),$$

so that

$$\begin{aligned} m &= \frac{ck}{a} e^{-c\psi} \sec \lambda \\ &= \frac{ck}{a} \frac{(1 - \sin \lambda)^{\frac{1}{2}(c-1)}}{(1 + \sin \lambda)^{\frac{1}{2}(c+1)}}. \end{aligned}$$

The meridians ($\phi = \text{constant}$) are the concurrent straight lines

$$y = x \tan c\phi;$$

the parallels of latitude ($\lambda = \text{constant}$) are the concentric circles

$$x^2 + y^2 = k^2 e^{-2c\psi} = k^2 \left(\frac{1 - \sin \lambda}{1 + \sin \lambda}\right)^c,$$

of course orthogonal to the concurrent meridian lines.

The representation is used for star-maps; and the constant c is determined, for any one map, by making the magnification the same at the parallels of highest and lowest latitude on the map. But these parallels must not be equidistant from the equator.

152. The conformal representation of a surface of revolution on a plane was first effected by Lagrange*. The general conformal representation of any surface upon any other surface is due† to Gauss, who obtained the necessary results for a number of problems and applied them to geodesy. There are many memoirs on the subject by other investigators; but the geometrical relations of the surfaces considered soon become merged in the analytical results, and the subject passes into the range of the theory of functions.

Some results are appended as examples.

Ex. 1. A plane map is made of a surface of revolution; shew that the curvature in the map of a meridian at a point ψ is $\dfrac{\partial}{\partial\psi}\left(\dfrac{1}{mr}\right)$, and that the curvature in the map of a parallel of latitude at a point ϕ is $\dfrac{\partial}{\partial\phi}\left(\dfrac{1}{mr}\right)$.

Ex. 2. A plane map is made of a surface of revolution, so that the meridians and the parallels of latitude are circles. Shew that, if r and θ are the polar coordinates of the point in the map which represents the point ϕ, ψ on the surface,

$$\frac{\cos\theta}{r} = -2ac\,\{ae^{2c\psi}\cos 2\,(c\phi+g)+b\cos(g+h)\},$$

$$\frac{\sin\theta}{r} = \quad 2ac\,\{ae^{2c\psi}\sin 2\,(c\phi+g)+b\sin(g+h)\},$$

where a, b, c, g, h are constants: and prove that the centres of the meridians and the centres of the parallels of latitude in the map lie on two perpendicular straight lines.

Ex. 3. Shew that, if x, y, z be a point on a sphere of radius a, every conformal representation of the sphere on the plane x', y' is given by

$$x'+iy'=f\left(\frac{x+iy}{a+z}\right),$$

for varying forms of the function f.

Can f be determined so that all great circles become straight lines in the map?

Ex. 4. Shew that rhumb lines of the meridians on a sphere become straight lines in Mercator's projection and equiangular spirals in a stereographic projection.

Ex. 5. In a star-map (§ 151), shew that the magnification is a minimum for the parallel of latitude $\sin^{-1}c$; and obtain an expression for the deviation of this parallel from the middle parallel of the map.

Ex. 6. A point on an oblate spheroid of eccentricity e is determined by its longitude ϕ and its geographical latitude λ. Shew that a map of the surface on the plane is given by the equation

$$x+iy=f(\phi+i\vartheta),$$

where

$$\vartheta=\operatorname{sech}^{-1}(\cos\lambda)-e\tanh^{-1}(e\sin\lambda).$$

Discuss the maps for the forms of f which correspond to the Mercator's projection, the stereographic projection, and the star-map for the sphere, especially in the cases where powers of e higher than the second can be neglected.

* See his collected works, vol. iv, pp. 635—692.

† *Ges. Werke*, t. iv, pp. 259—340.

153. It may be added that, for practical purposes, various other projections of the sphere are adopted; but they do not possess the conformal property. Three of them are more important than the rest*.

Thus there is a perspective projection. An origin, not on the surface of the sphere and not its centre, is joined to every point of a curve on the sphere; we obtain a perspective projection of the curve on a plane, when we take a section of the cone by the plane. (When the origin is taken on the sphere and the plane of section is parallel to the tangent plane at the origin, we have a stereographic projection.)

There is an orthographic projection. It is the special case of a perspective projection when the origin of perspective moves off to infinity: so that, in effect, we have projection by a cylinder.

There are central (or gnomonic) projections. The origin of projection is taken to be the centre of the sphere; the same construction as for perspective projection is made, and the plane of projection is a tangent plane. When the tangent plane is taken at the pole, the projection is called central polar. When the tangent plane is taken at a point on the equator, the projection is called central equatorial.

Geodesic Representation.

154. The fundamental property of conformal representation—that a surface and its map should be similar to one another in minute detail at every point, though the similarity cannot be secured for any full extent owing to the variation of the magnification—is not the only useful quality that may be required in the comparison of surfaces. Whether for charts, or for deformations of surfaces, or for other purposes of representation of a surface, it is manifestly desirable to know the possibilities of ranging two surfaces together in such a fashion that geodesics upon one of them correspond to geodesics upon the other—in general, that is to say, and not merely some special family. Thus to take the simplest instance, consider a central projection of a sphere, which has just been mentioned; the great circles, which are geodesics on the sphere, are projected into straight lines, which are geodesics on the tangent plane, a property that is of manifest importance in maps of the heavens. We thus are faced with the question of the geodesic representation of two surfaces upon one another, such that they correspond point by point; and the simplest form of the question arises when we seek for the representation of such surfaces (if any) as will allow the image of their geodesics to become straight lines upon a plane. In this limited form, the question was propounded by Beltrami; he shewed† that the only surfaces which can be thus represented are those of constant curvature. The result can be established as follows.

* A number are set out by Tissot, *Comptes Rendus*, t. l (1860), p. 475.

† *Ann. di Mat.*, t. vii (1866), p. 185.

Choose a family of geodesics and the orthogonal geodesic parallels as the parametric curves on the surface; then the arc-element has the form

$$ds^2 = dp^2 + D^2 dq^2,$$

and the general equation of geodesics becomes (§§ 68, 92)

$$\frac{d^2q}{dp^2} = -DD_1\left(\frac{dq}{dp}\right)^3 - \frac{D_2}{D}\left(\frac{dq}{dp}\right)^2 - 2\frac{D_1}{D}\frac{dq}{dp}.$$

Among the geodesics on the surface are the family $q =$ constant; and all the geodesics are to become straight lines on the plane of representation. Thus we can take q as one of the variables in this plane; if w denotes the other variable, the equation

$$Aw + Bq + C = 0$$

is to represent geodesics, that is, this equation is to be the primitive of the differential equation of all the geodesics for an appropriately determined magnitude w, as a function of p and q. Now

$$A\left(w_1 + w_2\frac{dq}{dp}\right) + B\frac{dq}{dp} = 0,$$

and therefore

$$\frac{d^2q}{dp^2}\left(w_1 + w_2\frac{dq}{dp}\right) = \frac{dq}{dp}\left\{w_{11} + 2w_{12}\frac{dq}{dp} + w_{22}\left(\frac{dq}{dp}\right)^2 + w_2\frac{d^2q}{dp^2}\right\},$$

that is,

$$\frac{d^2q}{dp^2} = \frac{w_{22}}{w_1}\left(\frac{dq}{dp}\right)^3 + 2\frac{w_{12}}{w_1}\left(\frac{dq}{dp}\right)^2 + \frac{w_{11}}{w_1}\frac{dq}{dp}.$$

This differential equation is equivalent to the postulated integral equation, and so it must be the same as the general equation of the geodesics. Hence

$$\frac{w_{22}}{w_1} = -DD_1, \quad 2\frac{w_{12}}{w_1} = -\frac{D_2}{D}, \quad \frac{w_{11}}{w_1} = -2\frac{D_1}{D}.$$

From the third of these relations we have

$$D^2 w_1 = \frac{1}{Q^3},$$

where Q is any function of q alone; and from the second of the relations we have

$$Dw_1^2 = \frac{1}{P^3},$$

where P is any function of p alone. Hence

$$D = \frac{P}{Q^2}, \quad w_1 = \frac{Q}{P^2};$$

and therefore

$$w = Q_1 + Q\int\frac{dp}{P^2},$$

where Q_1 is any function of q alone. In order that the first relation may be satisfied, we must have

$$Q_1'' + Q'' \int \frac{dp}{P^2} = w_{22}$$
$$= -DD_1 w_1$$
$$= -\frac{P'}{PQ^3};$$

and in this form the relation must be satisfied identically, for otherwise there would be a relation between the independent variables. Differentiating with respect to p, we have

$$\frac{Q''}{P^2} = -\frac{1}{Q^3}\frac{d}{dp}\left(\frac{P'}{P}\right),$$

that is,

$$Q^3 Q'' = -P^2 \frac{d}{dp}\left(\frac{P'}{P}\right),$$

and so each side of this equation must be equal to a constant, say a. Thus

$$P^2 \frac{d}{dp}\left(\frac{P'}{P}\right) = a,$$

and therefore

$$\left(\frac{P'}{P}\right)^2 = -\frac{a}{P^2} + b,$$

where b is a constant; hence

$$P'^2 = bP^2 - a,$$

and so

$$P'' = bP.$$

The Gauss measure of curvature of the surface is given by

$$K = -\frac{1}{D}\frac{\partial^2 D}{\partial p^2}$$
$$= -\frac{P''}{P} = -b,$$

and so is constant; hence we have Beltrami's result that *the only surfaces, which can be geodesically represented on a plane, are those with a constant measure of curvature.*

155. There are three cases to consider, according as the constant measure is zero, positive, or negative.

First, let the constant measure be zero. Then

$$P'' = 0,$$

and so

$$P = a'p,$$

where a' is a constant; no generality is lost by making the unexpressed additive constant of integration equal to zero. Also

$$Q^3Q'' = - PP'' + P'^2 = a'^2,$$

and therefore

$$Q'^2 = - \frac{a'^2}{Q^2} + c';$$

hence

$$(c'Q^2 - a'^2)^{\frac{1}{2}} = c'q,$$

again making the additive constant of integration equal to zero without any loss of generality. Hence the surface is

$$\begin{aligned} ds^2 &= dp^2 + \frac{c'^2a'^2p^2}{(c'^2q^2 + a'^2)^2} dq^2 \\ &= dp^2 + p^2 dq'^2, \end{aligned}$$

on changing the variable q. Also we had

$$Q_1'' + Q'' \int \frac{dp}{P^2} = - \frac{P'}{PQ^3},$$

which, on substitution for P and Q, gives

$$Q_1'' = 0,$$

so that

$$Q_1 = a''q + b''.$$

But Q_1 is an additive part of w, which appears in the equation

$$Aw + Bq + C = 0;$$

hence no generality is lost by taking $a'' = 0$, $b'' = 0$. Thus

$$w = Q \int \frac{dp}{P^2} = - \frac{Q}{a'^2 p}.$$

Now

$$\frac{c'a'dq}{c'^2q^2 + a^2} = dq',$$

that is,

$$c'q = a' \tan q',$$

and

$$c'Q^2 = a'^2 \sec^2 q'.$$

Hence, except as to constant factors,

$$w = \frac{1}{p} \sec q', \quad q = \tan q';$$

and therefore the geodesics on the surfaces, having their arc-element in the form

$$ds^2 = dp^2 + p^2 dq'^2,$$

are given by the equation

$$Aw + Bq + C = 0,$$

that is, by the equation

$$A'p \sin q' + B'p \cos q' + C' = 0.$$

Secondly, let the measure of curvature be positive and equal to a^{-2}. Then

$$\frac{1}{D}\frac{\partial^2 D}{\partial p^2} = -\frac{1}{a^2},$$

so that

$$D = Q_3 \sin\frac{p}{a} + Q_4 \cos\frac{p}{a}.$$

But

$$D = \frac{P}{Q^2};$$

no generality is lost by taking

$$Q_4 = 0, \quad Q_3 = \frac{1}{Q^2}, \quad P = \sin\frac{p}{a}.$$

Also

$$Q^3 Q'' = -PP'' + P'^2 = \frac{1}{a^2},$$

and therefore

$$Q'^2 = -\frac{1}{a^2 Q^2} + c',$$

leading to

$$\begin{aligned} a^2 c' Q^2 &= 1 + (ac'q + b')^2 \\ &= 1 + a^2 c'^2 q^2, \end{aligned}$$

without loss of generality. Let

$$dq' = \frac{1}{a}\frac{dq}{Q^2} = \frac{ac'dq}{1 + a^2 c'^2 q^2},$$

so that

$$\begin{aligned} ac'q &= \tan q', \\ a^2 c' Q^2 &= \sec^2 q'. \end{aligned}$$

Write

$$p = ap';$$

then the arc-element is

$$\begin{aligned} ds^2 &= dp^2 + \frac{P^2}{Q^4} dq^2 \\ &= a^2 (dp'^2 + \sin^2 p' dq'^2). \end{aligned}$$

Also

$$Q_1'' + Q'' \int \frac{dp}{P^2} = -\frac{P'}{PQ^3},$$

which, on substitution for P and Q, gives

$$Q_1'' = 0.$$

As Q_1 is an additive part of w, no loss of generality (so far as the geodesics are concerned) is incurred by taking $Q_1 = 0$. Then

$$\begin{aligned} w &= Q_1 + Q \int \frac{dp}{P^2} \\ &= -aQ \cot\frac{p}{a} = -aQ \cot p'. \end{aligned}$$

The geodesics on the surfaces, having their arc-element in the form

$$ds^2 = a^2(dp'^2 + \sin^2 p' dq'^2),$$

are given by the equation

$$A' \sin q' \sin p' + B' \cos q' \sin p' + C' \cos p' = 0.$$

The sphere, of course, is one of the included surfaces; the last equation shews that the geodesics on the sphere lie on planes through the centre, that is, are great circles.

Thirdly, let the measure of curvature be negative and equal to a^{-2}. The analysis is the same as for the second case, save that hyperbolic functions occur instead of circular functions. The result is that geodesics on the surfaces, having their arc-element in the form

$$ds^2 = a^2(dp'^2 + \sinh^2 p' dq'^2),$$

are given by an equation

$$A' \sin q' \sinh p' + B' \cos q' \sinh p' + C' \cosh p' = 0.$$

The Cartesian coordinates for the plane upon which the geodesics are represented as straight lines are

$$\begin{aligned} x &= p \sin q', & y &= p \cos q'; \\ x &= \sin q' \tan p', & y &= \cos q' \tan p'; \\ x &= \sin q' \tanh p', & y &= \cos q' \tanh p'; \end{aligned}$$

in the respective cases.

156. It thus appears that the variety of surfaces which can be represented geodesically upon a plane is gravely limited; and so it is natural to enquire what surfaces can be represented geodesically upon one another, without any restriction to a particular surface as that upon which the representation is to be effected. A solution of the problem, though initially not complete, was given* by Dini; a lacuna was supplied† by Lie; and another solution has been given‡ by Darboux.

In order to effect the geodesic representation of one surface upon another, it is necessary and sufficient to secure that the general equation of the geodesics, viz.,

$$\frac{d^2q}{dp^2} = \Gamma'' \left(\frac{dq}{dp}\right)^3 + (2\Gamma' - \Delta'') \left(\frac{dq}{dp}\right)^2 + (\Gamma - 2\Delta') \frac{dq}{dp} - \Delta,$$

should be the same for the two surfaces. This requires that the quantities

$$\Gamma'', \quad 2\Gamma' - \Delta'', \quad \Gamma - 2\Delta', \quad \Delta,$$

* *Ann. di Mat.*, 2a Ser., t. iii (1869), pp. 269—293.

† *Math. Ann.*, t. xx (1882), p. 421.

‡ In the chapter, pp. 40—65, of the third volume of his treatise.

should be the same for the two surfaces. The equations thus obtained are considerably simplified, if the same parametric curves are orthogonal on either surface; still greater is the simplification if those parametric curves are orthogonal on both surfaces.

But is this possible? The answer to the question is to be found in the following theorem, due* to Tissot:—

In any birational correspondence between the real points of two real surfaces, an orthogonal system on one surface exists having an orthogonal system on the other as its homologue; and the system is unique, unless the correspondence is conformal, or unless nul lines are homologous to nul lines.

Let the arc-elements on the two surfaces be

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2, \quad ds'^2 = E'dp^2 + 2F'dpdq + G'dq^2.$$

When the correspondence between the surfaces is a conformal representation, we must have

$$ds'^2 = m^2 ds^2,$$

where m is independent of the differential elements; hence, in that case, we should have

$$\frac{E'}{E} = \frac{F'}{F} = \frac{G'}{G},$$

or, if we write

$$F'G - FG' = A, \quad G'E - GE' = B, \quad E'F - EF' = C,$$

the conditions for conformal representation are

$$A = 0, \quad B = 0, \quad C = 0.$$

When the nul lines of one family on the first surface are homologues of the nul lines of one family on the second surface, the equations $ds^2 = 0$ and $ds'^2 = 0$ have one root dp/dq common; its value is given by

$$\frac{dp^2}{F'G - FG'} = \frac{2dpdq}{G'E - GE'} = \frac{dq^2}{E'F - EF'},$$

that is, by

$$\frac{dp^2}{A} = \frac{2dpdq}{B} = \frac{dq^2}{C};$$

and the condition is

$$B^2 - 4AC = 0.$$

We need not consider the case when both families of nul lines are homologous with both families of nul lines; for then

$$Edp^2 + 2Fdpdq + Gdq^2 = 0, \quad E'dp^2 + 2F'dpdq + G'dq^2 = 0,$$

would be the same equations, and we should have

$$\frac{E'}{E} = \frac{F'}{F} = \frac{G'}{G},$$

that is, we should have the preceding case of conformal representation.

* *Nouv. Ann. Math.*, 2$^{\text{me}}$ Sér., t. xvii (1878), p. 151.

If possible, then, let δp, δq; and $\delta' p$, $\delta' q$; represent an orthogonal pair of directions on both surfaces. We have

$$E\,\delta p\delta' p + F\,(\delta p\delta' q + \delta q\delta' p) + G\,\delta q\delta' q = 0,$$
$$E'\delta p\delta' p + F'(\delta p\delta' q + \delta q\delta' p) + G'\delta q\delta' q = 0;$$

and therefore

$$\delta p\delta' p = \theta A,$$
$$\delta p\delta' q + \delta q\delta' p = \theta B,$$
$$\delta q\delta' q = \theta C.$$

Hence the directions $\delta q/\delta p$ and $\delta' q/\delta' p$ are the roots of the quadratic

$$At^2 - Bt + C = 0.$$

We thus have a unique pair of orthogonal corresponding lines, unless either the quadratic is evanescent so that A, B, C vanish, or the quadratic has equal roots so that $B^2 = 4AC$. The former exception gives rise to conformal representation. The latter requires that one set of nul lines should be homologous, a correspondence that is imaginary for real surfaces. Hence we have Tissot's theorem.

157. Deferring for the moment the two possible exceptions, let us assume that the two surfaces have, in common, a unique system of orthogonal curves. We take them as parametric curves, so that the arc-elements on the two surfaces are

$$ds^2 = E\,dp^2 + G\,dq^2, \quad ds'^2 = E'dp^2 + G'dq^2.$$

The general equation of geodesics on the first surface is

$$\frac{d^2q}{dp^2} = \left(\frac{dq}{dp}\right)^3 \frac{-G_1}{2E} + \left(\frac{dq}{dp}\right)^2 \left(\frac{E_2}{E} - \frac{G_2}{2G}\right) + \frac{dq}{dp}\left(\frac{E_1}{2E} - \frac{G_1}{G}\right) + \frac{E_2}{2G};$$

and the general equation of geodesics on the second surface has the same form. If the two surfaces can be represented geodesically upon one another, the two general equations must be the same; so the necessary and sufficient conditions are

$$\frac{G_1}{E} = \frac{G_1'}{E'},$$
$$\frac{E_2}{E} - \frac{G_2}{2G} = \frac{E_2'}{E'} - \frac{G_2'}{2G'},$$
$$\frac{E_1}{2E} - \frac{G_1}{G} = \frac{E_1'}{2E'} - \frac{G_1'}{G'},$$
$$\frac{E_2}{G} = \frac{E_2'}{G'}.$$

From the second of these, we have

$$\frac{E^2}{G} = \frac{E'^2}{G'}P^2,$$

where P is a function of p only; and from the third, we have

$$\frac{G^2}{E} = \frac{G'^2}{E'} Q^3,$$

where Q is a function of q only. Thus

$$E = E'P^2Q, \quad G = G'PQ^2.$$

When we substitute these values of E' and G' in the first condition, it becomes

$$G_1(P - Q) = GP_1;$$

and then

$$G = \bar{Q}^2(P - Q),$$

where $\bar{Q}$ is a function of q only. When the values of E' and G' are substituted in the last of the conditions, it becomes

$$E_2(Q - P) = EQ_2;$$

and then

$$E = \bar{P}^2(P - Q),$$

where $\bar{P}$ is a function of p only.

Hence the two surfaces are

$$ds^2 = (P - Q)(\bar{P}^2 dp^2 + \bar{Q}^2 dq^2),$$

$$ds'^2 = \left(\frac{1}{Q} - \frac{1}{P}\right)\left(\frac{\bar{P}^2}{P} dp^2 + \frac{\bar{Q}^2}{Q} dq^2\right);$$

and these are Liouville surfaces (§§ 117, 121). Consequently, *a Liouville surface can be represented geodesically upon an associated Liouville surface.*

We have seen (p. 171) that geodesics on the first are given by

$$\bar{P}(P - a)^{-\frac{1}{2}} dp - \bar{Q}(a - Q)^{-\frac{1}{2}} dq = 0.$$

This equation is unaltered if we change P into $-P^{-1}$, Q into $-Q^{-1}$, $\bar{P}$ into $\bar{P}P^{-\frac{1}{2}}$, $\bar{Q}$ into $\bar{Q}Q^{-\frac{1}{2}}$, a into $-1/a'$. These changes turn the first surface into the second; and so there is a direct verification that the two Liouville surfaces can be represented geodesically upon one another.

158. We have to deal with the two exceptions to which Tissot's theorem does not apply.

In the first of them, there is conformal representation, so that

$$ds' = m\,ds.$$

Thus the nul lines on the two surfaces are the same. Let them be chosen as the parametric curves; then

$$ds^2 = 4\lambda\, dp\, dq, \quad ds'^2 = 4\lambda' dp\, dq,$$

and therefore

$$\lambda' = m^2 \lambda.$$

The general equation of geodesics on the former surface is

$$\frac{d^2q}{dp^2}=\frac{\lambda_1}{\lambda}\frac{dq}{dp}-\frac{\lambda_2}{\lambda}\left(\frac{dq}{dp}\right)^2;$$

and on the latter it is

$$\frac{d^2q}{dp^2}=\frac{\lambda_1'}{\lambda'}\frac{dq}{dp}-\frac{\lambda_2'}{\lambda'}\left(\frac{dq}{dp}\right)^2.$$

When the two surfaces can be represented geodesically on one another, these two general equations must be the same; hence

$$\frac{\lambda_1'}{\lambda'}=\frac{\lambda_1}{\lambda},\quad \frac{\lambda_2'}{\lambda'}=\frac{\lambda_2}{\lambda},$$

and therefore

$$\frac{\lambda'}{\lambda}=\text{constant},$$

that is, m is constant. Thus the two surfaces are similar to one another in a constant magnification.

In the second of the exceptions not included in Tissot's theorem, the homologues of a family of nul lines on one surface are a family of nul lines on the other.

Let one surface be referred to its nul lines as parametric curves; its arc-element is

$$ds^2 = 4\lambda\,dp\,dq.$$

Let $q=$ constant be the family of nul lines, of which the homologues are a family of nul lines on the other surface; then its arc-element has the form

$$ds'^2 = 2F\,dp\,dq + G\,dq^2.$$

For the latter surface, we have

$$\Gamma = \frac{F_1}{F}, \qquad \Delta = 0,$$

$$\Gamma' = \frac{G_1}{2F}, \qquad \Delta' = 0,$$

$$\Gamma'' = \frac{G_2}{2F} - \frac{G}{F^2}(F_2 - \tfrac{1}{2}G_1), \quad \Delta'' = \frac{1}{F}(F_2 - \tfrac{1}{2}G_1);$$

and the general equation of geodesics is

$$\frac{d^2q}{dp^2} = \Gamma''\left(\frac{dq}{dp}\right)^3 + (2\Gamma' - \Delta'')\left(\frac{dq}{dp}\right)^2 + (\Gamma - 2\Delta')\frac{dq}{dp} - \Delta.$$

On the former surface, the general equation of geodesics is

$$\frac{d^2q}{dp^2} = -\frac{\lambda_2}{\lambda}\left(\frac{dq}{dp}\right)^2 + \frac{\lambda_1}{\lambda}\frac{dq}{dp}.$$

When the two surfaces can be represented geodesically on one another, these two general equations are the same; hence

$$\frac{G_2}{2F} - \frac{G}{F^2}(F_2 - \tfrac{1}{2}G_1) = 0,$$

$$\frac{G_1}{F} - \frac{1}{F}(F_2 - \tfrac{1}{2}G_1) = -\frac{\lambda_2}{\lambda}.$$

$$\frac{F_1}{F} = \frac{\lambda_1}{\lambda}.$$

The third of these conditions yields the relation

$$F = \lambda Q,$$

where Q is a function of q only. When this relation is substituted in the second condition, the latter becomes

$$G_1 = \tfrac{2}{3}\lambda Q'.$$

The first condition now gives

$$\frac{G_2}{2G} = \frac{\lambda_2}{\lambda} + \frac{2}{3}\frac{Q'}{Q},$$

so that

$$G = a\lambda^2 Q^{\frac{4}{3}} P^{-2},$$

where, so far as the condition is concerned, P is a function of p only, and a is a disposable constant. Substituting this value of G in the modified form of the second condition, we find

$$\frac{\lambda_1}{P} - \lambda\frac{P'}{P^2} = \frac{1}{3a}Q'Q^{-\frac{4}{3}}P;$$

hence

$$\frac{\lambda}{P} = \bar{Q} + \frac{1}{3a}Q'Q^{-\frac{4}{3}}\int P\,dp,$$

where $\bar{Q}$ is a function of q only.

Now let

$$P\,dp = du, \quad \lambda = \mu P, \quad Q = R^{-3},$$

so that R is a function of q only, and u is a new variable; then

$$\mu = \bar{Q} - \frac{1}{a}uR',$$

or, choosing $a = -1$,

$$\mu = \bar{Q} + uR'.$$

Also

$$F = \mu P R^{-3},$$

$$G = -\mu^2 R^{-4}.$$

Thus the first surface is

$$ds^2 = 4\mu\,du\,dq$$

$$= 4(\bar{Q} + uR')\,du\,dq,$$

where $\bar{Q}$ and R are any functions of q alone; and the second surface* is

$$ds'^2 = 2\mu R^{-3} du dq - \mu^2 R^{-4} dq^2,$$

where

$$\mu = \bar{Q} + uR'.$$

These two surfaces can be represented upon one another so that their geodesics correspond. Now (§ 121) the geodesics on the surface

$$ds^2 = 4(\bar{Q} + uR') du dq$$

are given by the equation

$$\frac{u}{(a+2R)^{\frac{1}{2}}} - 2\int \frac{\bar{Q}dq}{(a+2R)^{\frac{3}{2}}} = c,$$

where a and c are arbitrary constants; this equation therefore gives also the geodesics upon the geodesically associated surface.

Spherical Representation; Tangential Coordinates.

159. We now come to the representation of a surface on a sphere, already indicated in § 147; it frequently is called the *spherical representation* of the surface, and it is due to Gauss originally. We take a sphere of radius unity and through its centre draw a line parallel to the positive direction of the normal to the surface; the point on the surface has its image in the point where the sphere is cut by the line. It thus follows that, in the representation, we are partly considering the tangent plane to the surface; and so, in using the direction-cosines of the normal, we are in effect using three of the tangential coordinates of the surface. We shall therefore find it convenient to deal with equations, expressed as far as possible, in terms of tangential coordinates; for they are essential to the resolution of the question as to how far a surface is determined by a given spherical representation.

The coordinates of the spherical image of a point on the surface, where the direction-cosines of the normal are X, Y, Z, are themselves X, Y, Z, which are subject to the condition $X^2 + Y^2 + Z^2 = 1$. Let dS be an arc-element on the sphere; then

$$dS^2 = dX^2 + dY^2 + dZ^2 = edp^2 + 2fdpdq + gdq^2,$$

where

$$\begin{aligned} e &= X_1^2 + Y_1^2 + Z_1^2 = -EK + LH, \\ f &= X_1X_2 + Y_1Y_2 + Z_1Z_2 = -FK + MH, \\ g &= X_2^2 + Y_2^2 + Z_2^2 = -GK + NH, \end{aligned}$$

on substituting the values of the derivatives of X, Y, Z given in § 29. We have

$$eg - f^2 = V^2K^2;$$

* The result is due to Lie; see § 156.

and therefore, as $eg - f^2 > 0$, the surface to be represented cannot be a developable surface.

Manifestly we have

$$dS^2 = -Kds^2 + H\frac{ds^2}{\rho},$$

with the usual notation for the curvature of the normal section. Now

$$\begin{aligned} K - \frac{H}{\rho} &= \frac{1}{\alpha\beta} - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\left(\frac{\cos^2\psi}{\alpha} + \frac{\sin^2\psi}{\beta}\right) \\ &= -\frac{\cos^2\psi}{\alpha^2} - \frac{\sin^2\psi}{\beta^2}, \end{aligned}$$

where ψ is the angle between the tangent to the curve and a line of curvature; and therefore

$$dS^2 = \left(\frac{\cos^2\psi}{\alpha^2} + \frac{\sin^2\psi}{\beta^2}\right) ds^2.$$

Hence *the spherical image usually is not a conformal representation.* But there are two classes of surfaces for which spherical representation is conformal. For one class, we have $\alpha = \beta$; its Cartesian equation, in the most general range, is

$$\left.\begin{aligned} y + \mu x &= \phi(\mu) \\ z - ix(1+\mu^2)^{\frac{1}{2}} &= \psi(\mu) \end{aligned}\right\},$$

where ϕ and ψ are arbitrary functions. For the other class, we have $\alpha + \beta = 0$; they are minimal surfaces, and will be discussed later (in Chap. VIII).

160. Some simple properties can be established at once.

I. When the parametric curves on the surface are orthogonal, we have

$$F = 0.$$

When they are orthogonal on the sphere also, then

$$f = 0;$$

and therefore, unless $H = 0$, we have $M = 0$.

Further, when $F = 0$, $M = 0$, we have $f = 0$ whether H vanishes or not. Hence the spherical image of the lines of curvature is an orthogonal system; and the lines of curvature are the only orthogonal system whose spherical image is orthogonal, unless the original surface is a minimal surface—in which case the spherical representation happens to be conformal also, so that any orthogonal system remains orthogonal in the representation.

II. But further, the spherical image of a line of curvature is parallel (directly or reversely) to the line; and if the spherical image of a curve is parallel to the curve, then the curve is a line of curvature.

For the first part, take the lines of curvature as the parametric curves, so that

$$F = 0, \quad M = 0.$$

We then have (§ 29), for the respective lines,

$$\frac{X_1}{x_1} = \frac{Y_1}{y_1} = \frac{Z_1}{z_1} = -\frac{L}{E},$$

$$\frac{X_2}{x_2} = \frac{Y_2}{y_2} = \frac{Z_2}{z_2} = -\frac{N}{G},$$

proving the statement.

For the second part, keep the parametric curves general; and suppose that the spherical image dX, dY, dZ of dx, dy, dz is parallel to it. Then

$$\frac{X_1 dp + X_2 dq}{x_1 dp + x_2 dq} = \frac{Y_1 dp + Y_2 dq}{y_1 dp + y_2 dq} = \frac{Z_1 dp + Z_2 dq}{z_1 dp + z_2 dq}.$$

Let the common value of these fractions be μ; then

$$X_1 dp + X_2 dq = \mu (x_1 dp + x_2 dq),$$

$$Y_1 dp + Y_2 dq = \mu (y_1 dp + y_2 dq),$$

$$Z_1 dp + Z_2 dq = \mu (z_1 dp + z_2 dq).$$

Multiply these equations by x_1, y_1, z_1 respectively, and add; and by x_2, y_2, z_2 respectively, and add; we have

$$-L dp - M dq = \mu (E dp + F dq),$$

$$-M dp - N dq = \mu (F dp + G dq);$$

and therefore

$$(EM - FL)\, dp^2 + (EN - GL)\, dp dq + (FN - GM)\, dq^2 = 0,$$

giving the directions of the lines of curvature.

III. A direction dx', dy', dz' on the original surface, which is conjugate to a given direction dp, dq, is such (§ 47) that

$$dx' dX + dy' dY + dz' dZ = 0,$$

where dX, dY, dZ are determined by dp, dq, that is, are the spherical image of the given direction. It therefore follows that, when two directions are conjugate on the surface, the spherical image of each direction is perpendicular to the other direction.

Moreover, as an asymptotic line is self-conjugate, it follows that the spherical image of an asymptotic line is perpendicular to the line.

IV. The preceding result, relating to conjugate lines, can be stated in another form, viz. the inclination of the spherical images of two conjugate lines is either equal to, or supplementary to, the inclination of the conjugate lines.

This can be established independently as follows. Taking the conjugate lines as the parametric curves, we have $M = 0$; and therefore

$$f = -FK.$$

As usual, denote by ω the inclination of the parametric curves on the surface which now are conjugate; and denote by ω' the inclination of their spherical images. Then (§ 25)

$$\cot \omega = F/V, \quad \cot \omega' = f(eg - f^2)^{-\frac{1}{2}}.$$

Now

$$v^2 = eg - f^2 = V^2K^2,$$

and we take v positive, just as V has been taken positive (§ 24); hence

$$v = \pm VK,$$

the upper or the lower sign being used, according as the surface is synclastic or anticlastic. Thus

$$\cot \omega' = \pm f/VK,$$

that is,

$$\cot \omega' = \mp \cot \omega,$$

which gives the property in question.

V. We can, by means of the spherical representation, prove Joachimsthal's theorems (§ 128) as to plane lines of curvature and spherical lines of curvature.

In the case of a plane line of curvature, let a, b, c be the direction-cosines of the normal to the plane. If at any point, ϖ is the angle between the normal to the surface and the principal normal to the curve, we have

$$\sin \varpi = aX + bY + cZ;$$

and therefore

$$\frac{d}{ds}(\sin \varpi) = \left(a\frac{dX}{dS} + b\frac{dY}{dS} + c\frac{dZ}{dS}\right)\frac{dS}{ds}$$

$$= \pm\left(a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds}\right)\frac{dS}{ds},$$

because the spherical image of a line of curvature is parallel to the line. Now a, b, c are the direction-cosines of the normal to the plane in which the line of curvature lies; thus

$$a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds} = 0.$$

Consequently, $\sin \varpi =$ constant; or the plane and the surface cut everywhere at a constant angle—which is the theorem as to plane lines of curvature. Moreover, the equation

$$aX + bY + cZ = \sin \varpi$$

now shews that the spherical image of the line of curvature is a small circle, unless the line of curvature is also a geodesic, in which case its spherical image is a great circle.

Also, if a family of lines of curvature is composed of plane curves, their spherical image is a family of small circles. If both families of lines of curvature are plane curves, then (because the spherical image of two lines of curvature is two orthogonal lines) the spherical images are two families of orthogonal small circles. We shall return later (§§ 197, 198) to this matter.

Next, consider a spherical line of curvature. Let the sphere be of radius r and have its centre at x_0, y_0, z_0; then, denoting by χ the angle between the normal to the surface and the normal to the sphere at a point on the line of curvature, we have

$$\cos\chi = X\frac{x-x_0}{r} + Y\frac{y-y_0}{r} + Z\frac{z-z_0}{r}.$$

Consequently

$$\frac{d}{ds}(\cos\chi) = \frac{1}{r}\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) + \frac{dS}{ds}\left(\frac{dX}{dS}\frac{x-x_0}{r} + \frac{dY}{dS}\frac{y-y_0}{r} + \frac{dZ}{dS}\frac{z-z_0}{r}\right)$$
$$= \pm\frac{dS}{ds}\left(\frac{x-x_0}{r}\frac{dx}{ds} + \frac{y-y_0}{r}\frac{dy}{ds} + \frac{z-z_0}{r}\frac{dz}{ds}\right),$$

because the spherical image of a line of curvature is parallel to the line. The quantity within the brackets on the right-hand side is zero because the line of curvature lies on the sphere; hence

$$\cos\chi = \text{constant},$$

and therefore the sphere and the surface cut at a constant angle along the line of curvature—which is the theorem as to spherical lines of curvature. (The angle χ is the angle $\varpi' - \varpi$ of § 128.)

161. The fundamental magnitudes of the second order, say L', M', N', for the sphere can be obtained simply. Denoting the direction-cosines of the positive direction of the normal to the sphere by X', Y', Z', we have (in accordance with §§ 27, 29)

$$X' = \frac{1}{v}(Y_1 Z_2 - Y_2 Z_1) = \frac{1}{v}KVX = \pm X,$$

according as the surface is synclastic or anticlastic; and similarly $Y' = \pm Y$, $Z' = \pm Z$, the upper signs being used together and the lower signs being used together. Thus

$$\begin{aligned} L' &= X'X_{11} + Y'Y_{11} + Z'Z_{11} \\ &= \pm(XX_{11} + YY_{11} + ZZ_{11}) \\ &= \pm\left\{\frac{\partial}{\partial p}(XX_1 + YY_1 + ZZ_1) - (X_1^2 + Y_1^2 + Z_1^2)\right\} \\ &= \mp e; \end{aligned}$$

and similarly $M' = \mp f$, $N' = \mp g$. The radius of curvature of any normal section of the sphere, being

$$\frac{edp^2 + 2fdpdq + gdq^2}{L'dp^2 + 2M'dpdq + N'dq^2},$$

is numerically equal to unity, as was to be expected; it is positive or negative, according as the normal to the positive side of the sphere has to be drawn inwards or outwards.

Thus only the fundamental magnitudes of the first order for the sphere need be taken into account. It is convenient to have the relations between them and the fundamental magnitudes of the surface. Substituting for K and H in the expressions for e, f, g, we find (see also § 29)

$$V^2 e = EM^2 - 2FLM + GL^2,$$
$$V^2 f = EMN - F(LN + M^2) + GLM,$$
$$V^2 g = EN^2 - 2FMN + GM^2;$$

and therefore

$$v^2 E = eM^2 - 2fLM + gL^2,$$
$$v^2 F = eMN - f(LN + M^2) + gLM,$$
$$v^2 G = eN^2 - 2fMN + gM^2.$$

We require quantities corresponding to Γ, Γ', Γ'', Δ, Δ', Δ''. Let

$$\mu = \tfrac{1}{2}e_1, \qquad \mu' = \tfrac{1}{2}e_2, \qquad \mu'' = f_2 - \tfrac{1}{2}g_1,$$
$$\nu = f_1 - \tfrac{1}{2}e_2, \qquad \nu' = \tfrac{1}{2}g_1, \qquad \nu'' = \tfrac{1}{2}g_2.$$

Then we take

$$\left.\begin{aligned} v^2\gamma &= \mu g - \nu f \\ v^2\gamma' &= \mu' g - \nu' f \\ v^2\gamma'' &= \mu'' g - \nu'' f \end{aligned}\right\}, \qquad \left.\begin{aligned} v^2\delta &= \nu e - \mu f \\ v^2\delta' &= \nu' e - \mu' f \\ v^2\delta'' &= \nu'' e - \mu'' f \end{aligned}\right\};$$

and we thus have the quantities required. Moreover, they give

$$\left.\begin{aligned} \mu &= e\gamma + f\delta \\ \mu' &= e\gamma' + f\delta' \\ \mu'' &= e\gamma'' + f\delta'' \end{aligned}\right\}, \qquad \left.\begin{aligned} \nu &= f\gamma + g\delta \\ \nu' &= f\gamma' + g\delta' \\ \nu'' &= f\gamma'' + g\delta'' \end{aligned}\right\},$$

corresponding to the relations in § 34.

As the values of the quantities e, f, g involve all the magnitudes E, F, G, L, M, N, these quantities $\mu, \mu', \mu'', \nu, \nu', \nu''$ must be expressible in terms of the derived magnitudes of the third order for the surface. We have

$$\frac{\partial e}{\partial p} = \frac{1}{V^2}\{2mM^2 - 2(n + m')LM + 2n'L^2\} + \frac{1}{V^2}\{2M_1(EM - FL) + 2L_1(GL - FM)\}$$
$$- \frac{2}{V^2}(\Gamma + \Delta')(EM^2 - 2FLM + GL^2)$$
$$= \frac{2}{V^2}\{(GL - FM)P + (EM - FL)Q + V^2(e\Gamma + f\Delta)\};$$

and similarly for the others. The aggregate of results is:—

$$\left.\begin{aligned}\mu &= \frac{1}{V^2}\{(GL - FM)P + (EM - FL)Q\} + e\Gamma + f\Delta \\ \mu' &= \frac{1}{V^2}\{(GL - FM)Q + (EM - FL)R\} + e\Gamma' + f\Delta' \\ \mu'' &= \frac{1}{V^2}\{(GL - FM)R + (EM - FL)S\} + e\Gamma'' + f\Delta''\end{aligned}\right\}.$$

$$\left.\begin{aligned}\nu &= \frac{1}{V^2}\{(GM - FN)P + (EN - FM)Q\} + f\Gamma + g\Delta \\ \nu' &= \frac{1}{V^2}\{(GM - FN)Q + (EN - FM)R\} + f\Gamma' + g\Delta' \\ \nu'' &= \frac{1}{V^2}\{(GM - FN)R + (EN - FM)S\} + f\Gamma'' + g\Delta''\end{aligned}\right\};$$

and it is easy to shew that

$$\left.\begin{aligned}V^2K(\gamma - \Gamma) &= PN - QM \\ V^2K(\gamma' - \Gamma') &= QN - RM \\ V^2K(\gamma'' - \Gamma'') &= RN - SM\end{aligned}\right\}, \qquad \left.\begin{aligned}V^2K(\delta - \Delta) &= -PM + QL \\ V^2K(\delta' - \Delta') &= -QM + RL \\ V^2K(\delta'' - \Delta'') &= -RM + SL\end{aligned}\right\}.$$

162. The tangential coordinates connected with the surface are defined, as usual, in connection with the tangent plane. Let x, y, z be the coordinates of any point on the surface, and let T (with, of course, a new significance for the symbol, different from the significance adopted in § 28) be the distance* of the tangent plane from the origin; then the equation of the plane is

$$xX + yY + zZ = T.$$

The quantities X, Y, Z, T are the *tangential coordinates* of the surface.

We require various quantities, and some of our established equations, expressed in terms of the tangential coordinates. For x, y, z, we have

$$\begin{aligned}xX + yY + zZ &= T, \\ xX_1 + yY_1 + zZ_1 &= T_1, \\ xX_2 + yY_2 + zZ_2 &= T_2.\end{aligned}$$

Now (§ 29)

$$\begin{vmatrix} X, & Y, & Z \\ X_1, & Y_1, & Z_1 \\ X_2, & Y_2, & Z_2 \end{vmatrix} = VK,$$

$$\begin{aligned}Y_1Z_2 - Y_2Z_1 &= VKX, \\ Y_2Z - YZ_2 &= \frac{1}{VK}(gX_1 - fX_2), \\ YZ_1 - Y_1Z &= \frac{1}{VK}(-fX_1 + eX_2);\end{aligned}$$

* Sometimes W is used (as in § 79, and regularly by Bianchi).

hence

$$\left.\begin{aligned} x &= TX + \frac{1}{v^2}\{eX_2T_2 - f(X_1T_2 + X_2T_1) + gX_1T_1\} \\ y &= TY + \frac{1}{v^2}\{eY_2T_2 - f(Y_1T_2 + Y_2T_1) + gY_1T_1\} \\ z &= TZ + \frac{1}{v^2}\{eZ_2T_2 - f(Z_1T_2 + Z_2T_1) + gZ_1T_1\} \end{aligned}\right\},$$

giving the point-coordinates of the surface in terms of the tangential coordinates.

It therefore follows that, when the tangential coordinates are given, the surface is completely determinate even as to position and orientation.

163. Again, we have (§ 161)

$$XX_{11} + YY_{11} + ZZ_{11} = -e.$$

Also, from the definitions of the quantities μ and ν, it follows that

$$X_1X_{11} + Y_1Y_{11} + Z_1Z_{11} = \mu,$$
$$X_2X_{11} + Y_2Y_{11} + Z_2Z_{11} = \nu;$$

hence, solving for X_{11}, Y_{11}, Z_{11}, we find

$$X_{11} = -eX + \gamma X_1 + \delta X_2;$$

and similarly for Y_{11}, Z_{11}. Thus

$$\left.\begin{aligned} X_{11} &= -eX + \gamma X_1 + \delta X_2 \\ Y_{11} &= -eY + \gamma Y_1 + \delta Y_2 \\ Z_{11} &= -eZ + \gamma Z_1 + \delta Z_2 \end{aligned}\right\}.$$

Similarly

$$\left.\begin{aligned} X_{12} &= -fX + \gamma' X_1 + \delta' X_2 \\ Y_{12} &= -fY + \gamma' Y_1 + \delta' Y_2 \\ Z_{12} &= -fZ + \gamma' Z_1 + \delta' Z_2 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} X_{22} &= -gX + \gamma'' X_1 + \delta'' X_2 \\ Y_{22} &= -gY + \gamma'' Y_1 + \delta'' Y_2 \\ Z_{22} &= -gZ + \gamma'' Z_1 + \delta'' Z_2 \end{aligned}\right\}.$$

These are the equations of the second order satisfied by X, Y, Z.

There are corresponding equations for T. We proceed from

$$T_1 = xX_1 + yY_1 + zZ_1;$$

then

$$\begin{aligned} T_{11} &= xX_{11} + yY_{11} + zZ_{11} + x_1X_1 + y_1Y_1 + z_1Z_1 \\ &= -eT + \gamma T_1 + \delta T_2 - L. \end{aligned}$$

Similarly for T_{12}, T_{22}; the results are

$$\left.\begin{aligned} T_{11} - \gamma\, T_1 - \delta\, T_2 + eT &= -L \\ T_{12} - \gamma'\, T_1 - \delta'\, T_2 + fT &= -M \\ T_{22} - \gamma'' T_1 - \delta'' T_2 + gT &= -N \end{aligned}\right\}.$$

It therefore follows that E, F, G, L, M, N are directly expressible, without any inverse operations, in terms of e, f, g, T, and their derivatives. Hence, when T and a spherical representation are known, the surface is uniquely determinate save as to orientation and position. The organic lines on the surface are expressible in terms of these quantities; it is easy to verify that the quantities A and W of § 125 are given by

$$-AVK = \begin{vmatrix} T_{11}p'^2 + 2T_{12}p'q' + T_{22}q'^2, & T_1, & T_2, & T \\ X_{11}p'^2 + 2X_{12}p'q' + X_{22}q'^2, & X_1, & X_2, & X \\ Y_{11}p'^2 + 2Y_{12}p'q' + Y_{22}q'^2, & Y_1, & Y_2, & Y \\ Z_{11}p'^2 + 2Z_{12}p'q' + Z_{22}q'^2, & Z_1, & Z_2, & Z \end{vmatrix},$$

$$v^2 W = \begin{vmatrix} T_{11}p' + T_{12}q', & T_{12}p' + T_{22}q', & T_1, & T_2 \\ X_{11}p' + X_{12}q', & X_{12}p' + X_{22}q', & X_1, & X_2 \\ Y_{11}p' + Y_{12}q', & Y_{12}p' + Y_{22}q', & Y_1, & Y_2 \\ Z_{11}p' + Z_{12}q', & Z_{12}p' + Z_{22}q', & Z_1, & Z_2 \end{vmatrix},$$

where $A = 0$ is the equation of the asymptotic lines, and $W = 0$ is the equation of the lines of curvature.

164. We have seen that, when the tangential coordinates X, Y, Z, T are known as functions of two parameters, the surface is completely determinate; and that when e, f, g, T are known, the surface is uniquely determinate save as to orientation and position. The data required for these inferences are—a knowledge of X, Y, Z, in the one case, and a knowledge of a spherical representation in the other case—together with a knowledge of T, which is quite independent of the sphere.

The question then arises as to how far a surface is defined by means solely of X, Y, Z, supposed given; or solely of a spherical representation, supposed given. In the one datum, we assume that X, Y, Z are known functions of p and q, subject to the condition $X^2 + Y^2 + Z^2 = 1$; in the other, we assume that e, f, g are known functions of p and q, subject to Gauss's characteristic equation when it gives unity as the measure of curvature. The answer to the question depends upon the determination of the quantity T.

When the values of L, M, N, which have just been obtained, are substituted in the expressions

$$\begin{aligned} v^2E &= eM^2 - 2fLM + gL^2, \\ v^2F &= eMN - f(LN + M^2) + gLM, \\ v^2G &= eN^2 - 2fMN + gM^2, \end{aligned}$$

the quantities E, F, G become functions of known quantities, of T, and of the derivatives of T up to the second order inclusive. Now, for every surface, the six fundamental magnitudes must satisfy the Gauss characteristic equation and the two Mainardi-Codazzi relations, viz.

$$LN - M^2 = -\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11})$$
$$\qquad - \frac{1}{V^2}\{E(nn'' - n'^2) - F(nm'' - 2n'm' + n''m) + G(mm'' - m'^2)\},$$

$$L_2 + \Gamma M + \Delta N = M_1 + \Gamma' L + \Delta' M,$$
$$M_2 + \Gamma' M + \Delta' N = N_1 + \Gamma'' L + \Delta'' M.$$

When the foregoing values of E, F, G are substituted in these equations—the algebra is exceedingly laborious—and then, when substitution is made for L, M, N in terms of the quantity T and the given magnitudes, the first of these relations gives a partial differential equation for T which is of the third order. (Owing to the presence of $E_{22} - 2F_{12} + G_{11}$, the equation might have been expected to be of the fourth order; but the terms of that order cancel.) The second of the relations gives another partial differential equation for T of the third order; and the last of them gives yet another partial differential equation for T of the third order. (In both of these equations, the terms of the third order disappear from $L_2 - M_1$, and $M_2 - N_1$, respectively; but they arise from the values of Γ, Γ', Γ'', Δ, Δ', Δ'', and they do not disappear.)

Thus in general, when a spherical representation of a surface is given, it is necessary to solve three simultaneous partial equations of the third order if the surface itself is to be determined thereby.

165. But simplifications of this complicated result can be secured by the assignment of particular conditions—these conditions really being limitations upon some of the arbitrary functions that occur in the primitive of the three simultaneous partial equations of the third order.

Let there be an assigned condition that the parametric curves on the sphere shall be the images of asymptotic lines on the surface. Instead of dealing with the specialised forms of the equations of the third order, it is simpler to deal with the original equations that are fundamental; so we assume the condition that the asymptotic lines are to be parametric, viz.,

$$L = 0, \quad N = 0,$$

and then the Mainardi-Codazzi relations are

$$(\Gamma - \Delta')M = M_1, \quad (\Delta'' - \Gamma')M = M_2.$$

Moreover, we have

$$(\Gamma + \Delta')V = V_1, \quad (\Delta'' + \Gamma')V = V_2;$$

so that

$$2\Gamma = \frac{M_1}{M} + \frac{V_1}{V}, \quad 2\Delta'' = \frac{M_2}{M} + \frac{V_2}{V},$$

$$2\Delta' = -\frac{M_1}{M} + \frac{V_1}{V}, \quad 2\Gamma' = -\frac{M_2}{M} + \frac{V_2}{V}.$$

Consequently

$$\frac{\partial \Gamma}{\partial q} = \frac{\partial \Delta''}{\partial p}, \quad \frac{\partial \Delta'}{\partial q} = \frac{\partial \Gamma'}{\partial p},$$

differential relations which, of course, are isolated under the special system of parametric curves.

Having regard to these curves, the values of P, Q, R, S (as given in § 40) are

$$P = -2M\Delta, \quad Q = -2M\Delta', \quad R = -2M\Gamma', \quad S = -2M\Gamma''.$$

Also we now have

$$v^2 E = eM^2, \quad v^2 F = -fM^2, \quad v^2 G = gM^2,$$

so that

$$vV = M^2,$$

and

$$K = \frac{-M^2}{V^2} = -\frac{v}{V} = -\frac{v^2}{M^2}.$$

As regards the quantities γ, γ', γ'', δ, δ', δ'', there are simple relations connecting them with Γ, Γ', Γ'', Δ, Δ', Δ'' under this system. We have (§ 161)

$$\gamma - \Gamma = -\frac{1}{V^2 K} QM = \frac{Q}{M} = -2\Delta',$$

$$\gamma' - \Gamma' = -\frac{1}{V^2 K} RM = \frac{R}{M} = -2\Gamma',$$

$$\gamma'' - \Gamma'' = -\frac{1}{V^2 K} SM = \frac{S}{M} = -2\Gamma'',$$

$$\delta - \Delta = -\frac{1}{V^2 K} PM = \frac{P}{M} = -2\Delta,$$

$$\delta' - \Delta' = -\frac{1}{V^2 K} QM = \frac{Q}{M} = -2\Delta',$$

$$\delta'' - \Delta'' = -\frac{1}{V^2 K} RM = \frac{R}{M} = -2\Gamma';$$

and therefore

$$\gamma = \Gamma - 2\Delta', \quad \delta'' = \Delta'' - 2\Gamma',$$

$$\gamma' = -\Gamma', \quad \gamma'' = -\Gamma'', \quad \delta = -\Delta, \quad \delta' = -\Delta'.$$

Consequently we also have (always under this system of parametric curves)

$$\frac{\partial \gamma}{\partial q} = \frac{\partial \delta''}{\partial p}, \quad \frac{\partial \gamma'}{\partial p} = \frac{\partial \delta'}{\partial q}.$$

as relations identically satisfied; thus the spherical representation cannot be chosen arbitrarily. Also

$$E = e\frac{M^2}{v^2} = -\frac{e}{K}, \quad F = \frac{f}{K}, \quad G = -\frac{g}{K};$$

therefore the arc-element on the surface is expressible in the form

$$ds^2 = -\frac{1}{K}(edp^2 - 2fdpdq + gdq^2).$$

And then the quantity T is an integral common to the two equations of the second order

$$\left.\begin{aligned} T_{11} - \gamma\ T_1 - \delta\ T_2 + eT = 0 \\ T_{22} - \gamma'' T_1 - \delta'' T_2 + gT = 0 \end{aligned}\right\}.$$

Ex. Let these results be applied to a pseudo-spherical surface. We have, for all surfaces referred to asymptotic lines as parametric curves (§ 42),

$$V^2 K_1 = -2MQ, \quad V^2 K_2 = -2MR;$$

and therefore, for a pseudo-sphere,

$$Q=0, \quad R=0.$$

Thus

$$\gamma' = \Gamma' = 0, \quad \delta' = \Delta' = 0, \quad \gamma = \Gamma, \quad \delta'' = \Delta''.$$

Now

$$2v^2\gamma' = e_2 g - g_1 f,$$

$$2V^2\Gamma' = E_2 G - G_1 F = \frac{1}{K^2}(e_2 g + g_1 f),$$

in this case; and so

$$e_2 = 0, \quad g_1 = 0.$$

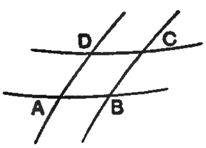
The same inferences follow from the conditions $\delta' = \Delta' = 0$. Hence e is a function of p only; it can therefore be made unity, because it can be absorbed into the term edp^2. Similarly g is a function of q only; it can therefore be made unity, because it can be absorbed into the term gdq^2. Thus the arc-element on the sphere is

$$dS^2 = dp^2 - 2dpdq\cos\omega + dq^2,$$

and on the pseudo-sphere is

$$ds^2 = -\frac{1}{K}(dp^2 - 2dpdq\cos\omega + dq^2).$$

The parametric curves are asymptotic lines; so that, over an ordinary region of the surface, asymptotic lines of the same family do not meet. We thus do not have an asymptotic triangle (like a spherical triangle on a sphere, or a geodesic triangle on any surface); but we do have an asymptotic quadrilateral. If ω denote the angle between the parametric curves, we have (§ 36) in general



$$-\frac{\partial^2\omega}{\partial p\,\partial q} = \frac{\partial}{\partial p}\left(\frac{V\Delta'}{E}\right) + \frac{\partial}{\partial q}\left(\frac{V\Gamma'}{G}\right) + VK;$$

and therefore, in the present case,

$$\frac{\partial^2\omega}{\partial p\,\partial q} = \sin\omega.$$

Now the area of the quadrilateral is

$$\begin{aligned}\iint -\frac{dp\,dq}{K}\sin\omega &= -\frac{1}{K}\iint \frac{\partial^2\omega}{\partial p\,\partial q}\,dp\,dq\\ &= -\frac{1}{K}\{\omega_C-(\pi-\omega_B)-(\pi-\omega_D)+\omega_A\}\\ &= -\frac{1}{K}(S-2\pi),\end{aligned}$$

where S is the sum of the internal angles of the quadrilateral on the surface.

Other simplifications can be suggested; the results are left for detailed working.

When the parametric curves are required to be orthogonal on the surface, we must have $F=0$, that is,

$$eMN-f(LN+M^2)+gLM=0.$$

When the values of L, M, N in terms of T and its derivatives are substituted, we have a non-linear partial equation for T of the second order.

When the parametric curves are required to be conjugate on the surface, we have $M=0$, that is,

$$T_{12}-\gamma' T_1-\delta' T_2+fT=0.$$

We have a linear partial equation of the second order; it is of the Laplace type*

When the parametric curves are required to be both orthogonal and conjugate on the surface, that is, are required to be lines of curvature on the surface, we have $F=0$, $M=0$. Then

$$f=0,\quad \gamma'=\frac{e_2}{2e},\quad \delta'=\frac{g_1}{2g};$$

and the linear partial equation for T is

$$T_{12}-\frac{e_2}{2e}T_1-\frac{g_1}{2g}T_2=0,$$

again of the Laplace type. When T is known, then

$$\begin{aligned}L&=T_{11}-\frac{e_1}{2e}T_1+\frac{e_2}{2g}T_2+eT,\\ N&=T_{22}+\frac{g_1}{2e}T_1-\frac{g_2}{2g}T_2+gT,\\ E&=\frac{L^2}{e},\\ G&=\frac{N^2}{g}.\end{aligned}$$

* See the author's *Theory of Differential Equations*, vol. vi, chap. xiii.

When the surface is to have its mean curvature constant, then the partial equation for T is

$$(LN - M^2) H = eN - 2fM + gL,$$

of the second order and the Monge-Ampère type.

In all these cases, we have to integrate a partial differential equation of the second order. Its primitive contains two arbitrary functions; and so there is, in each instance, a double family of surfaces with the assigned spherical representation.

EXAMPLES.

1. Any two stereographic projections of a sphere are inverses of each other, the origin of inversion in either being the origin of projection for the other.

2. Shew that the spherical images of the asymptotic lines on a minimal surface, as well as the asymptotic lines themselves, are an orthogonal isometric system.

3. Two surfaces have the same spherical representation. A family of spheres is drawn having their centres on one of the surfaces and the envelope of the spheres is constructed; shew that the other surface is normal to the chords of contact.

4. Shew that, if

$$X = \frac{u+v}{uv+1}, \quad Y = \frac{i(v-u)}{uv+1}, \quad Z = \frac{uv-1}{uv+1}, \quad t = T(uv+1),$$

the lines of curvature on the original surface are given by

$$du\, d\frac{\partial t}{\partial u} - dv\, d\frac{\partial t}{\partial v} = 0.$$

Also shew that, if another surface is given by the equations

$$x = \frac{\partial t}{\partial u}, \quad y = v, \quad z = -t + u\frac{\partial t}{\partial u},$$

its asymptotic lines are given by the foregoing differential equation.

5. Prove that, when e, f, g, T are known, the lines of curvature on the original surface are given by the equation

$$\begin{vmatrix} dq^2, & -dq\,dp, & dp^2 \\ e, & f, & g \\ T_{11} - \gamma T_1 - \delta T_2, & T_{12} - \gamma' T_1 - \delta' T_2, & T_{22} - \gamma'' T_1 - \delta'' T_2 \end{vmatrix} = 0.$$

6. Shew that the surfaces, which have one system of lines of curvature in parallel planes, are given by the equations

$$x = f(v)\cos v - f'(v)\sin v + \{g(u)\sin u + g'(u)\cos u\}\cos v,$$

$$y = f(v)\sin v + f'(v)\cos v + \{g(u)\sin u + g'(u)\cos u\}\sin v,$$

$$z = g(u)\cos u - g'(u)\sin u,$$

where f and g are any functions whatever.

CHAPTER VIII.

Minimal Surfaces.

The amount of mathematical literature, devoted to the subject of minimal surfaces, is of vast extent.

The theory was initiated by Lagrange, mainly in his non-geometrical treatment of the stationary values of double integrals. It attracted the fruitful attention of other great mathematicians such as Monge, who was the first to give a general solution of the question; and of Legendre, who first applied what now is called a contact-transformation to the partial differential equation of the second order that is characteristic of the surfaces. All this work belonged to the later part of the eighteenth century. Its progress continued intermittently in the earlier half of the nineteenth century until the researches of Bonnet, published in 1853 and later, which marked an entirely new development in the determination of real surfaces. Soon there followed the investigations of Weierstrass, who gave the useful forms to the equations obtained by Monge and from them constructed the generalities of the theory of minimal surfaces that are real and of surfaces that are algebraic; the significance of the theory of real surfaces being due to the fact that the analysis is bound up with functions of complex variables. Moreover, the researches of Weierstrass inspired the work of Schwarz who has contributed many important developments to the subject, on its geometrical side and its functional side. And Lie's work added substantially to the theory of algebraic minimal surfaces that are subjected to assigned conditions.

Mention also should be made of the memoirs of Beltrami who made notable additions to the subject and, in one of his early memoirs, gives a survey of the progress made down to 1860.

Above all, there is the section (Book iii, vol. i) in Darboux's treatise dealing with the whole matter, its history, its development, its later issues, problems half-solved or unsolved. That section is practically a complete treatise at the time of its publication (1887); whatever advances in detail may have been made since that date, Darboux's exposition should be studied carefully by every student of the subject.

The Critical Equation $H = 0$.

166. We now proceed to consider one particular class of special surfaces, usually called *minimal surfaces*. For many reasons, they are important. They are related to the calculus of variations, as providing the simplest significant example of a condition for the minimum of a double integral; it was in this relation, that they arose in investigations of Lagrange. They are related, in their analytical expression, to the theory of functions of a complex variable; implicitly beginning in results due to Monge, the association has been developed in many researches that have their foundation in some memoirs of Weierstrass, supplemented by the work of Schwarz and of Lie.

They are connected with problems in mathematical physics, of which the most picturesque is that of the soap-bubble.

Let surfaces be drawn so as to pass through an assigned closed curve (whether continuous in direction or not) and be required, along the curve, to touch an assigned developable surface passing through the curve. For the moment we are not concerned with the limitations imposed upon surfaces by these requirements, or with the extent of further condition that may be imposed simultaneously with the limitations. Among all these surfaces, let a surface be selected such that its area is a minimum—in the sense that, when small variations of any kind upon the surface are effected subject to the limitations, the result is to give an increase of area for the modified surface. The original surface is called *minimal*. It may or may not be unique. It may even be non-existent, owing to the complication of the conditions.

We have seen (§ 25) that the element of area on the surface can be represented by the quantity $Vdpdq$; and therefore the area of the surface bounded by some assigned curve is

$$\iint Vdpdq,$$

where the double integral is taken over the range limited by the curve. If then the area of the surface is to be a minimum among the areas of all surfaces which can be drawn through the curve, this double integral must be a minimum.

The conditions that the first variation should vanish (a condition which secures a stationary value for the double integral) are that the equations

$$\frac{\partial V}{\partial x} - \frac{\partial}{\partial p}\left(\frac{\partial V}{\partial x_1}\right) - \frac{\partial}{\partial q}\left(\frac{\partial V}{\partial x_2}\right) = 0,$$

$$\frac{\partial V}{\partial y} - \frac{\partial}{\partial p}\left(\frac{\partial V}{\partial y_1}\right) - \frac{\partial}{\partial q}\left(\frac{\partial V}{\partial y_2}\right) = 0,$$

$$\frac{\partial V}{\partial z} - \frac{\partial}{\partial p}\left(\frac{\partial V}{\partial z_1}\right) - \frac{\partial}{\partial q}\left(\frac{\partial V}{\partial z_2}\right) = 0,$$

should be satisfied. Now, as a function of the variables, V explicitly involves $x_1, x_2, y_1, y_2, z_1, z_2$; but it does not involve x, y, or z, so that

$$\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = 0, \quad \frac{\partial V}{\partial z} = 0.$$

Again (§ 27)

$$\frac{\partial V}{\partial x_1} = y_2 Z - z_2 Y, \quad \frac{\partial V}{\partial x_2} = -y_1 Z + z_1 Y;$$

hence the first equation becomes

$$y_2 Z_1 - z_2 Y_1 - y_1 Z_2 + z_1 Y_2 = 0,$$

that is, on substitution for the derivatives of Y and Z (§ 29),

$$VX(EN - 2FM + GL) = 0.$$

The second and the third equations similarly give

$$VY(EN-2FM+GL)=0, \quad VZ(EN-2FM+GL)=0;$$

hence all the conditions, required in order to make the first variation vanish, are satisfied by the single relation

$$EN-2FM+GL=0.$$

Thus H, the mean measure of curvature, vanishes; the two principal radii of curvature are equal and opposite, and therefore the indicatrix is a rectangular hyperbola.

When, instead of parametric curves, the coordinate axes are used for reference, the area is

$$\iint (1+p^2+q^2)^{\frac{1}{2}}\,dx\,dy,$$

where p and q now denote the derivatives of z. The condition for a stationary value is

$$\frac{\partial}{\partial x}\left\{\frac{p}{(1+p^2+q^2)^{\frac{1}{2}}}\right\}+\frac{\partial}{\partial y}\left\{\frac{q}{(1+p^2+q^2)^{\frac{1}{2}}}\right\}=0,$$

which becomes

$$(1+q^2)\,r-2pqs+(1+p^2)\,t=0,$$

in accordance with the preceding relation when the values (Ex. 3, p. 60) of the fundamental magnitudes are inserted.

167. The result can also be obtained without recourse to the general formulæ of the calculus of variations; and the process* leads to one condition, critical as regards a minimum, which is important for weak variations (§ 89) of the variables x, y, z of a point on the surface.

Let a length l, chosen as an arbitrary function of p and q, be measured along the normal to any surface; and suppose the surface referred to its lines of curvature. Then (§ 85) the quantity $\overline{V}$ for the surface, derived as the locus of the extremity of this length l, is given by

$$\overline{V}^2=\frac{EG}{\alpha^2\beta^2}(l-\alpha)^2(l-\beta)^2+l_1^2\left(\frac{l-\alpha}{\alpha}\right)^2G+l_2^2\left(\frac{l-\beta}{\beta}\right)^2E.$$

For the present purpose, the length l determines a small variation under which the surface is to be minimal; hence l itself is small, and (when we assume the small variation to be weak) the quantities l_1 and l_2 are small, of the same order as l. Expanding $\overline{V}$ in powers of the small quantities l, l_1, l_2, and neglecting powers of these quantities higher than the second, we have

$$\begin{aligned}\overline{V}^2&=V^2\left\{1-2l\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+l^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{4}{\alpha\beta}\right)+\frac{l_1^2}{E}+\frac{l_2^2}{G}\right\}\\&=V^2\left\{1-2lH+l^2(H^2+2K)+\frac{l_1^2}{E}+\frac{l_2^2}{G}\right\}.\end{aligned}$$

so that

$$\overline{V}=V\left\{1-lH+\frac{1}{2}\left(2Kl^2+\frac{l_1^2}{E}+\frac{l_2^2}{G}\right)\right\}.$$

* It is substantially due to Darboux, t. i, §§ 184, 185.

The area of the derived surface, corresponding to an area $\iint V dp dq$ of the original surface, is $\iint \overline{V} dp dq$; hence the variation of area, being

$$\iint \overline{V} dp dq - \iint V dp dq,$$

is given by the expression

$$-\iint l V H dp dq + \frac{1}{2}\iint \left(2Kl^2 + \frac{l_1^2}{E} + \frac{l_2^2}{G}\right) V dp dq.$$

If the original surface is to be minimal, this quantity must be positive for all arbitrary small quantities l, on the understanding that l_1 and l_2 also are small of the same magnitude as l. In the expression, the term of the first order governs the rest unless it vanishes; and when it does not vanish, we can make the sign of the term positive or negative at will by changing the sign of l, and then the condition for a minimum would not be satisfied. Hence the term of the first order must vanish; as l is arbitrary, this requirement can only be satisfied if the equation

$$H = 0$$

holds everywhere on the surface. (If H does not vanish everywhere, we can make the first term positive or negative at will, by choosing l everywhere of the same sign as H or everywhere of the opposite sign.) We thus have the former result as to the equation, which is characteristic of minimal surfaces.

Thus for weak variations of minimal surfaces, the most important term in the variation of the area (it is usually called the *second variation*) is

$$\frac{1}{2}\iint \left(2Kl^2 + \frac{l_1^2}{E} + \frac{l_2^2}{G}\right) V dp dq.$$

In this expression E, G, V are positive, while K is negative; in order that the surface may be a real minimum, the quantity must be positive for all non-vanishing weak variations. As l is arbitrary, subject only to the condition that it must vanish along the closed curve through which the minimal surface is bound to pass, the requirement of a positive sign for the second variation provides a test for a real minimum.

It is easy to prove that, when general parametric curves are selected instead of the lines of curvature, the second variation is

$$\frac{1}{2}\iint \left\{2Kl^2 + \frac{1}{V^2}(El_2^2 - 2Fl_1 l_2 + Gl_1^2)\right\} V dp dq.$$

Ex. Taking weak variations $x+\xi$, $y+\eta$, $z+\zeta$ of x, y, z, so that ξ, η, ζ and their derivatives are small, substituting and using the formulæ of § 27, shew that the second variation of the area can be expressed in the form

$$\frac{1}{2}\iint \frac{1}{V}(E\mu^2 - 2F\lambda\mu + G\lambda^2) dp dq + \iint (X d\eta d\zeta + Y d\zeta d\xi + Z d\xi d\eta),$$

where

$$\lambda = X\xi_1 + Y\eta_1 + Z\zeta_1, \quad \mu = X\xi_2 + Y\eta_2 + Z\zeta_2.$$

We shall see (in § 188) that the requirement of a positive sign for the second variation, in order to secure a real minimum, causes a limitation of the range over which the integration can extend; we shall have the conjugate of an initial curve, when it is associated with an initial tangent developable. But its discussion must be deferred until we have indicated other conditions which make a minimal surface precise.

When strong variations are taken into consideration (that is, variations which keep ξ, η, ζ small and do not demand that ξ_1, ξ_2, η_1, η_2, ζ_1, ζ_2 shall be small), it is necessary to construct the excess-function, as in § 89, IV.

Both investigations, in their general form, belong more to the domain of the calculus of variations* than to that of differential geometry; it may suffice to mention that the excess-function for a surface given by $H=0$ is positive, and so the minimal surface satisfies another test for a true minimum. For our purpose, the important property is that the relation

$$H = EN - 2FM + GL = 0$$

is satisfied at every point of a minimal surface.

Some General Properties.

168. Before proceeding to obtain integral equations, which shall be equivalent to the characteristic equation $H=0$, whether they give the surface intrinsically or express the coordinates of a point explicitly, it is worth while to notice some simple properties generally common to all minimal surfaces.

(i) The nul lines on a minimal surface are conjugate. Let them be taken as the parametric curves for the surface; then $E=0$, $G=0$, and F is not zero. But always

$$EN - 2FM + GL = 0;$$

hence, in this representation, $M=0$. Thus the parametric curves, being the nul lines, are conjugate.

(ii) The asymptotic lines on a minimal surface are perpendicular. Let them be taken as the parametric curves for the surface; then $L=0$, $N=0$, and M is not zero. Again, always

$$EN - 2FM + GL = 0;$$

* A full discussion for any double integral

$$\iint F(x, y, z, x_1, y_1, z_1, x_2, y_2, z_2)\, dp\, dq$$

was first given by Kobb, *Acta Math.*, t. xvi (1893), pp. 65—140. The particular result in the text agrees with Kobb's general result (*l.c.*, p. 114) when the integral is $\iint V dp dq$; his quantities F_1, F_2, F_3, F_4 then are

$$F_1 = G/V, \quad F_2 = E/V, \quad F_3 = -F/V, \quad F_4 = 2KV.$$

Also, the excess-function (*l.c.*, pp. 121—123, 139) becomes equal to $1-\cos\delta$, where δ denotes the angle at which the strong-variation surface cuts the minimal surface. Thus of the full tale of three tests—viz., the characteristic equation, the positive sign of the second variation, and the positive sign of the excess-function—there remains only the test as regards the second variation; it will be considered in § 188.

hence, in this representation, $F=0$. Thus the parametric curves, being the asymptotic lines, are perpendicular.

The property follows also from the fact that the asymptotic lines are always the asymptotes of the indicatrix which, in the case of a minimal surface, is a rectangular hyperbola.

(iii) The converses of the two preceding propositions are valid; that is, if the nul lines are conjugate, or if the asymptotic lines are perpendicular, the surface is minimal. The result is obtained by verifying that the relation

$$EN-2FM+GL=0$$

holds in each case.

(iv) Let the lines of curvature on a minimal surface be taken as the parametric curves. We then have $F=0$, $M=0$; and the characteristic equation of the surface becomes

$$EN+GL=0,$$

that is,

$$\frac{L}{E}+\frac{N}{G}=0.$$

Now, with this representation, the Mainardi-Codazzi relations are

$$L_2-\frac{1}{2}\left(\frac{L}{E}+\frac{N}{G}\right)E_2=0, \quad N_1-\frac{1}{2}\left(\frac{L}{E}+\frac{N}{G}\right)G_1=0.$$

Hence

$$L_2=0, \quad N_1=0;$$

that is, L is a function of p only and N is a function of q only. As

$$\frac{E}{L}=-\frac{G}{N},$$

we now have

$$\frac{\partial^2 \log E}{\partial p \partial q}=\frac{\partial^2 \log G}{\partial p \partial q},$$

which is the condition that the parametric curves are isometric (§ 63).

Thus the lines of curvature on a minimal surface are isometric.

But, as is known (§§ 62, 64), the converse is not valid; that is, a surface can have its lines of curvature an isometric system without being minimal. Thus it may be a surface of revolution, or a central quadric, or a surface of constant (non-zero) mean curvature.

169. Some properties of a simple character belong to the spherical representation of a minimal surface.

The fundamental quantities e, f, g in any spherical image are given by

$$e=-EK+LH, \quad f=-FK+MH, \quad g=-GK+NH;$$

hence, for the image of a minimal surface, we have

$$e=-EK, \quad f=-FK, \quad g=-GK.$$

The following properties may be noted.

(i) The spherical image of a minimal surface is a conformal representation of the surface (§ 159). For the arc-element on the surface is given by

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$

and the arc-element in the spherical image is given by

$$dS^2 = e\,dp^2 + 2f\,dp\,dq + g\,dq^2;$$

hence

$$dS^2 = -K\,ds^2.$$

Thus the magnification is the same in all directions at a point—which is the test for conformal representation.

(ii) The converse of the last proposition is partly valid; that is to say, if the spherical image of a surface is a conformal representation, the surface either is minimal or has its principal radii of curvature equal to one another.

When the spherical image of a surface is a conformal representation, we have

$$-EK + LH = \mu E, \quad -FK + MH = \mu F, \quad -GK + NH = \mu G,$$

where μ is independent of the differentials in the arc-elements; hence

$$LH = E(K + \mu), \quad MH = F(K + \mu), \quad NH = G(K + \mu).$$

Multiply by N, $-2M$, L, and add; we have

$$2KH = H(K + \mu),$$

that is,

$$H(\mu - K) = 0.$$

Multiply by G, $-2F$, E, and add; we have

$$H^2 = 2(K + \mu).$$

Hence either

$$H = 0, \quad \mu = -K;$$

so that either the surface is minimal, or

$$K = \mu, \quad H^2 = 4\mu = 4K,$$

and so

$$\alpha = \beta,$$

that is, the surface has its principal radii of curvature equal at every point.

The latter alternative follows at once from the original equations. For, when μ is not equal to $-K$, they give

$$\frac{L}{E} = \frac{M}{F} = \frac{N}{G};$$

and therefore, at every point, the curvature of the normal section is independent of the direction of the section. This can happen only when the

principal radii of curvature are equal to one another at every point of the surface. Its integral equation* has already (§ 159) been given in the form

$$\left.\begin{aligned} y + \mu x &= \phi(\mu) \\ z - ix(1+\mu^2)^{\frac{1}{2}} &= \psi(\mu) \end{aligned}\right\};$$

the only real surface with the property is the sphere, and it is given by taking

$$\phi(\mu) = b + a\mu + R(1+\mu^2)^{\frac{1}{2}}, \quad \psi(\mu) = c - iR\mu - ia(1+\mu^2)^{\frac{1}{2}},$$

where a, b, c, R are real.

Hence, if there be a restriction to real surfaces, we can declare that, when the spherical image of a surface is a conformal representation of the surface, the surface is either minimal or spherical.

(iii) The spherical images of nul lines on a minimal surface are nul lines on the spherical image.

Taking nul lines as parametric curves on the original surface, we have

$$E = 0, \quad G = 0.$$

Hence, in the spherical image of a minimal surface, we have

$$e = 0, \quad g = 0;$$

that is, the parametric curves are nul lines on the spherical image.

(iv) The nul lines are also asymptotic lines in the spherical image. For, taking them as parametric curves, we have (§ 161)

$$L' = \mp e = 0, \quad N' = \mp g = 0;$$

that is, the parametric curves (being the nul lines) are asymptotic lines in the spherical image.

(v) The converse of the proposition in (iii) is partly valid; that is to say, if the spherical images of nul lines on a surface are themselves nul lines, the surface either is minimal or has its principal radii of curvature equal to one another.

Take the nul lines on the surface as the parametric curves; then $E = 0$, $G = 0$. Now

$$e = -EK + LH, \quad g = -GK + NH;$$

hence, when these parametric curves are nul lines on the sphere, we have

$$LH = 0, \quad NH = 0.$$

We may have $H = 0$; the surface then is minimal. Or we may have H not zero; and then

$$L = 0, \quad N = 0.$$

* See a note by the author; *Messenger of Math.*, vol. xxvii (1898), pp. 129—137.

Thus

$$H = \frac{-2MF}{-F^2} = 2\frac{M}{F}, \quad K = \frac{-M^2}{-F^2} = \frac{M^2}{F^2};$$

hence

$$H^2 - 4K = 0,$$

and the principal radii of curvature are equal.

The same remark about the latter alternative, as was made at the end of the discussion of the property in (ii), holds in the present case.

(vi) The images of isometric lines on a minimal surface are themselves isometric lines.

Take the isometric lines as the parametric curves for the surface; then

$$F = 0, \quad \frac{E}{P} = \frac{G}{Q},$$

where P is a function of p only, and Q is a function of q only. Hence, in the spherical image of the minimal surface, we have

$$f = 0, \quad \frac{e}{P} = \frac{g}{Q};$$

that is, the parametric curves in the spherical image of the minimal surface are isometric lines.

The property also follows as an immediate consequence of the fact that the spherical image of a minimal surface is also a conformal representation of the surface.

(vii) The converse of the proposition in (vi) is partly valid; but the range of alternatives, when no extra condition is imposed, is wider than in the preceding converse propositions.

Suppose that the spherical images of isometric lines on a surface are themselves isometric lines. On the surface, take isometric lines as parametric curves; then

$$F = 0, \quad \frac{E}{P} = \frac{G}{Q},$$

where P is a function of p alone, and Q is a function of q alone. As the parametric curves are isometric in the spherical image, we have

$$f = 0, \quad \frac{e}{P_1} = \frac{g}{Q_1},$$

where P_1 is a function of p alone, and Q_1 is a function of q alone. Let

$$E = P\lambda, \quad G = Q\lambda, \quad e = P_1\mu, \quad g = Q_1\mu;$$

then we have

$$\begin{aligned} P_1\mu &= -P\lambda K + LH, \\ 0 &= MH, \\ Q_1\mu &= -Q\lambda K + NH. \end{aligned}$$

The middle relation can be satisfied by $H=0$; and the other two relations can then be satisfied by

$$\frac{P_1}{P}=\frac{Q_1}{Q}=\text{constant}.$$

The surface is minimal; but there must be a specialised relation between the isometric lines on the surface and the isometric quality of their image.

When the surface is not minimal, so that H is not zero, we must have $M=0$. As $F=0$, $M=0$, the isometric lines are lines of curvature on the surface; the surface accordingly belongs to the class of surfaces which have isometric lines of curvature (§ 64). The other two conditions remain; they impose limitations upon these surfaces.

As an illustration of the latter case, consider the specialised relation between the isometric lines on the surface and the isometric quality of their image given by

$$\frac{P_1}{P}=\frac{Q_1}{Q}=\text{constant};$$

we shall have once more the class of surfaces with their principal radii of curvature equal. For, choosing the special isometric system (§ 63) such that

$$P=1, \quad Q=1,$$

we then have

$$P_1=c, \quad Q_1=c,$$

where c is a constant. The two conditions now are

$$c\mu+\lambda K=LH, \quad c\mu+\lambda K=NH,$$

and, by the present hypothesis, H is not zero. Hence

$$L=N.$$

Thus $E=G$, $F=0$, $L=N$, $M=0$; and then the principal radii of curvature are equal.

In the last alternative, the same remark applies as in (ii) and in (v).

170. The general intrinsic equations of minimal surfaces can be deduced from some of the preceding results. After Bonnet's theorem, we know that any surface is determinate intrinsically (that is, save as to orientation and position) when the six fundamental magnitudes, satisfying the necessary equations of universal condition, are known.

Let the nul lines be taken as the parametric curves; then

$$E=0, \quad G=0,$$

and, because the surface is minimal,

$$M=0.$$

Then (§ 56) the Mainardi-Codazzi equations are

$$L_2 = 0, \quad N_1 = 0;$$

consequently

$$L = P, \quad N = Q,$$

where P is a function of p alone, and Q is a function of q alone. Also the specific curvature is

$$K = -\frac{1}{F}\frac{\partial^2}{\partial p \partial q}(\log F),$$

that is,

$$\frac{PQ}{F} = \frac{\partial^2}{\partial p \partial q}(\log F).$$

Let

$$F = PQ\Phi;$$

then the equation for Φ is

$$\Phi \frac{\partial^2 \log \Phi}{\partial p \partial q} = 1.$$

This is a well-known partial equation of the second order; its primitive (first given by Liouville) is

$$\frac{1}{\Phi} = -2\frac{P_1' Q_1'}{(P_1 + Q_1)^2},$$

where P_1 is any arbitrary function of p alone and Q_1 is any arbitrary function of q alone. Thus

$$F = -\tfrac{1}{2} PQ \frac{(P_1 + Q_1)^2}{P_1' Q_1'}.$$

We now have the values of E, F, G, L, M, N; hence the surface is intrinsically determinate.

The arc-element

$$= 2F dp dq$$

$$= -\frac{PQ}{P_1' Q_1'}(P_1 + Q_1)^2 dp dq.$$

The lines of curvature are given by the equation

$$P dp^2 - Q dq^2 = 0,$$

and the asymptotic lines are given by the equation

$$P dp^2 + Q dq^2 = 0;$$

in the case of both systems of lines, the integral equation is obtainable by quadrature.

Denoting by r and $-r$ the principal radii of curvature, we have

$$r^2 = \frac{F^2}{LN} = \tfrac{1}{4} PQ \frac{(P_1 + Q_1)^4}{P_1'^2 Q_1'^2},$$

so that

$$r = \frac{1}{2}\frac{(P_1+Q_1)^2}{P_1'Q_1'}(PQ)^{\frac{1}{2}}.$$

If we change the parametric variables so that

$$\frac{P}{P_1'}dp = id\xi, \quad \frac{Q}{Q_1'}dq = id\eta, \quad P_1 = \Xi, \quad Q_1 = \mathrm{H},$$

the arc-element is given by

$$ds^2 = (\Xi + \mathrm{H})^2 d\xi d\eta,$$

the form used by Bonnet.

Integral Equations, after Monge and Weierstrass.

171. We now proceed to a more explicit determination of the integral equations of the minimal surface, by obtaining expressions for the Cartesian coordinates of a point upon it in terms of two parameters. These expressions have a variety of useful forms.

Still taking the nul lines as parametric curves, we have

$$\Gamma' = 0, \quad \Delta' = 0;$$

and so three of the equations (§ 34) satisfied by the Cartesian coordinates are

$$x_{12} = MX, \quad y_{12} = MY, \quad z_{12} = MZ.$$

When the surface is minimal, and the nul lines are parametric,

$$M = 0;$$

and therefore the equations are

$$x_{12} = 0, \quad y_{12} = 0, \quad z_{12} = 0.$$

Hence

$$x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3,$$

where U_1, U_2, U_3 are functions of p alone, and V_1, V_2, V_3 are functions of q alone, all arbitrary so far as the particular equations of the second order are concerned. But we must have $E=0$, $G=0$, all the equations of the second order having been deduced from values of E, F, G among other relations; hence

$$U_1'^2 + U_2'^2 + U_3'^2 = 0,$$
$$V_1'^2 + V_2'^2 + V_3'^2 = 0.$$

Subject to these two relations, the functions U and V are arbitrary functions of p and q respectively.

These equations have already (§ 59) been obtained, though in the inverted sequence, during the establishment of Lie's theorem that a minimal surface is the locus of the middle point of a straight line joining any point on one nul line in space to any point on another nul line in space. That theorem, indeed, is the interpretation of the preceding equations and conditions.

The variables p and q are the parametric variables of the nul line; but we can take any function of p, instead of p itself, and any function of q, instead of q itself, and still have parametric variables of the nul lines. Thus p and q are not uniquely determinate quantities; and so some simplifications can be introduced by making the variables more precise. Accordingly, take

$$U_1 = p, \quad V_1 = q, \quad U_2 = \phi(p) = \phi, \quad V_2 = \psi(q) = \psi;$$

then the conditions are satisfied if

$$U_3' = i(1+\phi'^2)^{\frac{1}{2}}, \quad V_3' = i(1+\psi'^2)^{\frac{1}{2}}.$$

Then the integral equations become

$$\left.\begin{aligned} x &= p + q \\ y &= \phi + \psi \\ z &= i\int(1+\phi'^2)^{\frac{1}{2}}\,dp + i\int(1+\psi'^2)^{\frac{1}{2}}\,dq \end{aligned}\right\},$$

where ϕ is any function of p alone and ψ is any function of q alone, both of them arbitrary. This form of the integral equations of a minimal surface is usually associated with the name of Monge, by whom they were first obtained*.

172. Another method of satisfying the two conditions, to which the functions U and V are subject, is as follows. Let a new variable u be introduced, defined by the relation

$$U_1' + iU_2' = -uU_3';$$

manifestly u can be taken as the parametric variable for one set of nul lines. The condition among the functions U is satisfied if

$$U_1' - iU_2' = \frac{1}{u}U_3',$$

which accordingly can be used instead of the condition. From these two linear equations, we have

$$\frac{U_1'}{1-u^2} = \frac{U_2'}{i(1+u^2)} = \frac{U_3'}{2u} = \tfrac{1}{2}F(u)\frac{du}{dp},$$

say; as the one relation affecting the quantities U (which are arbitrary functions of p) is satisfied, the function $F(u)$ is arbitrary. Thus

$$U_1 = \tfrac{1}{2}\int(1-u^2)F(u)\,du,$$

$$U_2 = \tfrac{1}{2}i\int(1+u^2)F(u)\,du,$$

$$U_3 = \int uF(u)\,du.$$

* *Application de l'Analyse à la géométrie*, p. 211.

Next, let a new variable v, conjugate to u, be introduced by the relation

$$V_1' - iV_2' = -vV_3',$$

which is conjugate to the former relation; manifestly v can be taken as the parametric variable for the other set of nul lines. The condition governing the functions V is satisfied if

$$V_1' + iV_2' = \frac{1}{v} V_3',$$

which accordingly can be used instead of the condition. Proceeding as before, we have

$$V_1 = \tfrac{1}{2}\int (1-v^2)\, G(v)\, dv,$$

$$V_2 = -\tfrac{1}{2}i\int (1+v^2)\, G(v)\, dv,$$

$$V_3 = \int vG(v)\, dv,$$

where the function $G(v)$ is arbitrary. Hence the integral equations of the minimal surface become

$$\left.\begin{aligned} x &= \tfrac{1}{2}\int (1-u^2) F(u)\, du + \tfrac{1}{2}\int (1-v^2)\, G(v)\, dv \\ y &= \tfrac{1}{2}i\int (1+u^2) F(u)\, du - \tfrac{1}{2}i\int (1+v^2)\, G(v)\, dv \\ z &= \int uF(u)\, du + \int vG(v)\, dv \end{aligned}\right\},$$

where $F(u)$ is any arbitrary function of u alone, and $G(v)$ is any arbitrary function of v alone.

If x, y, z are to be real—that is, if we are to deal with only the real sheets of the surface—$G(v)$ must be the conjugate of $F(u)$. Denoting by Rw the real part of a complex variable w, we can write the foregoing equations in the form

$$x = R\int (1-u^2) F(u)\, du, \quad y = Ri\int (1+u^2) F(u)\, du, \quad z = R2\int uF(u)\, du.$$

Both forms suffer from the disadvantage of appearing to require quadratures; but the disadvantage can be removed by changing the arbitrary functions. Let

$$F(u) = f'''(u), \quad G(v) = g'''(v),$$

where $f(u)$ and $g(v)$ are new arbitrary functions of u alone and of v alone respectively; then the quadratures can be effected, with the result

$$\left.\begin{aligned} x &= \tfrac{1}{2}(1-u^2) f''(u) + uf'(u) - f(u) \\ &\qquad + \tfrac{1}{2}(1-v^2)\, g''(v) + vg'(v) - g(v) \\ y &= \tfrac{1}{2}i(1+u^2) f''(u) - iuf'(u) + if(u) \\ &\qquad - \tfrac{1}{2}i(1+v^2)\, g''(v) + ivg'(v) - ig(v) \\ z &= uf''(u) - f'(u) + vg''(v) - g'(v) \end{aligned}\right\}.$$

As before, if x, y, z are to be real, so that then we should be dealing with only the real sheets of the surface, $g(v)$ must be the conjugate of $f(u)$. In that case, the last set of equations can be written in the form

$$\left.\begin{aligned} x &= R\{(1-u^2)f''(u) + 2uf'(u) - 2f(u)\} \\ y &= R\{i(1+u^2)f''(u) - 2iuf'(u) + 2if(u)\} \\ z &= R\{2uf''(u) - 2f'(u)\} \end{aligned}\right\}.$$

All these forms are due* to Weierstrass, though the first suggestion of satisfying the conditions for the functions U and V in the preceding manner was made† by Enneper.

173. Before proceeding to use these forms of the integral equations of a minimal surface, it should be noticed that one assumption has tacitly been made and two possible exceptions have tacitly been ignored. It has been assumed

(i) that the nul lines are distinct;

(ii) that u, as defined, is variable and not constant;

(iii) that v, as defined, is variable and not constant.

Account must be taken of the cases, if any, in which these assumptions are not justified.

(i) Let us enquire whether it is possible to have a minimal surface on which the nul lines are coincident. When the arc-element, as usual, is

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2,$$

the condition that the nul lines should coincide is

$$EG - F^2 = 0.$$

Let this single direction be taken for the parametric curve $q = \text{constant}$; in order that this curve may be a nul line, we must have

$$E = 0.$$

The former condition thus gives

$$F = 0.$$

As the surface is minimal, we have

$$EN - 2FM + GL = 0;$$

and therefore, as G is not zero because the arc-element is given by

$$ds^2 = Gdq^2,$$

we have

$$L = 0;$$

that is,

$$E = 0, \quad F = 0, \quad L = 0.$$

* *Berl. Monatsber.*, (1866), pp. 612—625, 855—856.

† *Zeitschrift f. Math. u. Physik*, t. ix (1864), pp. 96—125.

Then $\Gamma = 0$, $\Gamma' = 0$, $\Gamma'' = -\frac{1}{2}GG_1$; $\Delta = 0$, $\Delta' = 0$, $\Delta'' = 0$; and so one of the sets of equations in § 34 becomes

$$x_{11} = 0, \quad y_{11} = 0, \quad z_{11} = 0.$$

Hence

$$x = pA_1 + A_2, \quad y = pB_1 + B_2, \quad z = pC_1 + C_2,$$

where the functions A, B, C are functions of q alone. But we are to have

$$E = 0, \quad F = 0;$$

hence

$$A_1^2 + B_1^2 + C_1^2 = 0,$$

$$A_1(pA_1' + A_2') + B_1(pB_1' + B_2') + C_1(pC_1' + C_2') = 0.$$

From the former we have

$$A_1A_1' + B_1B_1' + C_1C_1' = 0;$$

and so the latter becomes

$$A_1A_2' + B_1B_2' + C_1C_2' = 0.$$

By another of the sets of equations in § 34, we have, for the present case

$$\left.\begin{aligned} MX &= x_{12} = A_1' \\ MY &= y_{12} = B_1' \\ MZ &= z_{12} = C_1' \end{aligned}\right\},$$

so that the direction-cosines of the tangent plane to the surface are proportional to A_1', B_1', C_1'. Let the current Cartesian coordinates in space be momentarily denoted by ξ, η, ζ; then the Cartesian equation of the plane is

$$(\xi - x)A_1' + (\eta - y)B_1' + (\zeta - z)C_1' = 0,$$

that is,

$$\xi A_1' + \eta B_1' + \zeta C_1' = A_2A_1' + B_2B_1' + C_2C_1'.$$

Thus the equation to the tangent plane to the surface contains only one parameter. Hence the surface is a developable; and manifestly it is imaginary.

Also

$$\begin{aligned}(x_1y_2 - x_2y_1)^2 + (y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 &= V^2 \\ &= 0,\end{aligned}$$

in the present case; hence

$$X^2 + Y^2 + Z^2 = 0,$$

that is,

$$A_1'^2 + B_1'^2 + C_1'^2 = 0;$$

or the imaginary developable surface touches the circle at infinity*.

* See § 55, note.

(ii) Next, suppose that one (but not both) of the quantities u and v is constant. Let u be constant; then take

$$\frac{U_1'}{1-u^2}=\frac{U_2'}{i(1+u^2)}=\frac{U_3'}{2u}$$
$$=\frac{1}{2}\frac{dP}{dp},$$

where P is a function of p only. The integral equations of the minimal surface become

$$\left.\begin{aligned} x&=\tfrac{1}{2}(1-u^2)P+\tfrac{1}{2}\int(1-v^2)G(v)\,dv\\ y&=\tfrac{1}{2}i(1+u^2)P-\tfrac{1}{2}i\int(1+v^2)G(v)\,dv\\ z&=uP+\int vG(v)\,dv \end{aligned}\right\}.$$

The curves, $v=$ constant, on the surface are straight lines meeting the circle at infinity; the surface is an imaginary cylinder.

(iii) Lastly, if both u and v are constant, we find similarly

$$\left.\begin{aligned} x&=\tfrac{1}{2}(1-u^2)P+\tfrac{1}{2}(1-v^2)Q\\ y&=\tfrac{1}{2}i(1+u^2)P-\tfrac{1}{2}i(1+v^2)Q\\ z&=uP+vQ \end{aligned}\right\}.$$

The surface manifestly is a plane.

174. These exceptions may now be set aside. We return to the general integral equations of a minimal surface; when it is referred to nul lines as parametric curves, these equations are

$$\left.\begin{aligned} x&=\tfrac{1}{2}(1-u^2)f''(u)+uf'(u)-f(u)\\ &\qquad+\tfrac{1}{2}(1-v^2)g''(v)+vg'(v)-g(v)\\ y&=\tfrac{1}{2}i(1+u^2)f''(u)-iuf'(u)+if(u)\\ &\qquad-\tfrac{1}{2}i(1+v^2)g''(v)+ivg'(v)-ig(v)\\ z&=uf''(u)-f'(u)+vg''(v)-g'(v) \end{aligned}\right\},$$

where, for the present, the arbitrary functions $f(u)$ and $g(v)$ will not be limited by the condition of being conjugate to one another.

We write x_1 for $\partial x/\partial u$, x_2 for $\partial x/\partial v$, and so for all the derivatives. We have

$$x_1=\tfrac{1}{2}(1-u^2)f''',\quad y_1=\tfrac{1}{2}i(1+u^2)f''',\quad z_1=uf''',$$
$$x_2=\tfrac{1}{2}(1-v^2)g''',\quad y_2=-\tfrac{1}{2}i(1+v^2)g''',\quad z_2=vg''';$$

and therefore

$$E=0,\quad F=\tfrac{1}{2}(1+uv)^2f'''g''',\quad G=0,\quad V=iF;$$

thus the arc-element is

$$ds^2 = (1+uv)^2 f'''g''' du dv.$$

Further,

$$X = \frac{u+v}{1+uv}, \quad Y = i\frac{v-u}{1+uv}, \quad Z = \frac{uv-1}{1+uv};$$

and therefore

$$u = \frac{X+iY}{1-Z}, \quad v = \frac{X-iY}{1-Z},$$

giving another significance to u and v in connection with the normal to the surface*.

The fundamental magnitudes of the second order are

$$\left.\begin{aligned} L &= Xx_{11} + Yy_{11} + Zz_{11} = -f''' \\ M &= Xx_{12} + Yy_{12} + Zz_{12} = 0 \\ N &= Xx_{22} + Yy_{22} + Zz_{22} = -g''' \end{aligned}\right\}.$$

Also

$$\left.\begin{aligned} \Gamma &= \frac{F_1}{F} = \frac{2v}{1+uv} + \frac{f''''}{f'''}, \quad \Gamma' = 0, \quad \Gamma'' = 0 \\ \Delta'' &= \frac{F_2}{F} = \frac{2u}{1+uv} + \frac{g''''}{g'''}, \quad \Delta' = 0, \quad \Delta = 0 \end{aligned}\right\}.$$

The derived magnitudes of the third order are

$$\left.\begin{aligned} P &= \frac{4v}{1+uv} f''' + f'''' \\ Q &= 0 \\ R &= 0 \\ S &= \frac{4u}{1+uv} g''' + g'''' \end{aligned}\right\}.$$

175. The lines of curvature on the surface, being

$$\begin{vmatrix} Edu + Fdv, & Fdu + Gdv \\ Ldu + Mdv, & Mdu + Ndv \end{vmatrix} = 0$$

in general, now are

$$f''' du^2 - g''' dv^2 = 0.$$

The asymptotic lines are

$$f''' du^2 + g''' dv^2 = 0,$$

and manifestly are perpendicular to one another.

The nul lines are the parametric curves.

* These are the conjugate complex combinations already mentioned in § 17.

The geodesics on the surface are given (§ 118) by

$$\frac{d^2u}{ds^2}+\frac{F_1}{F}\left(\frac{du}{ds}\right)^2=0,\quad \frac{d^2v}{ds^2}+\frac{F_2}{F}\left(\frac{dv}{ds}\right)^2=0,$$

$$\frac{d^2v}{du^2}=-\frac{F_2}{F}\left(\frac{dv}{du}\right)^2+\frac{F_1}{F}\frac{dv}{du}.$$

(The equations are satisfied by

$$u=\text{constant},\quad v=\text{constant},$$

thus verifying the theorem (§ 92) that the nul lines satisfy the equations for geodesics.) When the value of F is inserted, the third of the equations becomes

$$\frac{d^2v}{du^2}=-\left(\frac{2u}{1+uv}+\frac{g''''}{g'''}\right)\left(\frac{dv}{du}\right)^2+\left(\frac{2v}{1+uv}+\frac{f''''}{f'''}\right)\frac{dv}{du}.$$

The lines of hyperosculation are

$$\left(\frac{4v}{1+uv}f'''+f''''\right)du^3+\left(\frac{4u}{1+uv}g'''+g''''\right)dv^3=0.$$

176. Three of the tangential coordinates, X, Y, Z, have been obtained in terms of u and v. For the remaining coordinate T, we have

$$T=Xx+Yy+Zz$$

$$=f'+g'-2\frac{vf+ug}{1+uv},$$

on substitution and reduction.

For the spherical representation of the minimal surface, we have

$$K=\frac{LN}{-F^2}=-\frac{4}{(1+uv)^4f'''g'''},\quad H=0;$$

and therefore constructing the coefficients in dS^2, which gives the element of arc on the sphere, we find

$$dS^2=\frac{4}{(1+uv)^2}du\,dv.$$

The spherical representation is manifestly conformal, as is known; the magnification m of the surface on the sphere, being $(-K)^{\frac{1}{2}}$, is such that

$$\frac{1}{m^2}=\tfrac{1}{4}(1+uv)^4f'''g'''.$$

Also

$$\gamma=-\frac{2v}{1+uv},\quad \gamma'=0,\quad \gamma''=0,$$

$$\delta=0,\quad \delta'=0,\quad \delta''=-\frac{2u}{1+uv};$$

and so (§ 163) X, Y, Z, T are four solutions of the equation

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{2}{(1+uv)^2}\theta = 0.$$

Since the foregoing expression for T involves two arbitrary functions, the primitive of this equation is given by $\theta = T$. The quantities X, Y, Z are special solutions, derivable by assigning special forms to the arbitrary functions f and g in T; thus

T becomes X, when $f = -\frac{1}{2}$, $\quad g = -\frac{1}{2}$,

.... Y, ... $f = -\frac{1}{2}i$, $\quad g = \ \frac{1}{2}i$,

.... Z, ... $f = -\frac{1}{2}u$, $\quad g = -\frac{1}{2}v$.

Moreover, the tangential equation of the minimal surface can be obtained at once; for

$$u = \frac{X+iY}{1-Z}, \quad v = \frac{X-iY}{1-Z},$$

so that

$$T = f'\left(\frac{X+iY}{1-Z}\right) + g'\left(\frac{X-iY}{1-Z}\right)$$
$$- (X-iY) f\left(\frac{X+iY}{1-Z}\right) - (X+iY) g\left(\frac{X-iY}{1-Z}\right),$$

being the tangential equation in question.

When we deal with only the real sheets of real surfaces, u and v are conjugate, while $f(u)$ and $g(v)$ also are conjugate; and then some simplification arises in the expression of the tangential equation. Thus for Enneper's surface (§ 177), given by

$$f = u^3, \quad g = v^3,$$

we have

$$T = \frac{4-2Z}{(1-Z)^2}(X^2 - Y^2),$$

an equation of the sixth class, when made homogeneous and rational; for Henneberg's surface (§ 177), given by

$$f = \frac{1}{u}(1-u^2)^2, \quad g = \frac{1}{v}(1-v^2)^2,$$

we have

$$(T - 4Z)(X^2 + Y^2)^2 = 4Z(X^2 - Y^2)(3X^2 + 3Y^2 + 2Z^2),$$

an equation of the fifth class.

177. Some special examples of minimal surfaces may be taken in illustration of the formulæ.

Ex. 1. Enneper's surface* has already (§ 59) been mentioned. We take

$$f=u^3, \quad g=v^3;$$

and so

$$\left.\begin{aligned} x&=3u-u^3+3v-v^3\\ y&=i(3u+u^3)-i(3v+v^3)\\ z&=3(u^2+v^2)\end{aligned}\right\}.$$

Since the expressions for x, y, z in terms of the parameters are algebraic and rational, the surface is algebraic and unicursal. When the parameters are eliminated, the Cartesian equation of the surface is found to be

$$2(2z^3-27x^2+27y^2+216z)^3=27z\{27(x^2+y^2)z+24z^3+162(x^2-y^2)+864z\}^2;$$

and the surface (known to be of the sixth class) is clearly of the ninth order.

The equation of the lines of curvature is $du^2-dv^2=0$; hence when we write

$$u=a+i\beta, \quad v=a-i\beta,$$

the quantities a and β are the parameters of the lines of curvature. We then have

$$\left.\begin{aligned} x&=6a+6a\beta^2-2a^3\\ -y&=6\beta+6a^2\beta-2\beta^3\\ z&=6a^2-6\beta^2\end{aligned}\right\}.$$

These give

$$\left.\begin{aligned} x+az&=6a+4a^3\\ y+\beta z&=-6\beta-4\beta^3\end{aligned}\right\};$$

these are the equations of the lines of curvature, which are plane.

The equation of the tangent plane is

$$2ax+2\beta y+(a^2+\beta^2-1)z=2a^4-2\beta^4+6a^2-6\beta^2;$$

and therefore

$$X=\frac{2a}{a^2+\beta^2+1}, \quad Y=\frac{2\beta}{a^2+\beta^2+1}, \quad Z=\frac{a^2+\beta^2-1}{a^2+\beta^2+1}.$$

Taking the plane lines of curvature as parametric curves, we find

$$E=G=36(1+a^2+\beta^2)^2, \quad F=0;$$

$$L=-12, \quad M=0, \quad N=12.$$

The asymptotic lines are given by $du^2+dv^2=0$, that is, by

$$a+\beta=c_1, \quad a-\beta=c_2,$$

where c_1 and c_2 are constants; along the former, we have

$$\begin{aligned} x&=6c_1-2c_1^3-6(1-c_1^2)\beta-4\beta^3,\\ -y&=6(1+c_1^2)\beta-12c_1\beta^2+4\beta^3,\\ z&=6c_1^2-12c_1\beta,\end{aligned}$$

so that the line is a twisted cubic, and similarly for the other; and their spherical images are small circles

$$X+Y=c_1(1-Z), \quad X-Y=c_2(1-Z).$$

The spherical images of the lines of curvature are the small circles

$$X=a(1-Z), \quad Y=\beta(1-Z).$$

* *Zeitschrift f. Math. u. Physik*, t. ix (1864), p. 108.

Ex. 2. Henneberg's surface* is given by

$$f(u)=\frac{1}{u}(1-u^2)^2, \quad g(v)=\frac{1}{v}(1-v^2)^2;$$

and the integral equations are

$$\left.\begin{aligned} x&=\frac{1}{u^3}(1-u^2)^3+\frac{1}{v^3}(1-v^2)^3\\ y&=\frac{i}{u^3}(1+u^2)^3-\frac{i}{v^3}(1+v^2)^3\\ z&=3\left(\frac{1}{u^2}+u^2\right)+3\left(\frac{1}{v^2}+v^2\right)\end{aligned}\right\}.$$

The surface is manifestly algebraical. Its fundamental magnitudes are

$$E=0, \quad G=0, \quad F=18(1-u^4)(1-v^4)(1+uv)^2u^{-4}v^{-4},$$

$$L=6\left(\frac{1}{u^4}-1\right), \quad M=0, \quad N=6\left(\frac{1}{v^4}-1\right).$$

The lines of curvature are algebraical, being given by the algebraical equation which is the equivalent of the differential equation

$$(1-u^4)^{\frac{1}{2}}\frac{du}{u^2}\pm(1-v^4)^{\frac{1}{2}}\frac{dv}{v^2}=0;$$

and the asymptotic lines also are algebraical, being given by the algebraical equation which is the equivalent of the differential equation

$$(1-u^4)^{\frac{1}{2}}\frac{du}{u^2}\pm(v^4-1)^{\frac{1}{2}}\frac{dv}{v^2}=0.$$

Ex. 3. Prove that the order of Henneberg's surface is 15.

Ex. 4. As another particular surface, let

$$f'''(u)=F(u)=e^{ia}u^{-2}, \quad g'''(v)=G(v)=e^{-ia}v^{-2};$$

and, assuming u and v to be conjugate, write

$$u=re^{i\theta}, \quad v=re^{-i\theta}.$$

Then

$$-x=\tfrac{1}{2}e^{ia}\left(\frac{1}{u}+u\right)+\tfrac{1}{2}e^{-ia}\left(\frac{1}{v}+v\right)$$

$$\left.\begin{aligned} &=r\cos(\theta+a)+\frac{1}{r}\cos(\theta-a)\\ -y&=r\sin(\theta+a)+\frac{1}{r}\sin(\theta-a)\\ -z&=2\theta\sin a-2(\log r)\cos a\end{aligned}\right\},$$

giving a helicoid†. We have

$$ds^2=\left(1+\frac{1}{r^2}\right)^2(dr^2+r^2d\theta^2),$$

so that the arc-element is independent of a; consequently, the surfaces in the family, constituted by all parametric values of a, are deformable into one another. Also

$$\frac{X}{2\cos\theta}=\frac{Y}{2\sin\theta}=\frac{Z}{r-\frac{1}{r}}=\frac{1}{r+\frac{1}{r}};$$

* *Ann. di Mat.*, 2ᵈᵃ Ser., t. ix (1878, 9), pp. 54—57.

† Frost, *Solid Geometry*, (3rd ed., 1886), p. 218.

so that, at corresponding points determined by the same values of r and θ on the family of surfaces, the tangent planes are parallel; and so the surfaces have the same spherical representation. Also

$$L = -\frac{2}{r^2}\cos a, \quad M = \frac{2}{r}\sin a, \quad N = 2\cos a;$$

and therefore, as

$$LN - M^2 = -\frac{4}{r^2},$$

the Gaussian measure of curvature is the same for all the surfaces at corresponding points—which will appear as a property of surfaces deformable into one another.

The lines of curvature are given by

$$\frac{dr}{r} + (\cot a \pm \operatorname{cosec} a)\, d\theta = 0,$$

and the asymptotic lines by

$$\frac{dr}{r} - (\cot a \pm \operatorname{cosec} a)\, d\theta = 0.$$

Note. Among the family of surfaces, there are two important special members. When $a = \frac{1}{2}\pi$, the surface is

$$-z = 2\tan^{-1}\frac{x}{y}.$$

When $a = 0$, the surface is

$$(x^2 + y^2)^{\frac{1}{2}} = 2\cosh z,$$

the catenoid; it is a surface of revolution.

Ex. 5. The catenoid is the only minimal surface of revolution. For any surface of revolution, we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = R,$$

where R is a function of r only; so

$$E = 1 + R'^2, \qquad F = 0, \quad G = r^2,$$
$$L = R''(1 + R'^2)^{-\frac{1}{2}}, \quad M = 0, \quad N = rR'(1 + R'^2)^{-\frac{1}{2}}.$$

When the surface is minimal, we have

$$rR'(1 + R'^2)^{\frac{1}{2}} + r^2 R''(1 + R'^2)^{-\frac{1}{2}} = 0.$$

Then

$$R' = a(r^2 - a^2)^{-\frac{1}{2}},$$

where a is an arbitrary constant; and so

$$r = a\cosh(R - c) = a\cosh(z - c),$$

where c is an arbitrary constant. This surface is the catenoid in question.

Real Surfaces; Algebraic Surfaces.

178. The analytical connection, between the formulæ giving a minimal surface and the general formulæ in the theory of functions of a complex variable, is too obvious to require any laboured discussion. Two initial questions, to which in special cases some special answers have been given, present themselves. In what circumstances is a minimal surface algebraic? What are the conditions that it should be real?

Two arbitrary (and therefore disposable) functions occur in Weierstrass's formulæ for a minimal surface. If these functions f and g are algebraic, the formulæ express x, y, z as algebraic functions of u and v; when the parameters are eliminated between the three equations, the eliminant is an algebraic relation between x, y, z; that is, the minimal surface is algebraic.

The converse also is true; that is to say, when a minimal surface is algebraic, the functions f and g are algebraic. Consider the nul lines on the algebraic surface; they are given by

$$dx^2 + dy^2 + dz^2 = 0, \quad dz = p\,dx + q\,dy.$$

Now p and q are algebraic functions of x, y, z, that is, owing to the equation of the surface, they are algebraic functions of x and y; hence these equations for the nul lines determine two sets of values for $dx : dy : dz$, each of which is composed of algebraic functions of x and y. But the surface is also minimal; so we have

$$\frac{x_1 + iy_1}{z_1} = -u, \quad \frac{x_1 - iy_1}{z_1} = \frac{1}{u}, \quad \frac{x_2 - iy_2}{z_2} = -v, \quad \frac{x_2 + iy_2}{z_2} = \frac{1}{v},$$

from the Weierstrass equations*. One direction of nul lines is given by

$$x_1 du : y_1 du : z_1 du,$$

that is, by

$$x_1 : y_1 : z_1.$$

The direction has just been proved to be expressible by algebraic functions of x and y; hence u is an algebraic function of x and y. Similarly for v. Thus u and v are algebraic functions of x and y; consequently x and y (and therefore z also, owing to the equation of the surface) are algebraic functions of u and v.

Now each of the coordinates x, y, z is expressed, by Weierstrass's formulæ in a form

$$\theta(u) + \vartheta(v);$$

hence, as each of them is an algebraic function of u and v, we have a relation

$$A\,\{\theta(u) + \vartheta(v), u, v\} = 0,$$

where A is algebraic. In this equation, let any constant value be assigned to v; then $\vartheta(v)$ also is constant; and so the equation determines $\theta(u)$ as an algebraic function of u. Similarly it determines $\vartheta(v)$ as an algebraic function of v.

The quantities $\theta(u)$ in the expressions for x, y, z respectively are

$$\xi = \tfrac{1}{2}(1 - u^2)f'' + uf' - f,$$
$$\eta = \tfrac{1}{2}i(1 + u^2)f'' - iuf' + if,$$
$$\zeta = uf'' - f';$$

* As $(x_1 + iy_1)/z_1$ has the same value whatever parameter of the nul line is used, being $(dx + idy)/dz$ for the line, the expression determines the actual value of u for a given minimal surface. Similarly for v.

and each of these is an algebraic function of u. But

$$f = -\tfrac{1}{2}\{(1-u^2)\xi + i(1+u^2)\eta + 2u\zeta\};$$

and therefore f is an algebraic function of u. Similarly g is an algebraic function of v.

Hence in order to have an algebraic minimal surface, it is necessary that the functions f and g should be algebraic functions of their arguments.

179. To discuss the reality of a minimal surface, it is simplest to proceed from the equations

$$x = \tfrac{1}{2}\int(1-u^2)F(u)\,du + \tfrac{1}{2}\int(1-v^2)G(v)\,dv,$$

$$y = \tfrac{1}{2}i\int(1+u^2)F(u)\,du - \tfrac{1}{2}i\int(1+v^2)G(v)\,dv,$$

$$z = \int uF(u)\,du + \int vG(v)\,dv.$$

When the paths of integration for u and for v are such as to give conjugate complex variables at corresponding points, and when $F(u)$ and $G(v)$ are conjugate, then x, y, z are real and the surface is real. The converse is true. The nul directions, as given by

$$dx^2 + dy^2 + dz^2 = 0, \quad dz = p\,dx + q\,dy,$$

are given by conjugate complex variables on a real surface; as they also are given by

$$\frac{x_1 + iy_1}{z_1} = -u, \quad \frac{x_2 - iy_2}{z_2} = -v,$$

it follows that u and v are conjugate. Also

$$x_1 - iy_1 = F(u), \quad x_2 + iy_2 = G(v),$$

and $x_1 - iy_1$, $x_2 + iy_2$ are conjugate; hence $F(u)$ and $G(v)$ are conjugate, shewing that the conditions for reality are sufficient.

The reason, why it is simpler to discuss the last matter through the functions F and G rather than through f and g, is that, as the functions are defined by the relations

$$f'''(u) = F(u), \quad g'''(v) = G(v),$$

the functions $f(u)$ and $g(v)$ are not definite but are subject to additive terms

$$au^2 + 2bu + c, \quad a'v^2 + 2b'v + c',$$

respectively. The effect of such additive terms is to add to x, y, z respectively the constants

$$a - c + a' - c', \quad \tfrac{1}{2}i(a + c - a' - c'), \quad -2b - 2b';$$

and these can be zero without making $au^2 + 2bu + c$ and $a'v^2 + 2b'v + c'$ conjugate, that is, without keeping $f(u)$ and $g(v)$ conjugate.

180. It is clear that, when functions $F(u)$ and $G(v)$ are given, the values of x, y, z are determinate and unique save as to additive arbitrary constants that arise in the quadratures; hence given functions F and G determine a minimal surface uniquely save as to its position in space.

It is not, however, the fact that a minimal surface leads to a unique determination of functions $F(u)$ and $G(v)$, in connection with nul lines as parametric curves. The quantities u and v are determined as a pair of magnitudes, being the joint parameters of the nul lines. Accordingly, let u' and v' be another pair of magnitudes as parameters of nul lines, and let $A(u')$ and $B(v')$ be the corresponding functions in the expressions for x, y, z. Then we have two cases:—(i) when u' is a function of u, and v' is a function of v; (ii) when u' is a function of v, and v' is a function of u.

In the former case, we have

$$\begin{aligned}(1-u^2)F(u)\,du &= (1-u'^2)A(u')\,du', & (1-v^2)G(v)\,dv &= (1-v'^2)B(v')\,dv',\\ (1+u^2)F(u)\,du &= (1+u'^2)A(u')\,du', & (1+v^2)G(v)\,dv &= (1+v'^2)B(v')\,dv',\\ uF(u)\,du &= u'A(u')\,du', & vG(v)\,dv &= v'B(v')\,dv';\end{aligned}$$

and these relations can only be satisfied if

$$u=u',\quad F(u)=A(u'),\quad v=v',\quad G(v)=B(v').$$

No new expressions for x, y, z are given in this case.

In the latter case, we have

$$\begin{aligned}(1-u^2)F(u)\,du &= (1-v'^2)B(v')\,dv', & (1-v^2)G(v)\,dv &= (1-u'^2)A(u')\,du',\\ (1+u^2)F(u)\,du &= -(1+v'^2)B(v')\,dv', & (1+v^2)G(v)\,dv &= -(1+u'^2)A(u')\,du',\\ uF(u)\,du &= v'B(v')\,dv', & vG(v)\,dv &= u'A(u')\,du';\end{aligned}$$

and these relations can only be satisfied if

$$uv'=-1,\quad F(u)=-v'^4B(v'),\quad u'v=-1,\quad G(v)=-u'^4A(u').$$

When the surfaces are real, F and G are conjugate, and A and B are conjugate; and

$$\begin{aligned}A(u') &= -\frac{1}{u'^4}G(v)\\ &= -\frac{1}{u'^4}G\left(-\frac{1}{u'}\right).\end{aligned}$$

Thus there are two forms of function, $F(u)$ with its conjugate, and $-\dfrac{1}{u'^4}G\left(-\dfrac{1}{u'}\right)$ with its conjugate, for the expressions of x, y, z as a point on a given real minimal surface.

Consider, further, this double analytical representation of a real minimal surface. The direction-cosines of the normal in the first representation are given by

$$X = \frac{u+v}{1+uv}, \quad Y = i\frac{v-u}{1+uv}, \quad Z = \frac{uv-1}{1+uv};$$

and in the second representation the direction-cosines of the normal are given by

$$X' = \frac{u'+v'}{1+u'v'}, \quad Y' = i\frac{v'-u'}{1+u'v'}, \quad Z' = \frac{u'v'-1}{1+u'v'}.$$

But

$$uv' = -1, \quad u'v = -1;$$

hence

$$X' = -X, \quad Y' = -Y, \quad Z' = -Z.$$

Consequently, the normals are in opposite directions in the two representations.

Double Surfaces.

181. One interesting set of surfaces arises when the functions in the expressions for a real minimal surface are such that

$$F(t) = -\frac{1}{t^4}G\left(-\frac{1}{t}\right), \quad G(t) = -\frac{1}{t^4}F\left(-\frac{1}{t}\right),$$

being of course only a single relation. The first representation then gives

$$\left.\begin{aligned} 2dx &= (1-u^2)F(u)\,du + (1-v^2)G(v)\,dv \\ 2dy &= i(1+u^2)F(u)\,du - i(1+v^2)G(v)\,dv \\ dz &= uF(u)\,du + vG(v)\,dv \end{aligned}\right\}.$$

The second representation then gives

$$\begin{aligned} 2dx &= (1-u'^2)A(u')\,du' + (1-v'^2)B(v')\,dv' \\ &= (1-u'^2)\left\{-\frac{1}{u'^4}G\left(-\frac{1}{u'}\right)\right\}du' + (1-v'^2)\left\{-\frac{1}{v'^4}F\left(-\frac{1}{v'}\right)\right\}dv' \\ &= (1-u'^2)F(u')\,du' + (1-v'^2)G(v')\,dv', \end{aligned}$$

and similarly for the others; that is, the second representation gives

$$\left.\begin{aligned} 2dx &= (1-u'^2)F(u')\,du' + (1-v'^2)G(v')\,dv' \\ 2dy &= i(1+u'^2)F(u')\,du' - i(1+v'^2)G(v')\,dv' \\ dz &= u'F(u')\,du' + v'G(v')\,dv' \end{aligned}\right\}.$$

Now

$$u' = -\frac{1}{v}, \quad v' = -\frac{1}{u};$$

and therefore the surface, in the vicinity of the point u, v, has exactly the

same variations as in the vicinity of the point $-\frac{1}{v}, -\frac{1}{u}$. The values of the parameters at any point are determinate functions of the position of the point; hence, when the integration for x, y, z is effected, either

(i) the values of x, y, z in the first representation differ from those in the second by constants; or

(ii) the values of x, y, z in the first representation are the same as those in the second.

In the first case, a suitable bodily translation (determined by the constants) will make the two sets of values of x, y, z the same; that is to say, a suitable translation of the surface will bring the part of the surface in the vicinity of the point u, v to coincide with the part of the surface in the vicinity of the point $-\frac{1}{v}, -\frac{1}{u}$. Such a surface is periodic and therefore not algebraical.

In the second case, the part of the surface in the vicinity of the point u, v coincides (without any translation) with the part of the surface in the vicinity of the point $-\frac{1}{v}, -\frac{1}{u}$. When the function F is algebraical, such a surface F is algebraical.

Now the normals at these two different parametric points, which geometrically coincide on the surface, lie in opposite senses on the same line. Accordingly if we trace a path on the surface from the point u, v to the point $-\frac{1}{v}, -\frac{1}{u}$, we return to the same geometrical position on the surface while, at the end of the path, the normal assumes a position directly opposite to its initial position. Thus it is possible, without any breach of continuity, to pass from any position to the same position as though the surface were pierced at that place; in other words, the surface has only one side*, instead of the familiar two sides. The notion of these minimal surfaces is due to Lie† who called them *double surfaces*. The test that a surface should be double is that, if F and G are conjugate functions in the quadrature expressions for the coordinates of a point on a real minimal surface, the relation

$$F(t) = -\frac{1}{t^4} G\left(-\frac{1}{t}\right)$$

should be satisfied identically.

* The simplest example, in model form, of a one-sided surface occurs when a long rectangular strip of paper $ABCD$ (of which AC and BD are the diagonals) is twisted once, or an odd number of times, and then joined into a twisted ring by making the edge AB coincide with the edge CD so that A coincides with C and B with D.

† *Math. Ann.*, t. xiv (1878), pp. 331—416; *ib.*, t. xv (1879), pp. 465—506.

182. Special examples of double surfaces can be obtained directly by solving (either generally or specially) this functional equation. Let

$$t^2 F(t) = \phi(t),$$

and let ϕ_0 be the function conjugate to ϕ; then the equation is

$$\phi(t) = -\phi_0\left(-\frac{1}{t}\right).$$

A solution of this equation is given by

$$\phi(t) = ia,$$

where a is a real constant; then

$$F(t) = \frac{ia}{t^2},$$

and we have the helicoid (§ 177, Ex. 4), a periodic surface.

Another solution is given by

$$\phi(t) = a\left(t + \frac{1}{t}\right),$$

where a is a real constant; then

$$F(t) = a\left(\frac{1}{t} + \frac{1}{t^3}\right),$$

and we have another periodic surface.

Another solution is given by

$$\phi(t) = a\left(t^2 - \frac{1}{t^2}\right),$$

where a is a real constant; then

$$F(t) = a\left(1 - \frac{1}{t^4}\right),$$

and then we have Henneberg's algebraic surface (§ 177, Ex. 2).

The general solution is given by

$$\phi(t) = ia_0 + \sum_{m=0} c_{2m+1}\left(t^{2m+1}e^{ia_{2m+1}} + t^{-2m-1}e^{-ia_{2m+1}}\right)$$
$$+ \sum_{m=1} c_{2m}\left(t^{2m}e^{ia_{2m}} - t^{-2m}e^{-ia_{2m}}\right),$$

where the quantities c and a are real.

183. We have already (§ 59) dealt with Lie's method of generating minimal surfaces by taking them as the locus of the middle point of the chord joining any point on one nul line in space to any point on another nul line in space. This method of generation (which really is an interpretation of the Monge formulæ and the Weierstrass formulæ) is the foundation of Lie's researches on minimal surfaces.

When the nul lines are one and the same, the chord comes to be a chord joining any two points on the nul line in space; the locus of its middle point still is a minimal surface; and it is the fact that this minimal surface is a one-sided or double surface. The proof of this theorem, which is due to Lie, is left as an exercise.

Deformation of minimal surfaces.

184. The general discussion of the deformation of surfaces has been reserved for a separate chapter. But the deformation of minimal surfaces, limited by the restriction that the surface is to remain minimal, is so particular that it may fitly be discussed here, especially as the detailed results lead to other issues.

Accordingly, let a minimal surface be deformed without stretching or tearing so as to remain minimal if that be possible. The arc-element must remain unaltered; and therefore, if u_1 and v_1 be the parameters of the nul lines in any deformed configuration, we must have

$$(1+uv)^2 FG\,du\,dv = (1+u_1v_1)^2 F_1G_1\,du_1\,dv_1,$$

where F and G are the functions in the Weierstrass equations, being functions of u alone and v alone, respectively, and likewise for F_1 and G_1 with regard to u_1 and v_1 respectively. Now

$$du_1 = \frac{\partial u_1}{\partial u}du + \frac{\partial u_1}{\partial v}dv, \quad dv_1 = \frac{\partial v_1}{\partial u}du + \frac{\partial v_1}{\partial v}dv;$$

and therefore

$$\frac{\partial u_1}{\partial u}\frac{\partial v_1}{\partial u} = 0, \quad \frac{\partial u_1}{\partial v}\frac{\partial v_1}{\partial v} = 0.$$

Hence either u_1 is a function of u only and v_1 is a function of v only, or u_1 is a function of v only and v_1 is a function of u only. The alternatives are effectively the same; so we take

$$u_1 = \lambda(u) = \lambda, \quad v_1 = \mu(v) = \mu,$$

and then

$$(1+uv)^2 FG = (1+\lambda\mu)^2 F_1G_1\lambda'\mu'.$$

Taking logarithms of both sides and then operating with $\frac{\partial^2}{\partial u \partial v}$, we find

$$\frac{1}{(1+uv)^2} = \frac{\lambda'\mu'}{(1+\lambda\mu)^2},$$

that is,

$$4\frac{du\,dv}{(1+uv)^2} = 4\frac{du_1\,dv_1}{(1+u_1v_1)^2}.$$

Hence the arc-elements in the spherical representations of the minimal surface in its different stages are the same; and so the spherical representations either are equal to one another or are symmetrical. But the deformation is continuous and the spherical representations begin by being the same;

hence the spherical representation at any stage is equal to the initial spherical representation. Consequently, choosing an appropriate location of the two forms of the minimal surface, we have

$$X_1 = X, \quad Y_1 = Y, \quad Z_1 = Z,$$

that is,

$$u_1 = u, \quad v_1 = v.$$

Then

$$FG = F_1 G_1.$$

Now F and F_1 are functions of u alone, while G and G_1 are functions of v alone; hence

$$\frac{F_1}{F} = \frac{G}{G_1} = \text{constant} = e^{i\alpha},$$

say, where α is any constant; that is,

$$F_1 = Fe^{i\alpha}, \quad G_1 = Ge^{-i\alpha}.$$

Minimal surfaces thus determined are called surfaces *associated* with the minimal surface; and so we have Bonnet's theorem that the only minimal surfaces, which can be deformed into a given minimal surface, are its associated surfaces.

185. Among the associates of a minimal surface, there is one of special importance. It is given by taking $\alpha = \frac{1}{2}\pi$, so that

$$F_1 = iF, \quad G_1 = -iG;$$

and it is called the *adjoint* surface (sometimes Bonnet's adjoint surface). Let x_0, y_0, z_0 be the point on it which corresponds to the point x, y, z on the original minimal surface; then, writing

$$x = A(u) + A'(v), \qquad y = B(u) + B'(v), \qquad z = C(u) + C'(v),$$

we have

$$x_0 = iA(u) - iA'(v), \quad y_0 = iB(u) - iB'(v), \quad z_0 = iC(u) - iC'(v).$$

When the original minimal surface is real, the adjoint surface is real. The two surfaces are algebraical together. And the same holds for every associate of a minimal surface.

The adjoint of the adjoint is not the original minimal surface; it is symmetrical with that original through the origin of coordinates.

The adjoint surface is not definite in position. For we can write $A(u) + a$ and $A'(v) - a$ in place of $A(u)$ and $A'(v)$, without altering the original surface; but the effect is to add a term $2ia$ to x_0. Similarly for y_0 and z_0. And the same holds for every associate.

We have

$$x - ix_0 = 2A(u), \quad y - iy_0 = 2B(u), \quad z - iz_0 = 2C(u),$$
$$x + ix_0 = 2A'(v), \quad y + iy_0 = 2B'(v), \quad z + iz_0 = 2C'(v);$$

and therefore, if ξ, η, ζ are the coordinates of the point which, on the associate determined by α, corresponds to x, y, z, we have

$$\left.\begin{aligned}\xi &= A(u)e^{i\alpha} + A'(v)e^{-i\alpha}\\ &= x\cos\alpha + x_0\sin\alpha\\ \eta &= y\cos\alpha + y_0\sin\alpha\\ \zeta &= z\cos\alpha + z_0\sin\alpha\end{aligned}\right\}.$$

Again, we have

$$dx_0 = ix_1du - ix_2dv.$$

But (§ 27) we have, in general,

$$Yz_1 - Zy_1 = (x_2E - x_1F)V^{-1}, \quad Yz_2 - Zy_2 = (x_2F - x_1G)V^{-1};$$

and therefore, in the present case, as $E=0$, $G=0$, $V=iF$,

$$Yz_1 - Zy_1 = ix_1, \quad Yz_2 - Zy_2 = -ix_2.$$

Consequently

$$\begin{aligned}dx_0 &= (Yz_1 - Zy_1)du + (Yz_2 - Zy_2)dv\\ &= Ydz - Zdy,\end{aligned}$$

and similarly for the others; that is, we have

$$\left.\begin{aligned}dx_0 &= Ydz - Zdy\\ dy_0 &= Zdx - Xdz\\ dz_0 &= Xdy - Ydx\end{aligned}\right\}.$$

These results are due to Schwarz; and they again shew that the adjoint surface, being obtainable through quadratures, is not definite in position.

Further, we have

$$\frac{\partial x_0}{\partial u} = ix_1, \quad \frac{\partial x_0}{\partial v} = ix_2,$$

and similarly for the other coordinates; hence the direction-cosines of the normal to the adjoint surface are the same as those of the normal to the original surface, that is, the tangent planes to the two surfaces are parallel. Also

$$dxdx_0 + dydy_0 + dzdz_0 = 0,$$

on substituting the values of dx_0, dy_0, dz_0; that is, corresponding curves on a minimal surface and its adjoint are perpendicular to one another at corresponding points.

The first of these results (but not the second) holds for any associate surface. For

$$\frac{\partial\xi}{\partial u} = e^{i\alpha}\frac{\partial A}{\partial u} = x_1e^{i\alpha}, \quad \frac{\partial\xi}{\partial v} = e^{-i\alpha}\frac{\partial A'}{\partial v} = x_2e^{-i\alpha},$$

and similarly for the other coordinates; thus the direction-cosines of the

normal to the associate are the same as for the original minimal surface, and so the tangent planes are parallel. But

$$dx\,d\xi + dy\,d\eta + dz\,d\zeta = (\Sigma\, dx^2)\cos\alpha + (\Sigma\, dx\,dx_0)\sin\alpha$$
$$= ds^2\cos\alpha,$$

which vanishes only if $\alpha = \frac{1}{2}\pi$; and

$$dx_0\,d\xi + dy_0\,d\eta + dz_0\,d\zeta = ds_0{}^2\sin\alpha$$
$$= ds^2\sin\alpha,$$

which vanishes only if α is 0 or π.

The lines of curvature on the original minimal surface are

$$F\,du^2 - G\,dv^2 = 0$$

(F and G being the functions in Weierstrass's expressions), and its asymptotic lines are

$$F\,du^2 + G\,dv^2 = 0.$$

On the adjoint surface the lines of curvature are

$$iF\,du^2 - (-iG)\,dv^2 = 0,$$

which therefore correspond to the asymptotic lines of the original; and the asymptotic lines are

$$iF\,du^2 - iG\,dv^2 = 0,$$

which therefore correspond to the lines of curvature of the original. And so for other properties.

Minimal surfaces under assigned conditions.

186. The special results just proved and due to Schwarz, which relate to the adjoint surface and determine it by a process of quadrature, have been applied by him to a problem of greater importance in the theory of minimal surfaces.

From any of the integral equations of a minimal surface, it appears that they contain a couple of arbitrary functions in their expression; and it is natural to consider alike the character and the extent of the conditions which the functions can help to satisfy. On the other hand, we know that the differential equation of minimal surfaces is substantially a partial differential equation of the second order, whether it occurs in the intrinsic form

$$EN - 2FM + GL = 0,$$

or in the explicit form

$$(1 + q^2)\,r - 2pqs + (1 + p^2)\,t = 0,$$

with the customary notation.

Now as regards such partial equations of the second order, the fundamental existence-theorem* (due to Cauchy) establishes the result that a unique uniform integral z of the partial equation exists which is determined by the conditions:—

(i) that, along any assigned curve (not being one of the 'characteristic' curves), the quantity z assumes an assigned value, and

(ii) that, along the same assigned curve, one of the derivatives of z assumes an assigned value.

In the case of the partial equation for minimal surfaces, the characteristic curves are given by

$$(1+q^2)\,dy^2 + 2pq\,dx\,dy + (1+p^2)\,dx^2 = 0,$$

that is, by

$$dx^2 + dy^2 + dz^2 = 0,$$

which are the nul lines on the surface. Therefore, in applying Cauchy's theorem, it is necessary to exclude the nul lines from the curves along which external conditions can be assigned. Further, along any non-excluded curve the value of z is given, as also is that of one of its derivatives, say p; thus, as we everywhere have

$$dz = p\,dx + q\,dy,$$

and as we are given z and p along the curve, we know q also; that is, we know p and q along the curve. Now at any point on the surface, the quantities p and q determine the direction-cosines of the normal to the surface, and therefore determine the tangent plane. We therefore can restate Cauchy's general theorem for partial equations of the second order, when it is applied to the equation of minimal surfaces, as follows:—

A minimal surface is uniquely determinate by the condition that it passes through an assigned curve and that, along the curve, it touches an assigned developable surface through the curve, provided the curve is not a nul line upon the surface.

To have the surface explicitly determinate, it is necessary to find the forms of the arbitrary functions which shall satisfy the assigned conditions; for that end, Schwarz's results can be used.

187. Without entering into all the cases, let us assume that the assigned curve is such that, along its range, the coordinates of a point x, y, z and the direction-cosines X, Y, Z of the tangent plane to the assigned developable surface, can be expressed in terms of a current parameter. We have

$$x - ix_0 = 2A, \quad x + ix_0 = 2A',$$

* See the author's *Theory of Differential Equations*, vol. vi, chaps. xii, xx.

that is,

$$x - i\int (Ydz - Zdy) = 2A, \quad x + i\int (Ydz - Zdy) = 2A';$$

and similarly

$$y - i\int (Zdx - Xdz) = 2B, \quad y + i\int (Zdx - Xdz) = 2B',$$

$$z - i\int (Xdy - Ydx) = 2C, \quad z + i\int (Xdy - Ydx) = 2C'.$$

These equations, when substitution is made for the values of x, y, z, X, Y, Z along the curve, determine the forms of the functions A, B, C, A', B', C'; and then taking two parameters p and q, conjugate for real surfaces, we have the integral equations of the minimal surface in the form

$$x = A(p) + A'(q), \quad y = B(p) + B'(q), \quad z = C(p) + C'(q),$$

that is,

$$\left.\begin{aligned} 2x &= \{x(p) + x(q)\} - i\int_q^p (Ydz - Zdy) \\ 2y &= \{y(p) + y(q)\} - i\int_q^p (Zdx - Xdz) \\ 2z &= \{z(p) + z(q)\} - i\int_q^p (Xdy - Ydx) \end{aligned}\right\}.$$

One remark, by way of warning, must be made, because the analysis will not be developed further. The nul lines can remain as parametric curves, when any arbitrary functions of the parameters are substituted for the respective parameters; and it must not therefore be assumed (it is not the actual fact) that the variables p and q in the preceding analysis are the variables u and v in the Weierstrass equations for a minimal surface.

Some examples will illustrate the working in detail. But it soon appears that the determination of a minimal surface in connection with assigned conditions becomes a problem in the theory of functions and differential equations; a full exposition is given in Darboux's treatise.

Ex. 1. Let it be required to find the minimal surface, which passes through a circle of radius unity lying on a right circular cone of semi-vertical angle a and touches the cone along that circle.

Along the circle, we have

$$x = \cos\theta, \qquad y = \sin\theta, \qquad z = \cot a,$$

$$X = \cos\theta\cos a, \quad Y = \sin\theta\cos a, \quad Z = -\sin a;$$

and therefore

$$Ydz - Zdy = \sin a\cos\theta\, d\theta,$$

$$Zdx - Xdz = \sin a\sin\theta\, d\theta,$$

$$Xdy - Ydx = \cos a\, d\theta.$$

Hence

$$2A = x - i\int (Ydz - Zdy) = \cos\theta - i\sin\alpha\sin\theta,$$

$$2A' = x + i\int (Ydz - Zdy) = \cos\theta + i\sin\alpha\sin\theta,$$

$$2B = y - i\int (Zdx - Xdz) = \sin\theta + i\sin\alpha\cos\theta,$$

$$2B' = y + i\int (Zdx - Xdz) = \sin\theta - i\sin\alpha\cos\theta,$$

$$2C = z - i\int (Xdy - Ydx) = \cot\alpha - i\theta\cos\alpha,$$

$$2C' = z + i\int (Xdy - Ydx) = \cot\alpha + i\theta\cos\alpha.$$

Thus

$$\begin{aligned} x &= A(p) + A'(q) \\ &= \tfrac{1}{2}(\cos p + \cos q) + \tfrac{1}{2}i\sin\alpha(\sin q - \sin p) \\ &= \cos\tfrac{1}{2}(p+q)\{\cos\tfrac{1}{2}(q-p) + i\sin\alpha\sin\tfrac{1}{2}(q-p)\}, \\ y &= B(p) + B'(q) \\ &= \sin\tfrac{1}{2}(p+q)\{\cos\tfrac{1}{2}(q-p) + i\sin\alpha\sin\tfrac{1}{2}(q-p)\}, \\ z &= C(p) + C'(q) \\ &= \cot\alpha + \tfrac{1}{2}i(q-p)\cos\alpha. \end{aligned}$$

When p and q are eliminated between these three equations, the resulting equation (being that of the minimal surface) is

$$\log\left\{\frac{(x^2+y^2)^{\frac{1}{2}} + (x^2+y^2-\cos^2\alpha)^{\frac{1}{2}}}{1+\sin\alpha}\right\} = \frac{z-\cot\alpha}{\cos\alpha}.$$

The surface is a catenoid.

Ex. 2. Find the minimal surface which touches an ellipsoid along a line of curvature.

Take the line of curvature as given by

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1, \quad \frac{x^2}{a+p} + \frac{y^2}{b+p} + \frac{z^2}{c+p} = 1.$$

Along the line in question, the quantities x, y, z, X, Y, Z are the same for the minimal surface as for the ellipsoid; hence, writing

$$\frac{a(a+p)}{-\beta\gamma} = \mathrm{a}, \quad \frac{b(b+p)}{-\gamma\alpha} = \mathrm{b}, \quad \frac{c(c+p)}{-\alpha\beta} = \mathrm{c},$$

$$\frac{bc(a+p)}{-p\beta\gamma} = \mathrm{a}', \quad \frac{ca(b+p)}{-p\gamma\alpha} = \mathrm{b}', \quad \frac{ab(c+p)}{-p\alpha\beta} = \mathrm{c}',$$

we have (§ 78)

$$x = \{\mathrm{a}(a+q)\}^{\frac{1}{2}}, \quad y = \{\mathrm{b}(b+q)\}^{\frac{1}{2}}, \quad z = \{\mathrm{c}(c+q)\}^{\frac{1}{2}},$$

$$X = \left\{\mathrm{a}'\frac{a+q}{q}\right\}^{\frac{1}{2}}, \quad Y = \left\{\mathrm{b}'\frac{b+q}{q}\right\}^{\frac{1}{2}}, \quad Z = \left\{\mathrm{c}'\frac{c+q}{q}\right\}^{\frac{1}{2}}.$$

Then

$$Ydz - Zdy = -\tfrac{1}{2}\left\{\frac{a(b+p)(c+p)}{p\beta\gamma}\right\}^{\frac{1}{2}}\left\{\frac{q}{(b+q)(c+q)}\right\}^{\frac{1}{2}} dq,$$

$$Zdx - Xdz = -\tfrac{1}{2}\left\{\frac{b(c+p)(a+p)}{p\gamma\alpha}\right\}^{\frac{1}{2}}\left\{\frac{q}{(c+q)(a+q)}\right\}^{\frac{1}{2}} dq,$$

$$Xdy - Ydx = -\tfrac{1}{2}\left\{\frac{c(a+p)(b+p)}{p\alpha\beta}\right\}^{\frac{1}{2}}\left\{\frac{q}{(a+q)(b+q)}\right\}^{\frac{1}{2}} dq;$$

and therefore, after the investigation in the text, the coordinates of the current point on the required minimal surface are given by

$$\left.\begin{aligned}
2x&=\{\mathrm{a}\,(a+u)\}^{\frac{1}{2}}+\{\mathrm{a}\,(a+v)\}^{\frac{1}{2}}-\tfrac{1}{2}i\left\{\frac{a\,(b+p)\,(c+p)}{p\beta\gamma}\right\}^{\frac{1}{2}}\int_v^u\left\{\frac{q}{(b+q)\,(c+q)}\right\}^{\frac{1}{2}}dq\\
2y&=\{\mathrm{b}\,(b+u)\}^{\frac{1}{2}}+\{\mathrm{b}\,(b+v)\}^{\frac{1}{2}}-\tfrac{1}{2}i\left\{\frac{b\,(c+p)\,(a+p)}{p\gamma a}\right\}^{\frac{1}{2}}\int_v^u\left\{\frac{q}{(c+q)\,(a+q)}\right\}^{\frac{1}{2}}dq\\
2z&=\{\mathrm{c}\,(c+u)\}^{\frac{1}{2}}+\{\mathrm{c}\,(c+v)\}^{\frac{1}{2}}-\tfrac{1}{2}i\left\{\frac{c\,(a+p)\,(b+p)}{pa\beta}\right\}^{\frac{1}{2}}\int_v^u\left\{\frac{q}{(a+q)\,(b+q)}\right\}^{\frac{1}{2}}dq
\end{aligned}\right\}.$$

Ex. 3. A minimal surface is drawn through a helix of pitch $\tan^{-1}c$ upon a circular cylinder of radius unity having its axis along the axis of z; and the minimal surface touches the cylinder along the helix. Prove that its equation can be expressed in the form

$$z=c\tan^{-1}\frac{y}{x}-c\tan^{-1}\left\{c\left(\frac{x^2+y^2-1}{x^2+y^2+c^2}\right)^{\frac{1}{2}}\right\}-\tanh^{-1}\left\{\left(\frac{x^2+y^2-1}{x^2+y^2+c^2}\right)^{\frac{1}{2}}\right\}.$$

Ex. 4. Suppose that a minimal surface is such that a real straight line can be drawn upon it.

Take the straight line for axis of z; then along this line we have

$$x=0,\quad y=0,\quad Z=0;$$

and so the equations of the minimal surface are

$$\left.\begin{aligned}
x&=-\tfrac{1}{2}i\int_v^u Ydz\\
y&=\ \tfrac{1}{2}i\int_v^u Xdz\\
z&=\ \tfrac{1}{2}(u+v)
\end{aligned}\right\},$$

where X and Y are appropriate functions of z subject to the relation

$$X^2+Y^2=1.$$

When u and v are interchanged, the value of z remains unaltered, while x and y change their signs but otherwise are unaltered; hence the axis of z is an axis of symmetry for the surface. In other words, when a straight line can be drawn upon a minimal surface, it is an axis of symmetry—a result due to Schwarz.

Ex. 5. As another example—(the investigation is due to Lie)—consider the possibility of a minimal surface having a plane line of curvature. We know (§ 128) that the plane cuts the surface at an angle that is constant along the line; and that, conversely, if the angle be constant, the line of intersection is a line of curvature. Let this constant angle be denoted by a.

Take the plane for the plane of x, y. The values of X, Y, Z along the curve are

$$X=-\frac{dy}{ds}\sin a,\quad Y=\frac{dx}{ds}\sin a,\quad Z=\cos a;$$

and along the curve, we have

$$x=x(s),\quad y=y(s),\quad z=0.$$

Then the equations of the minimal surface are

$$\left.\begin{aligned} x &= \tfrac{1}{2}\{x(u)+x(v)\}+\tfrac{1}{2}i\int_v^u \cos a\,dy \\ &= \tfrac{1}{2}\{x(u)+x(v)\}+\tfrac{1}{2}i\{y(u)-y(v)\}\cos a \\ y &= \tfrac{1}{2}\{y(u)+y(v)\}-\tfrac{1}{2}i\{x(u)-x(v)\}\cos a \\ z &= \tfrac{1}{2}i\int_v^u ds\sin a=\tfrac{1}{2}i(u-v)\sin a \end{aligned}\right\}.$$

The surface is algebraical if, and only if, $x(s)$ and $y(s)$ are algebraical.

Note on the range of a minimal surface.

188. We now proceed to the deferred consideration of the single remaining test (§ 167) that applies to the second variation. The test must be satisfied if the minimal surface, which passes through an assigned curve and touches an assigned developable along the curve, is to provide an actual minimum. For the complete consideration of this criterion, some of the laborious analysis in the calculus of variations would be needed; here, the discussion will be restricted to the case of weak variations, so that we shall require a positive sign for the value of u, where u denotes the second variation. We have

$$2u=\iint\left\{-2\frac{l^2}{\alpha^2}+\frac{1}{V^2}(El_2^2-2Fl_1l_2+Gl_1^2)\right\}Vdpdq,$$

where K, now necessarily negative, is denoted by $-1/\alpha^2$; the length l (measured normal to the surface) is an arbitrary function of p and q, subject to the condition of vanishing along the assigned curve. It will be proved that the requirement of a positive value for u imposes a possible limitation upon the range over which the surface provides an actual minimum, just as there is a possible limitation upon the range for which a geodesic (§ 89) provides an actually shortest distance on a surface.

The expression for u must be modified. We take any two variable quantities A and B, functions of p and q, reserving their assignment for subsequent use. The value of the double integral

$$\iint\left\{\frac{\partial}{\partial p}(Al^2)+\frac{\partial}{\partial q}(Bl^2)\right\}dpdq,$$

extended over a region of the variables bounded by the assigned curve at one limit, and by any other curve at some other limit (the latter merely indicating a range of the minimal surface to be considered), is zero; because, for the weak variations adopted, we assume that l vanishes at each boundary of the range. Adding this zero integral to $2u$, we have

$$2u=\iint \mathbf{U}\,dpdq,$$

where

$$\mathbf{U} = \frac{1}{V}(El_2^2 - 2Fl_1l_2 + Gl_1^2) + l^2\left(A_1 + B_2 - 2\frac{V}{\alpha^2}\right) + 2All_1 + 2Bll_2;$$

and $\mathbf{U}$ is expressible in the form

$$\mathbf{U} = \frac{1}{V}\left\{E\left(l_2 - \frac{\lambda_2}{\lambda}l\right)^2 - 2F\left(l_1 - \frac{\lambda_1}{\lambda}l\right)\left(l_2 - \frac{\lambda_2}{\lambda}l\right) + G\left(l_1 - \frac{\lambda_1}{\lambda}l\right)^2\right\},$$

provided the quantities λ, A, B satisfy the relations

$$A = \frac{F}{V}\frac{\lambda_2}{\lambda} - \frac{G}{V}\frac{\lambda_1}{\lambda},$$

$$B = -\frac{E}{V}\frac{\lambda_2}{\lambda} + \frac{F}{V}\frac{\lambda_1}{\lambda},$$

$$A_1 + B_2 - 2\frac{V}{\alpha^2} = \frac{E}{V}\frac{\lambda_2^2}{\lambda^2} - 2\frac{F}{V}\frac{\lambda_1\lambda_2}{\lambda^2} + \frac{G}{V}\frac{\lambda_1^2}{\lambda^2}.$$

Obviously A and B can be regarded as known, when λ is known. Eliminating A and B between the three relations, we have the equation for λ in the form

$$\frac{\partial}{\partial p}\left(\frac{F}{V}\frac{\lambda_2}{\lambda} - \frac{G}{V}\frac{\lambda_1}{\lambda}\right) + \frac{\partial}{\partial q}\left(-\frac{E}{V}\frac{\lambda_2}{\lambda} + \frac{F}{V}\frac{\lambda_1}{\lambda}\right) - 2\frac{V}{\alpha^2} = \frac{E}{V}\frac{\lambda_2^2}{\lambda^2} - 2\frac{F}{V}\frac{\lambda_1\lambda_2}{\lambda^2} + \frac{G}{V}\frac{\lambda_1^2}{\lambda^2},$$

that is,

$$E\lambda_{22} - 2F\lambda_{12} + G\lambda_{11} - (E\Gamma'' - 2F\Gamma' + G\Gamma)\lambda_1 \\ - (E\Delta'' - 2F\Delta' + G\Delta)\lambda_2 + 2\frac{V^2}{\alpha^2}\lambda = 0,$$

a partial differential equation of the second order.

The characteristics for the solution of this equation are

$$Edp^2 + 2Fdpdq + Gdq^2 = 0,$$

that is, are the nul lines of the surface, neither of which (§ 186) can be an assigned boundary of the surface. Hence, by Cauchy's theorem already quoted (§ 186), a unique regular integral of this equation exists, satisfying the conditions:—

(i) the magnitude λ shall, like l, vanish along the assigned curve through which the minimal surface must pass;

(ii) along the assigned curve, λ_1 and λ_2 shall differ from l_1 and l_2 respectively by relatively infinitesimal quantities.

When λ is thus determined, the equation $\lambda = \chi(p, q)$, for parametric values of λ, gives curves on the surface, one of them coinciding with the assigned curve when $\lambda = 0$.

The subject $\mathbf{U}$, in the modified integration for the second variation, is everywhere positive for real surfaces, because

$$E > 0, \quad G > 0, \quad V > 0,$$

unless it should happen that the quantities

$$l_1 - \frac{\lambda_1}{\lambda} l, \quad l_2 - \frac{\lambda_2}{\lambda} l,$$

could vanish together, that is, unless the relation

$$l = c\lambda$$

(where c is a pure constant) could hold, for variations l over the considered range of the surface. The relation holds at the initial stage of the range, because both l and λ vanish there. If, therefore, after the initial stage, λ could again vanish either at or before the final stage, the relation could hold over the whole considered range of the surface. The second variation then would be zero, for an assumed choice $l = c\lambda$; disregarding variations of higher orders, we could not declare that the included range of the minimal surface provides an actual minimum area.

Accordingly, we trace upon the surface the family of curves

$$\chi(p, q) = \lambda,$$

for parametric values of λ; we call the assigned curve, given by $\lambda = 0$ at the boundary of the integral, the initial curve. As λ varies, positively and negatively, it may again assume a zero value upon the surface; we call the curves, nearest to the initial curve in either sense along the surface, *conjugate* to the initial curve. We therefore infer the result:—

In order that an actual minimum area may be provided by a minimal surface, which is required to pass through an assigned curve and to touch an assigned developable along the curve, the range of the surface must not extend so far as the conjugate (if any) of the assigned curve on the surface.

It follows therefore that the range of a minimal surface must not extend so far as the conjugate of any curve upon it, if the area of the surface is to be an actual minimum for small variations. If only the descriptive property—that the mean curvature is zero—is required, it would be possessed by the surface over its whole extent; just as in the case of geodesics, the geodesic property—that its principal normal is the normal to its surface—is possessed along its whole course without any reference to conjugate points.

The more detailed consideration of the conjugate of any curve on a minimal surface belongs to the region of the calculus of variations.

EXAMPLES.

1. Shew that the surface

$$e^{az} = \frac{\cos ax}{\cos ay}$$

is minimal; that it is the locus of the middle point of a chord joining any two points on a particular nul curve in space; and that it is the only minimal surface such that

$$z = f(x) + g(y).$$

Obtain the equation of the adjoint surface in the form

$$\sin az = \sinh ax \sinh ay.$$

2. Two surfaces can be deformed into one another, and their tangent planes at corresponding points are parallel; shew that they are associated minimal surfaces.

3. Two surfaces can be deformed into one another and corresponding arc-elements are inclined to one another at a constant angle; shew that they are associated minimal surfaces.

4. Shew that for a minimal surface, given by the equations

$$\left.\begin{aligned} 2x(1-c^2)^{\frac{1}{2}} &= c(\theta+\phi)+\sin\theta+\sin\phi \\ 2y(1-c^2)^{\frac{1}{2}} &= i\{\theta-\phi+c(\sin\theta-\sin\phi)\} \\ 2z &= -\cos\theta-\cos\phi \end{aligned}\right\},$$

the lines of curvature become two families of circles in the spherical representation.

5. In Weierstrass's equations for a minimal surface, take

$$F(u) = au^k, \quad G(v) = a'v^k,$$

where k is a real constant, while a and a' are conjugate constants; shew that the surface can be deformed into a surface of revolution.

6. A minimal surface possesses a plane geodesic; shew that the plane of the geodesic is a plane of symmetry for the surface.

7. A minimal surface (Catalan's) is given by the equations

$$x = \sin^2 u + \sin^2 v,$$

$$y = 2i(\sin u - \sin v),$$

$$2z = 2u + \sin 2u + 2v + \sin 2v;$$

shew that it contains one geodesic which is a parabola, and another which is a cycloid.

8. Shew that the (Henry Smith) surface

$$z(x^2+y^2) = x^2$$

has only one side.

9. In Weierstrass's equations for a minimal surface, take

$$F(u) = \left(\frac{1}{u} - u\right)^{\alpha} \left(\frac{1}{u} + u\right)^{\beta} \frac{1}{u^2},$$

where β is an odd integer; shew that the minimal surface is a "double" surface.

10. Given two associated minimal surfaces; shew that the lines of curvature on either of them correspond to isogonal trajectories of the lines of curvature on the other.

11. On two adjoint surfaces, corresponding geodesics are drawn; shew that the circular curvature of one at any point is equal to the torsion of the other at the corresponding point.

CHAPTER IX.

SURFACES WITH PLANE OR SPHERICAL LINES OF CURVATURE; WEINGARTEN SURFACES.

THE present chapter is devoted to some special classes of surfaces, other than minimal surfaces. The vast variety of modern investigations leads to an extraordinary amount of detailed result. Here, we shall deal with only some of the principal classes of such surfaces.

Liouville surfaces have already been discussed, from the point of view of their most important property—that they can be geodesically represented upon one another, and that (for the explicit equation of their geodesics) they admit quadratic integrals of the critical equation of geodesics (§ 157).

Reference (to the extent of constructing the essential partial differential equation of the second order which serves for their construction) has also been made to surfaces having a constant measure of curvature—whether the Gauss measure, or the mean measure (§§ 54, 57).

We have also dealt, briefly, with surfaces which possess lines of curvature of the isometric type (§ 64). They will occur, later, under the discussion of triply orthogonal systems of surfaces in space.

Thus, for various reasons, a selection of two special systems of surfaces is made for the present chapter.

One of these systems is characterised by the property that the lines of curvature (in either or in both the sets) are composed of plane curves or of curves that lie upon a sphere. The special restriction to plane curves or to spherical curves is due to a theorem of Joachimsthal's (§ 128) which facilitates the construction of integral equations of the surfaces. The subject has been the cause of many investigations in the past; special note should be made of the memoirs by Serret*, Cayley†, Rouquet‡, of portions of Darboux's treatise§, and of Bianchi's treatise||. The literature of this part of the subject is so great that no attempt at a comprehensive bibliography can here be made; many references will be found in the authors just quoted.

* *Liouville's Journal*, t. xviii (1853), pp. 113—162.

† *Coll. Math. Papers*, vol. xii, pp. 601—638.

‡ *Mém. de l'Ac. des Sciences*, Toulouse, 8e Ser., t. ix (1887), t. x (1888).

§ Vol. i, pp. 114—118; vol. iv, pp. 198—266.

|| Vol. ii, chap. xxi.

The other of the systems of surfaces is characterised by the property that a functional relation—as arbitrary as can be chosen—exists between the principal radii of curvature. Such surfaces are called *Weingarten surfaces*; special instances, such as those which have one or other of the measures of curvature equal to a constant, are already known; the more general investigation of such surfaces is due to Weingarten, to whose memoirs (as to other investigations) detailed reference is given in Darboux's sections dealing with the subject*.

Surfaces with Plane or Spherical Lines of Curvature.

189. We have seen (§ 129) that, if a line of curvature on a surface is a plane curve, the plane cuts the surface at a constant angle; and that, if a line of curvature is a spherical curve (that is, if it lies on a sphere), the sphere and the surface cut at a constant angle; the two results being connected with one another owing to the property (§ 79) that inversion conserves lines of curvature. In each case, the constancy of the angle is maintained along the particular line of curvature. When there is a family of plane lines, or when there is a family of spherical lines, the angle that is constant along any one line can (and usually does) vary from one line to another. The simplest illustration is provided by surfaces of revolution.

The property, originally discovered by Joachimsthal, can be used to obtain a first integral of some associated differential equations of the surface; and the two cases—according as the lines of curvature are plane or are spherical—can be treated together analytically.

Let an equation

$$k(x^2+y^2+z^2) = 2(ax+by+cz+u)$$

be taken; it represents a sphere if $k=1$, and a plane if $k=0$. It is to be the sphere or the plane, as the case may be, containing the line of curvature; and therefore the quantities a, b, c, u will be functions of one parameter, which will be constant along the line and will vary from one line to another. The property, that the sphere or the plane cuts the surface at a constant angle, is analytically expressed by a relation

$$(kx-a)X+(ky-b)Y+(kz-c)Z=l,$$

where l is constant along the line of curvature and usually varies from one line to another; that is, l also is a function of the parameter of the lines of curvature in the family.

* See his treatise, vol. iii, Book vii, chaps. vii, ix, x.

190. Before proceeding further with the discussion of the problem, we may note one reason (chiefly of manipulative ease) why consideration is restricted mainly to those classes of lines of curvature which are either plane or spherical. It is not inconceivable that a family of lines of curvature should be curves lying on a family of quadrics; thus they might be helices on a family of circular cylinders. The analysis, however, in all such cases becomes more complicated; for the first integral, similar to the Joachimsthal property for planes and spheres, appears to be unobtainable.

To see the distinction between the cases, let the surface be referred to its lines of curvature as parametric curves. We then have

$$F=0, \quad M=0;$$

and then (§ 29)

$$EX_1=-Lx_1, \quad EY_1=-Ly_1, \quad EZ_1=-Lz_1,$$

$$GX_2=-Nx_2, \quad GY_2=-Ny_2, \quad GZ_2=-Nz_2.$$

Now suppose that the line of curvature, given by $p=$ constant, lies upon a surface

$$\phi(x, y, z, p)=0.$$

The direction-cosines of the line at any point are proportional to x_2, y_2, z_2; so we have

$$\frac{\partial\phi}{\partial x}x_2+\frac{\partial\phi}{\partial y}y_2+\frac{\partial\phi}{\partial z}z_2=0$$

along the line, that is, we have

$$\frac{\partial\phi}{\partial x}X_2+\frac{\partial\phi}{\partial y}Y_2+\frac{\partial\phi}{\partial z}Z_2=0$$

along the line. What is required, to secure some progress in the investigation, is some less differentiated equivalent relation.

Let the surfaces $\phi(x, y, z, p)=0$ be a family of planes

$$ax+by+cz+u=0,$$

where a, b, c, u are functions of p only; the foregoing equation is

$$aX_2+bY_2+cZ_2=0,$$

and therefore an integral is

$$aX+bY+cZ=l,$$

where l is a function of p only. This gives Joachimsthal's theorem concerning plane lines of curvature.

Next, let the surfaces $\phi(x, y, z, p)=0$ be a family of spheres

$$x^2+y^2+z^2-2ax-2by-2cz-2u=0,$$

where again a, b, c, u are functions of p only; the foregoing general equation becomes

$$(x-a)X_2+(y-b)Y_2+(z-c)Z_2=0.$$

But we always have

$$Xx_2+Yy_2+Zz_2=0;$$

hence

$$\frac{\partial}{\partial q}\{(x-a)X+(y-b)Y+(z-c)Z\}=0,$$

and therefore

$$(x-a)X+(y-b)Y+(z-c)Z=l,$$

where l is a function of p only. This gives Joachimsthal's theorem concerning spherical lines of curvature.

In each case, the surface of which the line in question is a line of curvature, and the surface (plane or spherical) on which the line lies, cut at an angle that is constant along the line. If there were the same integral for any other surface, we should have

$$\left(X\frac{\partial\phi}{\partial x}+Y\frac{\partial\phi}{\partial y}+Z\frac{\partial\phi}{\partial z}\right)\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}^{-\frac{1}{2}}=\text{function of } p \text{ only},$$

where the factor $\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}^{-\frac{1}{2}}$ may be capable of simplification by the relation $\phi(x, y, z, p)=0$; but the plane and the sphere appear to be the only surfaces which allow the integral.

It is conceivable that we could have an integral

$$X\frac{\partial\phi}{\partial x}+Y\frac{\partial\phi}{\partial y}+Z\frac{\partial\phi}{\partial z}=\text{function of } p \text{ only},$$

equivalent to the general relation

$$X_2\frac{\partial\phi}{\partial x}+Y_2\frac{\partial\phi}{\partial y}+Z_2\frac{\partial\phi}{\partial z}=0.$$

In that case, we must have

$$\begin{aligned}X&\left(\frac{\partial^2\phi}{\partial x^2}x_2+\frac{\partial^2\phi}{\partial x\partial y}y_2+\frac{\partial^2\phi}{\partial x\partial z}z_2\right)\\&+Y\left(\frac{\partial^2\phi}{\partial x\partial y}x_2+\frac{\partial^2\phi}{\partial y^2}y_2+\frac{\partial^2\phi}{\partial y\partial z}z_2\right)\\&\quad+Z\left(\frac{\partial^2\phi}{\partial x\partial z}x_2+\frac{\partial^2\phi}{\partial y\partial z}y_2+\frac{\partial^2\phi}{\partial z^2}z_2\right)=0.\end{aligned}$$

Now this equation will be satisfied if a quantity ρ exists such that

$$\begin{aligned}X\frac{\partial^2\phi}{\partial x^2}+Y\frac{\partial^2\phi}{\partial x\partial y}+Z\frac{\partial^2\phi}{\partial x\partial z}&=2X\rho,\\X\frac{\partial^2\phi}{\partial x\partial y}+Y\frac{\partial^2\phi}{\partial y^2}+Z\frac{\partial^2\phi}{\partial y\partial z}&=2Y\rho,\\X\frac{\partial^2\phi}{\partial x\partial z}+Y\frac{\partial^2\phi}{\partial y\partial z}+Z\frac{\partial^2\phi}{\partial z^2}&=2Z\rho.\end{aligned}$$

When the surfaces $\phi(x, y, z, p) = 0$ are quadrics, say

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz + u = 0,$$

where the coefficients are functions of p alone, we have

$$\begin{vmatrix} a-\rho, & h, & g \\ h, & b-\rho, & f \\ g, & f, & c-\rho \end{vmatrix} = 0,$$

so that ρ is a function of p only. In the case of a plane, the three equations are evanescent, for $\rho = 0$. In the case of a sphere, we have

$$f = g = h = 0, \quad a = b = c = \rho;$$

the three equations are satisfied identically. In other cases, the three equations, combined with

$$X^2 + Y^2 + Z^2 = 1,$$

determine X, Y, Z as functions of p alone. Thus

$$X_2 = 0, \quad Y_2 = 0, \quad Z_2 = 0,$$

and so

$$Nx_2 = 0, \quad Ny_2 = 0, \quad Nz_2 = 0;$$

and therefore as we cannot have x_2, y_2, z_2 all zero (for the surface would then be a curve), we must have

$$N = 0,$$

in addition to $M = 0$. The Mainardi-Codazzi relations become

$$L_2 = \Gamma' L, \quad 0 = \Gamma'' L,$$

and we cannot now have $L = 0$; hence $\Gamma'' = 0$, that is, $G_1 = 0$ so that G is a function of q only which can easily be made unity. Thus the arc is

$$ds^2 = dq^2 + Edp^2,$$

and the surface is developable; the lines of curvature, p = constant, are geodesics and so are plane. There is no new case.

We thus, in the main, restrict ourselves for the present purpose to lines of curvature that are plane or spherical.

191. Two remarks may be made in passing.

In the case of a developable surface (but not in the case of any other ruled surface) one system of lines of curvature is made up of the generators, all of which touch the edge of regression; and the other system is made up of their orthogonal trajectories, which are the superficial involutes of that edge. A generator, however, does not lie in a definite plane; and so it is simpler to consider developable surfaces apart.

Again, one system of lines of curvature may be circles. When a circle is regarded as a plane curve, its plane is definite; when it is regarded as a

spherical curve, its sphere may be indefinite. Accordingly, unless the family of spheres is given, it is usually simpler to discuss circular lines of curvature as plane curves than to discuss them as spherical curves.

Serret-Cayley Treatment of the Two Cases.

192. We now resume the analysis of § 189; and we assume that the other system of lines of curvature also is composed of curves that are plane or spherical. Let the family of surfaces, upon which they lie, be

$$\kappa(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z-2v=0,$$

where κ is either 0 or 1, and where α, β, γ, v are functions of q alone; then we have

$$(\kappa x-\alpha)X+(\kappa y-\beta)Y+(\kappa z-\gamma)Z=\lambda,$$

where λ also is a function of q alone. Hence, for the whole surface, we have the equations

$$\left.\begin{aligned}
0 &= k(x^2+y^2+z^2)-2ax-2by-2cz-2u\\
l &= (kx-a)X+(ky-b)Y+(kz-c)Z\\
0 &= \kappa(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z-2v\\
\lambda &= (\kappa x-\alpha)X+(\kappa y-\beta)Y+(\kappa z-\gamma)Z\\
1 &= X^2+Y^2+Z^2\\
0 &= Xdx+Ydy+Zdz
\end{aligned}\right\},$$

where k and κ are 0 or 1 independently of one another, a, b, c, u, l are functions of p alone, and $\alpha, \beta, \gamma, v, \lambda$ are functions of q alone. The first five of these equations determine five of the quantities x, y, z, X, Y, Z, p, q in terms of the other three, say X, Y, Z, p, q in terms of x, y, z. When the values are substituted in the sixth and it is integrated—we shall prove that the "condition of integrability" (§ 30) is satisfied—we have a new equation $I=0$, say; we then have six equations and can regard them as determining x, y, z, X, Y, Z in terms of p and q. We thus require this integrated equation.

Let the direction-cosines of the two lines of curvature through a point x, y, z on the surface be proportional to dx, dy, dz for the line along which p is constant, and to $\delta x, \delta y, \delta z$ for the line along which q is constant. Then

$$(kx-a)\,dx+(ky-b)\,dy+(kz-c)\,dz=0,$$

$$Xdx+\quad Ydy+\quad Zdz=0,$$

and therefore

$$dx:dy:dz=\left\|\begin{matrix} kx-a, & ky-b, & kz-c\\ X\;, & Y\;, & Z\end{matrix}\right\|.$$

Similarly

$$\delta x : \delta y : \delta z = \left\| \begin{matrix} \kappa x - \alpha, & \kappa y - \beta, & \kappa z - \gamma \\ X \quad , & Y \quad , & Z \end{matrix} \right\|.$$

The two directions are perpendicular to one another; hence

$$\Sigma \begin{vmatrix} ky - b, & kz - c \\ Y \quad , & Z \end{vmatrix} \begin{vmatrix} \kappa y - \beta, & \kappa z - \gamma \\ Y \quad , & Z \end{vmatrix} = 0,$$

and therefore

$$(X^2 + Y^2 + Z^2)\{(kx - a)(\kappa x - \alpha) + (ky - b)(\kappa y - \beta) + (kz - c)(\kappa z - \gamma)\}$$
$$= \{X(kx - a) + Y(ky - b) + Z(kz - c)\}\{X(\kappa x - \alpha) + Y(\kappa y - \beta) + Z(\kappa z - \gamma)\}.$$

Consequently

$$\tfrac{1}{2}\kappa\{k(x^2 + y^2 + z^2) - 2ax - 2by - 2cz\}$$
$$+ \tfrac{1}{2}k\{\kappa(x^2 + y^2 + z^2) - 2\alpha x - 2\beta y - 2\gamma z\} + a\alpha + b\beta + c\gamma = l\lambda;$$

and so

$$a\alpha + b\beta + c\gamma + u\kappa + kv = l\lambda.$$

It therefore appears that the parametric coefficients in the equations of the two families of surfaces, upon which the lines of curvature lie, cannot all be taken arbitrarily.

193. As regards the integrability of the equation, when X, Y, Z are determined by the first five of the equations, we have

$$(kx - a)\,dX + (ky - b)\,dY + (kz - c)\,dZ = (Xa_1 + Yb_1 + Zc_1 + l_1)\,dp,$$

and

$$(kx - a)\,dx + (ky - b)\,dy + (kz - c)\,dz = (xa_1 + yb_1 + zc_1 + u_1)\,dp;$$

hence, writing

$$A = \frac{Xa_1 + Yb_1 + Zc_1 + l_1}{xa_1 + yb_1 + zc_1 + u_1},$$

we have

$$(kx - a)\,dX + (ky - b)\,dY + (kz - c)\,dZ$$
$$= A\{(kx - a)\,dx + (ky - b)\,dy + (kz - c)\,dz\}$$
$$= U,$$

say. Similarly we have

$$(\kappa x - \alpha)\,dX + (\kappa y - \beta)\,dY + (\kappa z - \gamma)\,dZ$$
$$= B\{(\kappa x - \alpha)\,dx + (\kappa y - \beta)\,dy + (\kappa z - \gamma)\,dz\}$$
$$= V,$$

say, where

$$B = \frac{X\alpha_2 + Y\beta_2 + Z\gamma_2 + \lambda_2}{x\alpha_2 + y\beta_2 + z\gamma_2 + v_2}.$$

And

$$XdX + YdY + ZdZ = 0.$$

Let

$$\Omega = \begin{vmatrix} kx - a, & ky - b, & kz - c \\ \kappa x - \alpha, & \kappa y - \beta, & \kappa z - \gamma \\ X\ , & Y\ , & Z \end{vmatrix},$$

where Ω is not zero (for otherwise the normal to the surface would be coplanar with the normals to the parametric surfaces); then

$$\Omega dX = U\{(\kappa y - \beta) Z - (\kappa z - \gamma) Y\} + V\{(kz - c) Y - (ky - b) Z\},$$
$$\Omega dY = U\{(\kappa z - \gamma) X - (\kappa x - \alpha) Z\} + V\{(kx - a) Z - (kz - c) X\},$$
$$\Omega dZ = U\{(\kappa x - \alpha) Y - (\kappa y - \beta) X\} + V\{(ky - b) X - (kx - a) Y\}.$$

Hence

$$\Omega \frac{\partial Y}{\partial z} = A(kz - c)\{(\kappa z - \gamma) X - (\kappa x - \alpha) Z\} + B(\kappa z - \gamma)\{(kx - a) Z - (kz - c) X\},$$

$$\Omega \frac{\partial Z}{\partial y} = A(ky - b)\{(\kappa x - \alpha) Y - (\kappa y - \beta) X\} + B(\kappa y - \beta)\{(ky - b) X - (kx - a) Y\}.$$

Let

$$P = (kx - a)(\kappa x - \alpha) + (ky - b)(\kappa y - \beta) + (kz - c)(\kappa z - \gamma)$$
$$= a\alpha + b\beta + c\gamma + kv + u\kappa,$$

as before; then

$$\Omega\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) = A\{PX - (\kappa x - \alpha) l\} - B\{PX - (kx - a)\lambda\}.$$

Similarly

$$\Omega\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) = A\{PY - (\kappa y - \beta) l\} - B\{PY - (ky - b)\lambda\},$$

$$\Omega\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = A\{PZ - (\kappa z - \gamma) l\} - B\{PZ - (kz - c)\lambda\}.$$

Multiplying by X, Y, Z respectively and adding, we have

$$\Omega\left\{X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)\right\} = (A - B)(P - l\lambda).$$

But the analytical expression of the orthogonality of the lines of curvature was shewn to be

$$P = l\lambda,$$

and Ω is not zero; hence

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0.$$

The condition of integrability (§ 30) of the equation $Xdx + Ydy + Zdz = 0$ in our set of six equations is therefore satisfied.

The foregoing analysis shews that the necessary relation

$$a\alpha + b\beta + c\gamma + kv + u\kappa = l\lambda$$

gives the orthogonality of the lines of curvature and the condition of integrability; also it shews that these two properties are analytically equivalent to one another.

It is the resolution of this relation, combined with the general equations, that gives rise to the various surfaces.

Surfaces with Two Plane Systems.

194. In the first place, consider surfaces having plane curves for both their systems of lines of curvature. Then

$$k = 0, \quad \kappa = 0;$$

the equations involving the determination of X, Y, Z are

$$ax + by + cz = u, \quad \alpha x + \beta y + \gamma z = v,$$

$$aX + bY + cZ = l, \quad \alpha X + \beta Y + \gamma Z = \lambda,$$

$$X^2 + Y^2 + Z^2 = 1,$$

$$a\alpha + b\beta + c\gamma = l\lambda,$$

while a differential equation of the surface as usual is

$$Xdx + Ydy + Zdz = 0.$$

Sometimes it will prove convenient to denote the first derivatives of z with regard to x and y by p and q; so a change of notation will be made. On the surface, we shall take m and μ as the current parameters of the lines of curvature; and we shall assume that a, b, c, u, l are functions of m alone, while α, β, γ, v, λ are functions of μ alone.

I. If possible, let $l = 0$, $\lambda = 0$, so that the plane of every line of curvature is perpendicular to the tangent plane of the surface; thus all the lines of curvature are geodesics. Hence there are two families of geodesics cutting at right angles; therefore (§ 114) the surface must be developable. Then

$$a\alpha + b\beta + c\gamma = 0.$$

Now a, b, c cannot all vanish; let a be different from zero, so we can make it unity, and the relation becomes

$$\alpha + b\beta + c\gamma = 0.$$

Also β and γ cannot both vanish, for then α would vanish also; so let β be different from zero. Then we can take* $\beta = 1$, and thus the relation becomes

$$\alpha + b + c\gamma = 0.$$

* In effect, we can divide by a in the former case and by β in the latter case; the homogeneous equations are substantially unchanged.

In this relation, b and c at the utmost are functions of m alone, while α and γ at the utmost are functions of μ alone. Hence:—either b is a pure constant, c is a pure constant, and $\alpha + c\gamma$ is a pure constant; or α and γ are pure constants, and $b + c\gamma$ is a pure constant. The alternatives are interchanged by an interchange of parameters; we choose the first. Thus one family of the planes is

$$x + by + cz = u,$$

that is, by a change of axes, it is

$$x = u,$$

so that we can take $b = 0$, $c = 0$. The relation now gives $\alpha = 0$; and so the other family of planes is

$$y + \gamma z = v,$$

where γ and v are functions of μ alone, or (what is the same thing) where v is a function of γ, say

$$v = (1 + \gamma^2)^{\frac{3}{2}} F'(\gamma).$$

The equations for X, Y, Z now are

$$X = 0, \quad Y + \gamma Z = 0;$$

hence the differential equation $Xdx + Ydy + Zdz = 0$ of the surface now is

$$\begin{aligned} dz &= \gamma dy \\ &= \gamma (dv - \gamma dz - z d\gamma), \end{aligned}$$

that is,

$$(1 + \gamma^2)\, dz + z\gamma d\gamma = \gamma dv.$$

Thus

$$\begin{aligned} z (1 + \gamma^2)^{\frac{1}{2}} &= \int \gamma (1 + \gamma^2)^{-\frac{1}{2}} dv \\ &= \gamma (1 + \gamma^2) F'(\gamma) - F(\gamma). \end{aligned}$$

The tangent plane to the surface is perpendicular to the planes

$$x = u, \quad y + \gamma z = v,$$

at the point; hence its equation is

$$\gamma y - z = (1 + \gamma^2)^{\frac{1}{2}} F(\gamma),$$

containing one parameter. The surface is a cylinder, having its generators perpendicular to the plane $x = 0$; its section by the plane $x = 0$, or by any plane parallel to $x = 0$, is the envelope of the straight line

$$z = \gamma y - (1 + \gamma^2)^{\frac{1}{2}} F(\gamma).$$

II. In the second place, let only one of the two quantities l and λ vanish. Let l be zero; so that the planes of the lines of curvature in that system contain the normal to the surface, and the lines of curvature in the

system are geodesics. The most obvious example is that of a surface of revolution.

The analysis of the preceding case applies in its initial stage; we can take $a = 1$, $\beta = 1$, and the critical relation is

$$a + b + c\gamma = 0.$$

Now consider the alternative rejected in the preceding case; without loss of generality, we take

$$a = 0, \quad \gamma = 0, \quad b = 0;$$

the equations become

$$x + cz = u, \qquad y = v,$$

$$X + cZ = 0, \qquad Y = \lambda,$$

where u and c are functions of m alone, while v and λ are functions of μ alone. Thus we can regard u as a function of c, and v as a function of λ, say

$$u = f(c) = (1 + c^2)^{\frac{3}{2}} F'(c), \quad v = g(\lambda) = (1 - \lambda^2)^{\frac{3}{2}} G'(\lambda).$$

Then

$$X = c\left(\frac{1 - \lambda^2}{1 + c^2}\right)^{\frac{1}{2}}, \quad Z = -\left(\frac{1 - \lambda^2}{1 + c^2}\right)^{\frac{1}{2}}, \quad Y = \lambda;$$

and so the equation of the surface, $Xdx + Ydy + Zdz = 0$, becomes

$$\frac{c\,dx - dz}{(1 + c^2)^{\frac{1}{2}}} + \frac{\lambda g'(\lambda)}{(1 - \lambda^2)^{\frac{1}{2}}} d\lambda = 0,$$

that is,

$$\frac{cf'(c)}{(1 + c^2)^{\frac{1}{2}}} dc - (1 + c^2)^{\frac{1}{2}} dz - z\frac{c\,dc}{(1 + c^2)^{\frac{1}{2}}} + \frac{\lambda g'(\lambda)}{(1 - \lambda^2)^{\frac{1}{2}}} d\lambda = 0,$$

on substituting from $x + cz = f(c)$. Integrating, we have

$$z(1 + c^2)^{\frac{1}{2}} = c(1 + c^2)F'(c) - F(c) + \lambda(1 - \lambda^2)G'(\lambda) - G(\lambda);$$

this equation, and

$$x + cz = f(c) = (1 + c^2)^{\frac{3}{2}} F'(c),$$

$$y = g(\lambda) = (1 - \lambda^2)^{\frac{3}{2}} G'(\lambda),$$

are the equations of the surface.

When we take

$$T = \frac{z - cx}{(1 + c^2)^{\frac{1}{2}}} - \frac{\lambda y}{(1 - \lambda^2)^{\frac{1}{2}}} + F(c) + G(\lambda),$$

the equations of the surface are

$$T = 0, \quad \frac{\partial T}{\partial c} = 0, \quad \frac{\partial T}{\partial \lambda} = 0,$$

giving it as the envelope of its tangent plane, the equation of which contains two parameters.

The special case, when $F(c) = 0$, gives the general surface of revolution.

Some exceptional cases must be noted. It might happen that λ is a constant. The equation of the surface then is

$$\frac{c\,dx - dz}{(1+c^2)^{\frac{1}{2}}} + \frac{\lambda}{(1-\lambda^2)^{\frac{1}{2}}}\,dy = 0,$$

leading to

$$z(1+c^2)^{\frac{1}{2}} = c(1+c^2)^{\frac{1}{2}} F'(c) - F(c) + \frac{\lambda}{(1-\lambda^2)^{\frac{1}{2}}}\,y,$$

together with

$$x + cz = (1+c^2)^{\frac{3}{2}} F'(c).$$

When we take

$$T' = \frac{z - cx}{(1+c^2)^{\frac{1}{2}}} + F(c) - \frac{\lambda}{(1-\lambda^2)^{\frac{1}{2}}}\,y,$$

the surface is given by

$$T' = 0, \quad \frac{\partial T'}{\partial c} = 0;$$

it is the envelope of a plane containing one parameter, and therefore it is a developable surface.

It might happen that c is a constant. The equation of the surface then is

$$\frac{c\,dx - dz}{(1+c^2)^{\frac{1}{2}}} + \frac{\lambda g'(\lambda)}{(1-\lambda^2)^{\frac{1}{2}}}\,d\lambda = 0,$$

leading to

$$\frac{cx - z}{(1+c^2)^{\frac{1}{2}}} + \lambda(1-\lambda^2)\,G'(\lambda) - G(\lambda) = 0,$$

together with

$$y = (1-\lambda^2)^{\frac{3}{2}} G'(\lambda).$$

When we take

$$T'' = \frac{cx - z}{(1+c^2)^{\frac{1}{2}}} + \frac{\lambda y}{(1-\lambda^2)^{\frac{1}{2}}} - G(\lambda),$$

the surface is given by

$$T'' = 0, \quad \frac{\partial T''}{\partial \lambda} = 0;$$

and so it is a developable surface.

III. Now suppose that neither l nor λ vanishes; the critical relation is

$$a\alpha + b\beta + c\gamma = l\lambda.$$

As a, b, c cannot all vanish, suppose that a is not zero; we can take it equal

to unity, as before. Also, as α, β, γ cannot all vanish, suppose that β is not zero; we can take it equal to unity as before. Thus

$$\alpha + b + c\gamma = l\lambda,$$

where b, c, l are functions of one parameter m, and α, γ, λ are functions of the other parameter μ.

We have

$$b' + c'\gamma = l'\lambda,$$
$$c'\gamma' = l'\lambda',$$

where b' is the derivative of b; and similarly for the other quantities. If c' is not zero and γ' is not zero, we have

$$l' = \frac{\gamma'}{\lambda'}c',$$

$$b' = \left(\lambda\frac{\gamma'}{\lambda'} - \gamma\right)c'.$$

Hence $\lambda\frac{\gamma'}{\lambda'} - \gamma$ is a constant or is zero, and so there is a linear relation between b and c. Thus either c is constant, or γ is constant, or there is a linear relation between b and c; that is, the planes of one of the families are parallel to a fixed line. Let it be the family determined by the parameter m, and take the fixed line for axis of y; then

$$b = 0,$$

and the critical relation becomes

$$\alpha + c\gamma = l\lambda.$$

Hence

$$c'\gamma = l'\lambda.$$

If c were constant, we should either have $\lambda = 0$, which is excluded, or $l' = 0$, so that the family of planes would be only a single plane; thus

$$\gamma = \frac{l'}{c'}\lambda,$$

$$\alpha = \left(l - \frac{l'}{c'}c\right)\lambda.$$

Hence

$$\frac{\alpha}{\gamma} = l\frac{c'}{l'} - c,$$

and each must therefore be a constant; so the planes of the second family are parallel to a line in the plane of xz. Take this line as the axis of x; then we have $\alpha = 0$, and the critical relation becomes

$$c\gamma = l\lambda,$$

that is,

$$\frac{\gamma}{\lambda}=\frac{l}{c}=g,$$

where g is a constant. Writing $f=\frac{1}{g}$, we now have the equations

$$x+cz=u, \qquad y+\gamma z=v,$$
$$X+cZ=gc, \quad Y+\gamma Z=f\gamma,$$

and

$$X^2+Y^2+Z^2=1.$$

Let

$$\frac{1}{C^2}=f+(f-g)c^2, \quad \frac{1}{\Gamma^2}=g+(g-f)\gamma^2;$$

the equation

$$c^2(g-Z)^2+\gamma^2(f-Z)^2+Z^2=1$$

gives

$$Z=\frac{fC-g\Gamma}{C-\Gamma};$$

and then

$$X=-cC\frac{f-g}{C-\Gamma},$$

$$Y=-\gamma\Gamma\frac{f-g}{C-\Gamma}.$$

Now

$$dx=-c\,dz-z\,dc+u'dc,$$
$$dy=-\gamma\,dz-z\,d\gamma+v'd\gamma;$$

and the differential equation of the surface is

$$X\,dx+Y\,dy+Z\,dz=0,$$

that is,

$$-cC\,dx-\gamma\Gamma\,dy+\frac{fC-g\Gamma}{f-g}\,dz=0.$$

Substituting for dx and dy, and reducing, we find

$$\left(\frac{1}{C}-\frac{1}{\Gamma}\right)dz+(f-g)\,z\,(cC\,dc+\gamma\Gamma\,d\gamma)-(f-g)(cCu'dc+\gamma\Gamma v'd\gamma)=0,$$

and therefore

$$z\left(\frac{1}{C}-\frac{1}{\Gamma}\right)-(f-g)\int(cCu'dc+\gamma\Gamma v'd\gamma)=0.$$

To have the integral free from quadratures, let

$$u=g\frac{F'}{C^3}, \quad v=f\frac{\Phi'}{\Gamma^3},$$

where F is any function of c alone, and Φ is any function of γ alone, so that the generality of u and of v is conserved; then we have

$$z\left(\frac{1}{C}-\frac{1}{\Gamma}\right)+(f-g)\left\{F+\Phi-gc\frac{F'}{C^2}-f\gamma\frac{\Phi'}{\Gamma^2}\right\}=0.$$

Substituting this value of z in the equations

$$x + cz = g\frac{F'}{C^3}, \quad y + \gamma z = f\frac{\Phi'}{\Gamma^3},$$

we find

$$x\left(\frac{1}{C} - \frac{1}{\Gamma}\right) + (f-g)\left\{-c(F+\Phi) + fc\gamma\frac{\Phi'}{\Gamma^2}\right\} + \left(-1 + \frac{g}{C\Gamma}\right)\frac{F'}{C^2} = 0,$$

$$y\left(\frac{1}{C} - \frac{1}{\Gamma}\right) + (f-g)\left\{-\gamma(F+\Phi) + gc\gamma\frac{F'}{C^2}\right\} + \quad \left(1 - \frac{f}{C\Gamma}\right)\frac{\Phi'}{\Gamma^2} = 0.$$

These three equations for x, y, z are the parametric equations of the surface.

Other forms can be given to them. Eliminating F', Φ'; and F, Φ'; and F', Φ; in turn, we have

$$P = -cCx - \gamma\Gamma y + \frac{fC - g\Gamma}{f-g}z + F + \Phi = 0,$$

$$\frac{\partial P}{\partial u} = -fC^3(x + cz) + F' = 0,$$

$$\frac{\partial P}{\partial v} = -g\Gamma^3(y + \gamma z) + \Phi' = 0,$$

equations which represent the surface as the envelope of the plane $P = 0$. Moreover, the equations

$$\frac{\partial P}{\partial u} = 0, \quad \frac{\partial P}{\partial v} = 0,$$

are the planes of the lines of curvature of the two systems; the inclination of the former to the tangent plane at x, y, z is

$$\cos^{-1}\frac{gu}{(1+u^2)^{\frac{1}{2}}},$$

and the inclination of the latter to that tangent plane is

$$\cos^{-1}\frac{fv}{(1+v^2)^{\frac{1}{2}}},$$

while the inclination of the two planes to one another is

$$\cos^{-1}\frac{uv}{(1+u^2)^{\frac{1}{2}}(1+v^2)^{\frac{1}{2}}}.$$

When we take new parameters α and β, and new functions A and B of them respectively, where

$$k\alpha = -cC, \quad k\beta = -\gamma\Gamma, \quad F = kA, \quad \Phi = kB,$$

$$k = (f-g)^{-\frac{1}{2}}, \quad \lambda k = f^{\frac{1}{2}},$$

the plane $P=0$ becomes

$$\alpha x+\beta y+z\{\lambda(1-\alpha^2)^{\frac{1}{2}}-(\lambda^2-1)^{\frac{1}{2}}(1-\beta^2)^{\frac{1}{2}}\}+A+B=0.$$

The surface is the envelope of this plane, α and β being the parameters.

The earlier form is the form obtained by Serret and Cayley; the later is the form obtained by Darboux.

Dupin's Cyclides.

195. One of the most interesting examples of surfaces, having both its systems of lines of curvature in the form of plane curves, is provided by Dupin's cyclide*. The name cyclide was originally given to surfaces all whose lines of curvature are circles; it now is given to all surfaces of the fourth order which have the circle at infinity for a double line, and to all surfaces of the third order which contain the circle at infinity.

Dupin's cyclide is defined as the envelope of a sphere which has its centre on one conic and passes through any one assigned point on another conic; the two conics are to lie in perpendicular planes, and each of them is to pass through the foci of the other. Also, the generation is double; either of the conics can be taken as the locus of the centre of the moving sphere, but there is a relation between the fixed points on the respective conics through which the moving spheres are required to pass.

That the envelope surface, under the double generation, has circular lines of curvature can easily be seen. Take either generation. Where the surface envelopes a sphere, the normals to both are the same; because they are normals to the sphere, they intersect; and so, as these normals to the surface meet one another, the curve of contact is a line of curvature. The curve of contact is the intersection of two consecutive spheres, and therefore it is a circle; and so the lines of curvature in the system are circular. Similarly for the lines of curvature in the other system.

The analysis is simple. Let the two conics be

$$\left.\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2}&=1\\ z&=0\end{aligned}\right\},\quad \left.\begin{aligned}\frac{x^2}{c^2}-\frac{z^2}{b^2}&=1\\ y&=0\end{aligned}\right\};$$

the condition that each of them passes through the foci of the other is

$$c^2=a^2-b^2.$$

* See Dupin's *Applications de géométrie et de méchanique*, p. 200; Cayley, *Coll. Math. Papers*, vol. IX, pp. 64—78; Darboux, *Leçons sur les systèmes orthogonaux*, 2me éd. (1910), pp. 484—498.

Denote by $a\cos\theta$, $b\sin\theta$, 0, the centre of the sphere on the first conic, and by α, 0, γ, the fixed point on the second conic through which the sphere is to pass; the equation of the sphere is

$$x^2+y^2+z^2-2(x-\alpha)\,a\cos\theta-2by\sin\theta-\alpha^2-\gamma^2=0.$$

The envelope of the sphere is

$$(x^2+y^2+z^2-\alpha^2-\gamma^2)^2=4a^2(x-\alpha)^2+4b^2y^2;$$

thus one system of the circular lines of curvature, being the intersections of these consecutive spheres, is given by the equations

$$\left.\begin{aligned} x^2+y^2+z^2-2(x-\alpha)\,a\cos\theta-2by\sin\theta-\alpha^2-\gamma^2=0\\ (x-\alpha)\,a\sin\theta-by\cos\theta=0\end{aligned}\right\}.$$

For the other generation of the cyclide, denote by $c\cos\phi$, 0, $ib\sin\phi$, the centre of the sphere on the second conic, and by α', β', 0, the fixed point on the first conic through which the sphere is to pass; the equation of the sphere is

$$x^2+y^2+z^2-2(x-\alpha')\,c\cos\phi-2ibz\sin\phi-\alpha'^2-\beta'^2=0.$$

The envelope of the sphere is

$$(x^2+y^2+z^2-\alpha'^2-\beta'^2)^2=4c^2(x-\alpha')^2-4b^2z^2;$$

thus the other system of circular lines of curvature, being the intersections of these consecutive spheres, is given by the equations

$$\left.\begin{aligned} x^2+y^2+z^2-2(x-\alpha')\,c\cos\phi-2ibz\sin\phi-\alpha'^2-\beta'^2=0\\ (x-\alpha')\,c\sin\phi-ibz\cos\phi=0\end{aligned}\right\}.$$

Moreover, among the constants, we have the relations

$$\frac{\alpha^2}{c^2}-\frac{\gamma^2}{b^2}=1,\quad \frac{\alpha'^2}{a^2}+\frac{\beta'^2}{b^2}=1,\quad c^2=a^2-b^2.$$

The two envelopes of the two sets of moving spheres are to be one and the same surface. When the two equations are compared and these relations are used, we find that the two equations are the same, provided the additional relation

$$a^2\alpha=c^2\alpha'$$

is satisfied. We take a new quantity μ such that

$$\alpha=\mu\frac{c}{a},\quad \alpha'=\mu\frac{a}{c};$$

and then the equation of the cyclide has the equivalent forms

$$\left.\begin{aligned} (x^2+y^2+z^2-\mu^2+b^2)^2=4(ax-c\mu)^2+4b^2y^2\\ (x^2+y^2+z^2-\mu^2-b^2)^2=4(cx-a\mu)^2-4b^2z^2\end{aligned}\right\}.$$

Let ρ_θ be the radius of the circle

$$x^2 + y^2 + z^2 - \mu^2 + b^2 - 2(ax - c\mu)\cos\theta - 2by\sin\theta = 0,$$
$$(ax - c\mu)\sin\theta - by\cos\theta = 0,$$

which are the equations of one system of lines of curvature; and let Θ be the inclination of the radius to the normal to the cyclide, Θ being constant (by Joachimsthal's theorem) along the line. Then

$$\rho_\theta = \frac{(\mu - c\cos\theta)\, b}{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{1}{2}}},$$

$$\frac{\sin\Theta}{c\sin\theta} = \frac{\cos\Theta}{b} = \frac{1}{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{1}{2}}};$$

and therefore the principal radius of curvature of the cyclide along this line of curvature is given by

$$R_\theta = \mu - c\cos\theta.$$

In the equations of the other system of lines of curvature, the quantity ϕ is imaginary; taking real arguments, write

$$\cos\phi = \sec\psi, \quad \sin\phi = i\tan\psi;$$

then the equations of the lines are

$$x^2 + y^2 + z^2 - \mu^2 - b^2 - 2(cx - a\mu)\sec\psi + 2bz\tan\psi = 0,$$
$$(cx - a\mu)\sin\psi - bz = 0.$$

Let ρ_ψ be the radius of this circle, and let Ψ be the inclination of its radius to the normal to the cyclide, Ψ being constant along the line; then

$$\rho_\psi = \frac{(\mu\cos\psi - a)\, b}{(a^2\sin^2\psi + b^2\cos^2\psi)^{\frac{1}{2}}},$$

$$\frac{\sin\Psi}{a\sin\psi} = \frac{\cos\Psi}{b\cos\psi} = \frac{1}{(a^2\sin^2\psi + b^2\cos^2\psi)^{\frac{1}{2}}};$$

and therefore the principal radius of curvature of the cyclide along this line of curvature is given by

$$R_\psi = \mu - a\sec\psi.$$

The coordinates of any point on the surface, given as the intersection of two lines of curvature, are

$$\left.\begin{aligned} x &= \frac{\mu a}{c} + \frac{b^2}{c}\,\frac{c\cos\theta - \mu}{a - c\cos\theta\cos\psi} \\ y &= \frac{b\,(a - \mu\cos\psi)}{a - c\cos\theta\cos\psi}\sin\theta \\ z &= \frac{b\,(c\cos\theta - \mu)}{a - c\cos\theta\cos\psi}\sin\psi \end{aligned}\right\}.$$

The fundamental quantities of the first order are

$$E=\left(\frac{\partial x}{\partial \theta}\right)^2+\left(\frac{\partial y}{\partial \theta}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=\frac{b^2(\mu\cos\psi-a)^2}{(a-c\cos\theta\cos\psi)^2},$$

$$G=\left(\frac{\partial x}{\partial \psi}\right)^2+\left(\frac{\partial y}{\partial \psi}\right)^2+\left(\frac{\partial z}{\partial \psi}\right)^2=\frac{b^2(\mu-c\cos\theta)^2}{(a-c\cos\theta\cos\psi)^2},$$

$$F=0;$$

and the fundamental quantities of the second order are

$$L=\frac{E}{R_\psi}=\frac{b^2(\mu\cos\psi-a)\cos\psi}{(a-c\cos\theta\cos\psi)^2},$$

$$N=\frac{G}{R_\theta}=\frac{b^2(\mu-c\cos\theta)}{(a-c\cos\theta\cos\psi)^2},$$

$$M=0.$$

The direction-cosines X, Y, Z of the normal to the cyclide, being the same as those of the enveloped sphere at the point, are

$$X=\frac{a\cos\theta\cos\psi-c}{a-c\cos\theta\cos\psi},$$

$$Y=\frac{b\sin\theta\cos\psi}{a-c\cos\theta\cos\psi},$$

$$Z=\frac{-b\sin\psi}{a-c\cos\theta\cos\psi}.$$

Two spheres of different systems touch; the centre of one of them is $a\cos\theta$, $b\sin\theta$, 0, and its radius is $|\mu-c\cos\theta|$; the centre of the other of them is $c\sec\psi$, 0, $-b\tan\psi$, and its radius is $|\mu-a\sec\psi|$; and so the distance between the centres is equal to the difference of the radii. The point of contact is a point on the surface, which therefore lies on the line joining the centres; and this line is normal to the spheres and therefore normal to the surface. (It is easy to verify that its direction-cosines are X, Y, Z.) Hence any straight line, meeting the initial ellipse and the initial hyperbola, is a normal to the cyclide.

For other properties of Dupin's cyclides, reference may be made to the authorities already quoted.

Ex. Shew that, for parametric values of μ, the Dupin cyclides are a family of parallel surfaces.

196. A limiting case of the preceding investigation has to be noted; and one case has not been included. The results will merely be stated, and their establishment left as an exercise.

The limiting case arises, when the ellipse becomes a circle and the hyperbola degenerates into the straight line through the centre of the circle perpendicular to its plane. The cyclide then becomes an anchor-ring, of which the circle is the central thread; and the only parametric element in the equation of the surface is the radius of the core.

The non-included case arises when the conics, which supply the foundation of the construction of the surface, are parabolas—of course, in perpendicular planes and each passing through the focus of the other. When the equations of these parabolas are taken in the form

$$\left.\begin{array}{l} y^2 = 4l(x+l) \\ z = 0 \end{array}\right\}, \quad \left.\begin{array}{l} z^2 = -4lx \\ y = 0 \end{array}\right\}.$$

the equation of the cyclide (with the double generation as before) is

$$x(x^2+y^2+z^2) + (x^2+y^2)(l-\mu) - z^2(l+\mu) - (x-l-\mu)(l+\mu)^2 = 0,$$

a surface of only the third order. The coordinates of a point on the surface can be expressed in the form

$$\begin{aligned} x(1+t^2+\theta^2) &= l(\theta^2 - t^2 + 1) + \mu(\theta^2 + t^2 + 1), \\ y(1+t^2+\theta^2) &= 2l(\theta^2+1)\,t + 2\mu t, \\ z(1+t^2+\theta^2) &= 2l\theta t^2 - 2\mu\theta, \end{aligned}$$

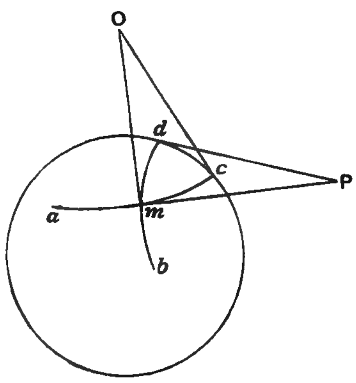
where t and θ are the parametric variables of the lines of curvature; and the principal radii of curvature of the surface are

$$\mu - lt^2, \quad \mu + l(1+\theta^2).$$

Rouquet's Method, by Spherical Representation.

197. Some of the foregoing results can be obtained* simply, from the properties of the spherical image of the surface when the latter has a double plane system of lines of curvature.

It has already been proved (§ 160) that the spherical image of a plane line of curvature is a small circle, and that the line of curvature and its image are parallel to one another at corresponding points; also that the latter property suffices to secure the result that the curves are lines of curvature. Hence, on the surface of the sphere, there are two series of small circles cutting one another orthogonally.



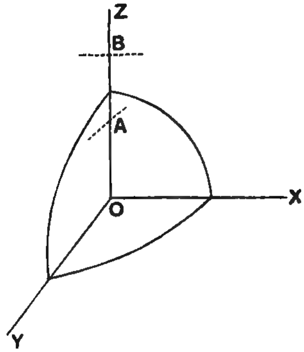
Consider two such circles, intersecting in m; let O and P be the vertices of the cones that circumscribe the sphere along the circles. Then mO is a tangent at m to the circle dmb; that is, the locus of O for all the circles amc lies in the plane dmb. Similarly, the locus of O for all the circles amc lies in the plane of any other small circle of the series to which the circle dmb belongs; and therefore it is a straight line. Hence all the planes of the series of small circles dmb pass through a straight line. Now the planes amc are polars of

* Rouquet, *Toul. Mém.*, 8e Sér., t. ix (1887), t. x (1888).

points O on this straight line; hence they all pass through the conjugate line. Thus the two systems of planes pass through two straight lines which are conjugate to one another; the latter are necessarily perpendicular to one another, and the product of their distances from the centre of the sphere is unity.

Take one of the lines in the plane of YZ, and let $OA = g$; then any plane through that line is

$$X + c(Z - g) = 0,$$



where c is a parameter varying from plane to plane, that is, it is the parameter of the spherical images of the lines of curvature in the family. Take the conjugate line in the plane of XZ, and let $OB = f$, so that

$$fg = 1;$$

then any plane through that line is

$$Y + \gamma(Z - f) = 0,$$

where γ is the parameter of the spherical images of the lines of curvature in the other plane family.

Any point on the sphere is thus given by the equations

$$X + c(Z - g) = 0, \quad Y + \gamma(Z - f) = 0, \quad X^2 + Y^2 + Z^2 = 1,$$

being effectively the same relations as in § 194; hence

$$X = -cC\frac{f-g}{C-\Gamma}, \quad Y = -\gamma\Gamma\frac{f-g}{C-\Gamma}, \quad Z = \frac{fC - g\Gamma}{C-\Gamma},$$

where

$$\frac{1}{C^2} = f + (f-g)c^2, \quad \frac{1}{\Gamma^2} = g + (g-f)\gamma^2.$$

Let

$$f = \cos\alpha, \quad g = \sec\alpha,$$

$$c\tan\alpha = \tanh u, \quad \gamma\sin\alpha = \tan v,$$

so that u and v are a couple of new parameters; then

$$C = (\cos\alpha)^{-\frac{1}{2}}\cosh u, \quad \Gamma = (\cos\alpha)^{\frac{1}{2}}\cos v.$$

Hence

$$X = \frac{\sin\alpha\sinh u}{\cosh u - \cos\alpha\cos v}, \quad Y = \frac{\sin\alpha\sin v}{\cosh u - \cos\alpha\cos v}, \quad Z = \frac{\cos\alpha\cosh u - \cos v}{\cosh u - \cos\alpha\cos v}.$$

The tangent plane to the sphere at the point, determined by u and v, has

$$X\sin\alpha\sinh u + Y\sin\alpha\sin v + Z(\cos\alpha\cosh u - \cos v) = \cosh u - \cos\alpha\cos v$$

for its equation; and therefore the equation of the tangent plane to the surface (which is parallel to this plane) is

$$x \sin \alpha \sinh u + y \sin \alpha \sin v + z (\cos \alpha \cosh u - \cos v) = F(u, v),$$

where $F(u, v)$ is some function of u and v. The point x, y, z on the surface is given by combining this relation with

$$x \sin \alpha \cosh u + z \cos \alpha \sinh u = \frac{\partial F}{\partial u},$$

$$y \sin \alpha \cos v + z \sin v = \frac{\partial F}{\partial v}.$$

But

$$Y_1 = -\sinh u \frac{\sin \alpha \sin v}{(\cosh u - \cos \alpha \cos v)^2}, \quad Z_1 = \frac{\sin^2 \alpha \cos v}{(\cosh u - \cos \alpha \cos v)^2} \sinh u,$$

so that

$$Y_1 \sin \alpha \cos v + Z_1 \sin v = 0.$$

The parametric curves are lines of curvature, so that

$$EX_1 = -Lx_1, \quad EY_1 = -Ly_1, \quad EZ_1 = -Lz_1;$$

hence

$$x_1 \sin \alpha \cos v + z_1 \sin v = 0.$$

Hence, from the third of the equations that give the values of x, y, z, we have

$$\frac{\partial^2 F}{\partial u \partial v} = 0;$$

and the same result follows from constructing X_2 and Z_2, and using the second of those equations. Thus

$$F = U + V,$$

where U is a function of u only, and V is a function of v only; and now the equations of the surface are

$$\left.\begin{aligned} x \sin \alpha \sinh u + y \sin \alpha \sin v + z (\cos \alpha \cosh u - \cos v) &= U + V \\ x \sin \alpha \cosh u + z \cos \alpha \sinh u &= U' \\ y \sin \alpha \cos v + z \sin v &= V' \end{aligned}\right\},$$

which are easily seen to be in accordance with the results previously obtained (§ 194).

The second and the third of these equations, taken separately, are the equations of the planes of the lines of curvature.

Note. We may also proceed from the equations for the tangential coordinates, as given in § 163.

It is easy to prove that

$$e = \frac{\sin^2 \alpha}{(\cosh u - \cos \alpha \cos v)^2} = g, \quad f = 0;$$

and so, with the equation

$$xX + yY + zZ = T$$

in general, and the equation

$$T_{12} - \gamma' T_1 - \delta' T_2 = 0$$

in this case, where

$$\gamma' = \frac{e_2}{2e}, \quad \delta' = \frac{g_1}{2g},$$

we have

$$\frac{\partial^2}{\partial u \partial v}(DT) = DT_{12} + D_2 T_1 + D_1 T_2 + TD_{12}$$
$$= 0,$$

when

$$D = \cosh u - \cos \alpha \cos v.$$

Thus

$$DT = U + V,$$

where U is a function of u only, and V is a function of v only; that is, the equation of the tangent plane to the surface is

$$x \sin \alpha \sinh u + y \sin \alpha \sin v + z(\cos \alpha \cosh u - \cos v) = U + V,$$

as before.

Ex. Shew that, if

$$U = a \sinh u + b \cosh u + c, \quad V = k \sin v + l \cos v,$$

where a, b, c, k, l are constants, the equation of the surface is

$$\{(b - z\cos\alpha)^2 - (x \sin \alpha - a)^2\}^{\frac{1}{2}} + \{(y \sin \alpha - k)^2 + (z + l)^2\}^{\frac{1}{2}} + c = 0.$$

Prove that the surface is a Dupin cyclide, having its centre at the point

$$a \operatorname{cosec} \alpha, \quad k \operatorname{cosec} \alpha, \quad (l + b \cos \alpha)\operatorname{cosec}^2 \alpha;$$

and find the smallest value of $|c|$ which allows the cyclide to be real. (When $c = 0$, the surface is a point-sphere duplicated.)

198. The general result is ineffective in the special case when

$$f = g = 1.$$

The equations then are

$$X + c(Z - 1) = 0, \quad Y + \gamma(Z - 1) = 0, \quad X^2 + Y^2 + Z^2 = 1;$$

so that

$$\frac{X}{2c} = \frac{Y}{2\gamma} = \frac{Z}{c^2 + \gamma^2 - 1} = \frac{1}{c^2 + \gamma^2 + 1}.$$

The tangent plane of the surface is

$$2cx + 2\gamma y + (c^2 + \gamma^2 - 1) z = F(c, \gamma),$$

where F is some function of c and γ. The coordinates of a point on the surface are obtainable by joining this equation to the other two equations

$$x + cz = \frac{1}{2}\frac{\partial F}{\partial c},$$

$$y + \gamma z = \frac{1}{2}\frac{\partial F}{\partial \gamma}.$$

Also

$$Y_1 = \frac{-4c\gamma}{(c^2+\gamma^2+1)^2}, \quad Z_1 = \frac{4c}{(c^2+\gamma^2+1)^2},$$

so that

$$Y_1 + \gamma Z_1 = 0.$$

Hence

$$y_1 + \gamma z_1 = 0,$$

and therefore

$$\frac{\partial^2 F}{\partial c \partial \gamma} = 0.$$

Hence F is of the form $C + \Gamma$, where C is any function of c alone, and Γ is any function of γ alone; and now the equations of the surface are

$$\left.\begin{aligned} 2cx + 2\gamma y + (c^2+\gamma^2-1)z &= C + \Gamma \\ x + cz &= \tfrac{1}{2}C' \\ y + \gamma z &= \tfrac{1}{2}\Gamma' \end{aligned}\right\}.$$

The second and third of these equations are, as before, the planes of the two systems of lines of curvature.

Ex. 1. Shew that the surface

$$\frac{x^2}{a-z} + \frac{y^2}{b-z} = z$$

has plane curves for its systems of lines of curvature.

Ex. 2. Shew that, in the general case for any values of C and of Γ as functions of c and of γ respectively, the surface can be generated as the envelope of spheres having their centres on the parabola

$$y=0, \quad x^2 = \tfrac{1}{2} - 2z,$$

and as the envelope of spheres having their centres on the parabola

$$x=0, \quad y^2 = \tfrac{1}{2} + 2z.$$

Obtain the relation between the two families of spheres when both generations are effective.

One Plane System and one Spherical System.

199. The preceding discussion of surfaces, when both systems of lines of curvature are plane curves, gives a sufficient indication of one of the methods of proceeding in the case of surfaces, having one or both systems of lines of curvature given as spherical curves. For the full detail of cases, reference may be made to the memoirs of Serret* and of Cayley†; the developments, naturally, are mainly of an analytical character.

* *Liouville's Journal*, t. xviii (1853), pp. 113—162.

† *Coll. Math. Papers*, vol. xii, pp. 601—638.

In particular, when we deal with surfaces having a set of plane curves for one system of lines of curvature and a set of spherical curves for the other system, there are seven substantially distinct cases to be set out, according to Serret's investigations. The fundamental equations (with merely changes of sign from the general case) are

$$\left.\begin{aligned} X^2+Y^2+Z^2&=1\\ aX+bY+cZ+l&=0\\ ax+by+cz+u&=0\\ x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z-2v&=0\\ (x-\alpha)X+(y-\beta)Y+(z-\gamma)Z-\lambda&=0 \end{aligned}\right\},$$

where a, b, c, l, u are functions of one parameter m, and α, β, γ, λ, v are functions of the other parameter μ; and the double condition, at once of orthogonality for the lines of curvature and of integrability for the equation of the surface, becomes

$$a\alpha+b\beta+c\gamma-l\lambda+u=0.$$

The seven cases of the critical equation just indicated are as follows, account always being taken of simplification without loss of generality:—

I, $l=0,\ u=0,\ \alpha=0,\ \beta=0,\ \gamma=0$;

II, $u=ml,\ \lambda=m$;

III, $l=-mc,\ u=0,\ \alpha=0,\ \beta=0,\ \gamma=m\lambda$;

IV, $c=0,\ u=ml,\ \alpha=0,\ \beta=0,\ \lambda=m$;

V, $c=0,\ l=0,\ u=0,\ \alpha=0,\ \beta=0$;

VI, $a=0,\ c=0,\ u=ml,\ \beta=0,\ \lambda=m$;

VII, $c=0,\ l=ma,\ u=0,\ \beta=0,\ \alpha=-m\lambda$;

where, throughout, m denotes an arbitrary constant, and so remains an arbitrary function of its argument.

200. Among these, consider specially the case where

(i) the quantities a, b, c, are unrestricted by conditions, while $u=0$;

(ii) the quantities α and β vanish.

The critical equation of orthogonality becomes

$$c\gamma=l\lambda,$$

and therefore we may take

$$\frac{\gamma}{\lambda}=\frac{l}{c}=k,$$

where k is an arbitrary constant. It may be zero, or it may be infinite; the latter case is merged in the former, by interchange of parameters. The lines of curvature of one system lie on concentric spheres.

I. Take the special sub-case, when the constant k is zero. The fundamental equations then are

$$X^2 + Y^2 + Z^2 = 1,$$

$$aX + bY + cZ = 0, \quad xX + yY + zZ = \lambda,$$

$$ax + by + cz = 0, \quad x^2 + y^2 + z^2 = v;$$

and the critical equation is satisfied.

The equations are homogeneous in a, b, c; so we may assume

$$a^2 + b^2 + c^2 = 1.$$

Also, a, b, c being functions of u alone, let

$$a' = a_1 (a_1^2 + b_1^2 + c_1^2)^{-\frac{1}{2}}, \quad b' = b_1 (a_1^2 + b_1^2 + c_1^2)^{-\frac{1}{2}}, \quad c' = c_1 (a_1^2 + b_1^2 + c_1^2)^{-\frac{1}{2}};$$

thus

$$aa' + bb' + cc' = 0,$$

$$a'^2 + b'^2 + c'^2 = 1.$$

Further, if

$$a'', b'', c'' = \left\| \begin{matrix} a, & b, & c \\ a', & b', & c' \end{matrix} \right\|,$$

we have

$$aa'' + bb'' + cc'' = 0,$$

$$a'a'' + b'b'' + c'c'' = 0,$$

$$a''^2 + b''^2 + c''^2 = 1.$$

The aggregate of relations is

$$X^2 + Y^2 + Z^2 = 1, \quad aX + bY + cZ = 0,$$

$$a''^2 + b''^2 + c''^2 = 1, \quad a''a' + b''b' + c''c' = 0,$$

$$a'^2 + b'^2 + c'^2 = 1, \quad a''a + b''b + c''c = 0,$$

$$a^2 + b^2 + c^2 = 1, \quad a'a + b'b + c'c = 0.$$

Hence we may take

$$X = a' \cos t - a'' \sin t, \quad Y = b' \cos t - b'' \sin t, \quad Z = c' \cos t - c'' \sin t,$$

where t is a new variable; these satisfy the two equations in which X, Y, Z occur. Take other three magnitudes

$$X' = a' \sin t + a'' \cos t, \quad Y' = b' \sin t + b'' \cos t, \quad Z' = c' \sin t + c'' \cos t,$$

which obviously are such that

$$X'^2 + Y'^2 + Z'^2 = 1, \quad aX' + bY' + cZ' = 0;$$

moreover,

$$XX' + YY' + ZZ' = 0.$$

Now consider quantities

$$\xi = X\alpha + X'\beta, \quad \eta = Y\alpha + Y'\beta, \quad \zeta = Z\alpha + Z'\beta:$$

then

$$\xi X + \eta Y + \zeta Z = \alpha,$$
$$\xi^2 + \eta^2 + \zeta^2 = \alpha^2 + \beta^2,$$
$$a\xi + b\eta + c\zeta = 0.$$

Comparing these with

$$xX + yY + zZ = \lambda,$$
$$x^2 + y^2 + z^2 = \upsilon,$$
$$ax + by + cz = 0,$$

and writing

$$\alpha = \lambda, \quad \alpha^2 + \beta^2 = \upsilon,$$

we have

$$x = \xi, \quad y = \eta, \quad z = \zeta.$$

Thus

$$x = X\lambda + X'(\upsilon - \lambda^2)^{\frac{1}{2}},$$
$$y = Y\lambda + Y'(\upsilon - \lambda^2)^{\frac{1}{2}},$$
$$z = Z\lambda + Z'(\upsilon - \lambda^2)^{\frac{1}{2}},$$

which are expressions for x, y, z involving three variables, viz. m (through the quantities a, b, c), μ (through λ and υ), and t (through X, Y, Z, X', Y', Z'). These three variables can be reduced to two as follows.

We have

$$a''da + b''db + c''dc = (a''a' + b''b' + c''c')(a_1^2 + b_1^2 + c_1^2)^{\frac{1}{2}} du$$
$$= 0;$$

hence, as $a''a + b''b + c''c = 0$, it follows that

$$ada'' + bdb'' + cdc'' = 0.$$

Also

$$a''da'' + b''db'' + c''dc'' = 0.$$

Consequently

$$\frac{da''}{bc'' - b''c} = \frac{db''}{ca'' - c''a} = \frac{dc''}{ab'' - a''b},$$

that is,

$$\frac{da''}{a'} = \frac{db''}{b'} = \frac{dc''}{c'}$$
$$= d\theta,$$

say, where θ is a function of m only. Thus

$$a'da'' + b'db'' + c'dc' = \quad d\theta;$$

and therefore

$$a''da' + b''db' + c''dc' = - d\theta.$$

Hence

$$\begin{aligned}\Sigma X da' &= d\theta \,.\, \sin t,\\ \Sigma X da'' &= d\theta \,.\, \cos t,\\ \Sigma X dX' &= \Sigma X\,(X dt + da' \,.\, \sin t + da'' \,.\, \cos t)\\ &= dt + d\theta.\end{aligned}$$

Now along the surface, we have

$$X dx + Y dy + Z dz = 0;$$

and the foregoing values of x, y, z must satisfy this relation. Substituting, we find

$$d\lambda + (v - \lambda^2)^{\frac{1}{2}}\,(dt + d\theta) = 0,$$

that is,

$$-\,dt = (v - \lambda^2)^{-\frac{1}{2}}\,d\lambda + d\theta.$$

With this value of t, the equations of the surface are

$$\left.\begin{aligned} x &= \{a''\,(v - \lambda^2)^{\frac{1}{2}} + a'\lambda\}\cos t + \{a'\,(v - \lambda^2)^{\frac{1}{2}} - a''\lambda\}\sin t\\ y &= \{b''\,(v - \lambda^2)^{\frac{1}{2}} + b'\lambda\}\cos t + \{b'\,(v - \lambda^2)^{\frac{1}{2}} - b''\lambda\}\sin t\\ z &= \{c''\,(v - \lambda^2)^{\frac{1}{2}} + c'\lambda\}\cos t + \{c'\,(v - \lambda^2)^{\frac{1}{2}} - c''\lambda\}\sin t \end{aligned}\right\}.$$

II. Now take the less special case, when the constant k is not zero. The equations are

$$\left.\begin{aligned} X^2 + Y^2 + Z^2 &= 1\\ aX + bY + cZ &= -\,ck\\ xX + yY + (z - k\lambda)\,Z &= \lambda\\ ax + by + cz &= 0\\ x^2 + y^2 + z^2 - 2k\lambda z &= 2v \end{aligned}\right\}.$$

The equations are homogeneous in a, b, c; so we can take $c = -1$. Following Serret*, let

$$X = -\,p\,(1 + p^2 + q^2)^{-\frac{1}{2}},\quad Y = -\,q\,(1 + p^2 + q^2)^{-\frac{1}{2}},\quad Z = (1 + p^2 + q^2)^{-\frac{1}{2}},$$

where p and q now denote the derivatives of z with respect to x and y. The first equation is satisfied identically. The remaining equations become

$$\left.\begin{aligned} ap + bq + 1 &= -\,k\,(1 + p^2 + q^2)^{\frac{1}{2}}\\ z - px - qy &= \{k + (1 + p^2 + q^2)^{\frac{1}{2}}\}\,\lambda\\ z - ax - by &= 0\\ x^2 + y^2 + z^2 - 2k\lambda z &= 2v \end{aligned}\right\},$$

where λ and v are functions of μ, so that λ is a function of v. From the second of these, we have

$$-\,x dp - y dq = \{k + (1 + p^2 + q^2)^{\frac{1}{2}}\}\,\lambda' dv + (1 + p^2 + q^2)^{-\frac{1}{2}}\lambda\,(p dp + q dq);$$

* *Liouville's Journal*, t. xviii (1853), p. 141.

so that, when p and q are taken as the parametric variables, we find

$$x = -\{k + (1+p^2+q^2)^{\frac{1}{2}}\}\lambda' \frac{\partial v}{\partial p} - p(1+p^2+q^2)^{-\frac{1}{2}}\lambda,$$

$$y = -\{k + (1+p^2+q^2)^{\frac{1}{2}}\}\lambda' \frac{\partial v}{\partial q} - q(1+p^2+q^2)^{-\frac{1}{2}}\lambda,$$

and therefore

$$z = -\{k + (1+p^2+q^2)^{\frac{1}{2}}\}\left(p\frac{\partial v}{\partial p} + q\frac{\partial v}{\partial q}\right)\lambda' + \{k + (1+p^2+q^2)^{-\frac{1}{2}}\}\lambda.$$

Substitute in

$$z = ax + by,$$

and take account of the first equation; then

$$(a-p)\frac{\partial v}{\partial p} + (b-q)\frac{\partial v}{\partial q} = 0.$$

Substitute also in

$$x^2 + y^2 + z^2 - 2k\lambda z = 2v;$$

then

$$\{k + (1+p^2+q^2)^{\frac{1}{2}}\}^2\left\{\left(\frac{\partial v}{\partial p}\right)^2 + \left(\frac{\partial v}{\partial q}\right)^2 + \left(p\frac{\partial v}{\partial p} + q\frac{\partial v}{\partial q}\right)^2\right\}\lambda'^2 = (k^2-1)\lambda^2 + 2v.$$

Consequently,

$$\frac{\partial v}{\partial p} = (b-q)\Delta, \quad \frac{\partial v}{\partial q} = -(a-p)\Delta,$$

where

$$\Delta\lambda' = \{(k^2-1)\lambda^2 + 2v\}^{\frac{1}{2}}(1+p^2+q^2)^{-\frac{1}{2}}(a^2+b^2+1-k^2)^{-\frac{1}{2}}\{k+(1+p^2+q^2)^{\frac{1}{2}}\}^{-1}.$$

The relation

$$dv = \frac{\partial v}{\partial p}dp + \frac{\partial v}{\partial q}dq,$$

for the determination of v, now becomes

$$\frac{\lambda' dv}{\{(k^2-1)\lambda^2 + 2v\}^{\frac{1}{2}}} = \frac{(b-q)\,dp - (a-p)\,dq}{(1+p^2+q^2)^{-\frac{1}{2}}(a^2+b^2+1-k^2)^{\frac{1}{2}}\{k+(1+p^2+q^2)^{\frac{1}{2}}\}},$$

together with

$$ap + bq + 1 = -k(1+p^2+q^2)^{\frac{1}{2}}.$$

As the left-hand side of the differential relation is a perfect differential, the right-hand side also must be a perfect differential. To evaluate it, write

$$\sin\phi = \left(\frac{a^2+b^2+1-k^2}{a^2+b^2}\right)^{\frac{1}{2}}\frac{1+k(1+p^2+q^2)^{\frac{1}{2}}}{k+(1+p^2+q^2)^{\frac{1}{2}}};$$

then, after reduction, we find

$$\frac{\lambda' dv}{\{(k^2-1)\lambda^2+2v\}^{\frac{1}{2}}} + \frac{(a'b-ab')\,dm}{(a^2+b^2)(a^2+b^2+1-k^2)^{\frac{1}{2}}} + (1-k^2)^{-\frac{1}{2}}d\phi = 0,$$

where the variables are separated. We thus have ϕ as a function of m and v; and then from the equation for $\sin\phi$, together with the relation between a, b, p, q, we have p and q as functions of m and v, that is, we have x, y, z expressed as functions of m and v, which are the parametric variables of the lines of curvature.

For a more detailed development of this result, reference may be made to the memoir by Serret already quoted* and to the memoir† by Cayley.

Ex. 1. Shew that, when $u=ml$, $\lambda=m$, $a=0$, $\beta=0$, $\gamma=0$, where m is a constant, the surface is developable.

Ex. 2. Shew that, when all the quantities c, l, u, a, β vanish, the surface is one of revolution.

Ex. 3. Shew that, when the relations

$$a=0, \quad c=0, \quad u=ml, \quad \beta=0, \quad \lambda=m,$$

are satisfied, the surface is tubular.

General Equations for Arbitrarily Assigned Curves.

201. In the preceding discussion of surfaces possessing assigned classes of curves as their lines of curvature, there has been a limitation to curves that are plane or spherical; the main reason (other than the comparative simplicity of the curves) for the limitation was that it facilitated the construction of integral results by the method of investigation adopted. It is at least worth while formulating the problem in its most general type, when the assigned lines of curvature are any two families of curves whatever, subject of course to such necessary conditions as are demanded by the equations.

Let the surface be referred to the lines of curvature as parametric curves, so that

$$F=0, \quad M=0;$$

then the Gauss characteristic equation is

$$4LNEG = E(E_2G_2 + G_1^2) + G(E_1G_1 + E_2^2) - 2EG(E_{22} + G_{11}),$$

while the Mainardi-Codazzi relations are

$$L_2 = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)E_2, \quad N_1 = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)G_1.$$

Let s denote the arc along the line of curvature, $p=$ constant, the arc being measured from some director curve; and let t denote a similarly measured arc along the line of curvature, $q=$ constant; then

$$ds = G^{\frac{1}{2}}dq, \quad dt = E^{\frac{1}{2}}dp.$$

* *l.c.*, p. 144.

† *Coll. Math. Papers*, vol. xii, p. 624.

The lines of curvature are to belong to known families of curves, so that (§ 19) it is sufficient to know the circular curvature $1/\rho$ and the torsion $1/\sigma$. Then ρ and σ will be functions of s and, as they may vary from line to line, they may be functions of p also; let

$$\rho = f(s, p), \quad \sigma = h(s, p).$$

Further, at the point in question, let ϖ denote the angle between the normal to the surface and the principal normal to the line of curvature; then (§ 126)

$$\frac{1}{\sigma} = -\frac{d\varpi}{ds}:$$

and therefore, as

$$\frac{ds}{dq} = G^{\frac{1}{2}} \quad \text{.................................(i)},$$

we have

$$-\frac{d\varpi}{dq} = \frac{G^{\frac{1}{2}}}{\sigma} = \frac{G^{\frac{1}{2}}}{h(s, p)} \quad \text{..........................(ii)}.$$

By the known results, giving the curvature of a normal section of the surface through the tangent to $p =$ constant and the geodesic curvature of the line $p =$ constant (§ 127), we have

$$\frac{N}{G} = \frac{\cos\varpi}{\rho} = \frac{\cos\varpi}{f(s, p)} \quad \text{.........................(iii)},$$

$$\frac{G_1}{2GE^{\frac{1}{2}}} = -\frac{\sin\varpi}{f(s, p)} \quad \text{...............................(iv)}.$$

Denoting the circular curvature and the torsion of the other family of lines of curvature by $1/\rho'$ and $1/\sigma'$, we have ρ' and σ' as functions of t and q, say

$$\rho' = g(t, q), \quad \sigma' = k(t, q).$$

Also, at the point in question, we denote by ϖ' the angle between the normal to the surface and the principal normal to the line of curvature; then we have the relations

$$\frac{dt}{dp} = E^{\frac{1}{2}} \quad \text{.................................(v)},$$

$$-\frac{d\varpi'}{dp} = \frac{E^{\frac{1}{2}}}{k(t, q)} \quad \text{............................(vi)},$$

$$\frac{L}{E} = \frac{\cos\varpi'}{g(t, q)} \quad \text{............................(vii)},$$

$$\frac{E_2}{2EG^{\frac{1}{2}}} = -\frac{\sin\varpi'}{g(t, q)} \quad \text{.........................(viii)}.$$

In the last eight equations, there are initially eight unknown quantities, viz. s, ϖ, t, ϖ', E, G, L, N, the independent variables being p and q; the equations

are potentially sufficient for the determination of the magnitudes, and will give expressions that involve arbitrary elements. But the quantities E, G, L, N must satisfy the characteristic equation and the Mainardi-Codazzi relations; and so there will be conditions to be satisfied not merely by the arbitrary elements, but also by the quantities f, h, g, k, ϖ, ϖ'. In the simplest instance, when the two families of lines of curvature are plane or spherical, we saw that the parameters and other magnitudes connected with the lines are certainly subject to one relation and so cannot all be taken arbitrarily; the relation is additional to the limitation that the curves are plane or spherical. *A fortiori* it is to be expected that, when curves are assigned initially without the specialising limitation, they will have to satisfy some condition or conditions, in order that they may provide the two systems of lines of curvature for a surface.

Ex. 1. To illustrate the analysis, let it be required to determine surfaces having circles for both sets of lines of curvature. Then

$$\rho=f(p), \quad \sigma^{-1}=0,$$

$$\rho'=g(q), \quad \sigma'^{-1}=0;$$

hence

$$\frac{d\varpi}{dq}=0, \quad \frac{d\varpi'}{dp}=0.$$

Consequently

$$\varpi=P, \quad \varpi'=Q,$$

where P is a function of p only that may be constant or zero, and Q is a function of q only that may be constant or zero; these two results are, of course, Joachimsthal's theorem on plane lines of curvature.

Suppose that neither P nor Q vanishes in general. The special equations (other than the intrinsic relations for all surfaces) are

$$\frac{N}{G}=\frac{\cos P}{f(p)} \qquad \text{(iii)}',$$

$$\frac{G_1}{2GE^{\frac{1}{2}}}=-\frac{\sin P}{f(p)} \qquad \text{(iv)}',$$

$$\frac{L}{E}=\frac{\cos Q}{g(q)} \qquad \text{(vii)}',$$

$$\frac{E_2}{2EG^{\frac{1}{2}}}=-\frac{\sin Q}{g(q)} \qquad \text{(viii)}',$$

potentially sufficient to determine L, N, E, G. The equations (iv)$'$ and (viii)$'$ suffice to determine E and G; their primitive is

$$E^{-\frac{1}{2}}H'=(H+K)\frac{\sin P}{f(p)},$$

$$G^{-\frac{1}{2}}K'=(H+K)\frac{\sin Q}{g(q)},$$

where H is an arbitrary function of p and H' is its derivative, while K is an arbitrary function of q and K' is its derivative. Then L is determined by (vii)$'$ and N by (iii)$'$.

These quantities are to satisfy the intrinsic equations common to all surfaces. Substituting in the Mainardi-Codazzi relation

$$L_2=\frac{1}{2}\left(\frac{L}{E}+\frac{N}{G}\right)E_2,$$

we have

$$\frac{\cos P}{f(p)}-\frac{\cos Q}{g(q)}=-\frac{H+K}{K'}\frac{d}{dq}\left\{\frac{\cos Q}{g(q)}\right\}.$$

As H is a function of p only and K of q only, while p and q are independent variables, we must have

$$-\frac{1}{K'}\frac{d}{dq}\left\{\frac{\cos Q}{g(q)}\right\}=\text{constant}=a^{-1},$$

; and the foregoing equation then gives

$$H-a\frac{\cos P}{f(p)}=-\left\{K+a\frac{\cos Q}{g(q)}\right\}.$$

Each side of this equation must be a constant, say $-\beta$; then

$$H=\quad a\frac{\cos P}{f(p)}-\beta,$$

$$K=-a\frac{\cos Q}{g(q)}+\beta,$$

which satisfy all these equations. The same result follows from substituting in the other Mainardi-Codazzi relation.

There remains the Gauss characteristic equation. Writing

$$\frac{\cos P}{f(p)}=P_1,\qquad \frac{\cos Q}{g(q)}=Q_1,$$

$$\frac{\sin P}{f(p)}=P_2,\qquad \frac{\sin Q}{g(q)}=Q_2,$$

so that

$$E^{-\frac{1}{2}}P_1'=(P_1-Q_1)P_2,\quad -G^{-\frac{1}{2}}Q_1'=(P_1-Q_1)Q_2,$$

and substituting in the Gauss equation, we have

$$P_1Q_1+P_1Q_2\frac{Q_2'}{Q_1'}+Q_1P_2\frac{P_2'}{P_1'}=A+B,$$

where

$$A=P_1P_2\frac{P_2'}{P_1'}-P_2^2,\qquad B=Q_1Q_2\frac{Q_2'}{Q_1'}-Q_2^2,$$

so that A is a function of p only, and B is a function of q only. Hence

$$P_1'\left(Q_1+Q_2\frac{Q_2'}{Q_1'}\right)+Q_1\frac{d}{dp}\left(P_2\frac{P_2'}{P_1'}\right)=A',$$

$$P_1\frac{d}{dq}\left(Q_1+Q_2\frac{Q_2'}{Q_1'}\right)+Q_1'P_2\frac{P_2'}{P_1'}=B',$$

$$P_1'\frac{d}{dq}\left(Q_1+Q_2\frac{Q_2'}{Q_1'}\right)+Q_1'\frac{d}{dp}\left(P_2\frac{P_2'}{P_1'}\right)=0.$$

From the last of these derived relations, we have

$$\frac{d}{dq}\left(Q_1+Q_2\frac{Q_2'}{Q_1'}\right)=a'Q_1',\qquad \frac{d}{dp}\left(P_2\frac{P_2'}{P_1'}\right)=-a'P_1',$$

where a' is a constant; and therefore

$$Q_1 + Q_2 \frac{Q_2'}{Q_1'} = a'Q_1 + b', \qquad P_2 \frac{P_2'}{P_1'} = -a'P_1 + c',$$

where b' and c' are constants. Consequently

$$Q_1^2 + Q_2^2 = a'Q_1^2 + 2b'Q_1 + m,$$
$$P_2^2 = -a'P_1^2 + 2c'P_1 + m'.$$

Instead of using the other derived relations, we substitute these values in the modified form of the Gauss equation; it becomes

$$b'P_1 + c'Q_1 = -c'P_1 - b'Q_1 - (m + m'),$$

so that

$$b' = -c', \quad m + m' = 0.$$

Accordingly, all the necessary equations are completely satisfied by the set of values

$$\left.\begin{aligned} Q_1^2 + Q_2^2 &= a'Q_1^2 - 2c'Q_1 + m \\ P_2^2 &= -a'P_1^2 + 2c'P_1 - m \\ H &= \alpha P_1 - \beta \\ K &= -\alpha Q_1 + \beta \end{aligned}\right\}.$$

These values are required for the expressions of E, G, L, N. When these are formed, α and β disappear; so that, in the forms obtained, three arbitrary constants a', c', m appear.

The expressions can be simplified by changing the independent variables. Let new variables ϕ and θ be introduced, defined by the equations

$$\frac{1}{P_1} = \mu - a\cos\phi, \qquad \frac{1}{Q_1} = \mu' - c\cos\theta;$$

then

$$\frac{P_2}{P_1} = \frac{a}{ib}\sin\phi, \qquad \frac{Q_2}{Q_1} = \frac{c}{b}\sin\theta,$$

provided

$$m = -\frac{1}{b^2}, \quad c' = m\mu = m\mu', \quad a' = m(\mu^2 - a^2) = 1 + m(\mu'^2 - c^2).$$

The last conditions are satisfied, provided

$$\mu = \mu', \quad c^2 = a^2 - b^2;$$

and thus, instead of the three constants a', c', μ, we have four constants a, b, c, μ, tied by the relation

$$c^2 = a^2 - b^2.$$

Let $\overline{E}$ and $\overline{G}$, $\overline{L}$ and $\overline{N}$, be the fundamental quantities when ϕ and θ are made the independent variables, instead of p and q; then

$$E = \overline{E}\left(\frac{d\phi}{dp}\right)^2, \qquad G = \overline{G}\left(\frac{d\theta}{dq}\right)^2,$$

and so for the others. After simple reduction, we find

$$\overline{E} = -b^2 \frac{(\mu - c\cos\theta)^2}{(a\cos\phi - c\cos\theta)^2},$$

$$\overline{G} = \frac{b^2(\mu - a\cos\phi)^2}{(a\cos\phi - c\cos\theta)^2},$$

$$\frac{\overline{L}}{\overline{E}} = \frac{L}{E} = \frac{1}{\mu - c\cos\theta},$$

$$\frac{\overline{N}}{\overline{G}} = \frac{N}{G} = \frac{1}{\mu - a\cos\phi}.$$

These (together with $F=0$, $M=0$) are the fundamental magnitudes, of the first order and the second order, belonging to a Dupin cyclide. Hence, by Bonnet's theorem (§ 37), the surface is a Dupin cyclide.

Ex. 2. In the preceding analysis, it has been assumed that both P and Q are generally different from zero.

Shew that, when one of the two quantities vanishes, say $P=0$, the surface is an anchor-ring, given by the intrinsic equations

$$\left.\begin{array}{lll} E=(c+\cos q)^2, & F=0, & G=1 \\ L=c+\cos q, & M=0, & N=1 \end{array}\right\}.$$

Ex. 3. Shew that when both P and Q vanish, the surface is a developable surface*.

202. On the main line of development, especially when one of the families of lines of curvature is plane, much simplification comes when $1/\sigma$ is zero, so that ϖ is constant and may be zero. The most direct illustration arises in connection with tubular surfaces; that is, surfaces which are the envelopes of spheres, having their centres on any given curve in space, and having their radii any assigned continuous function of the arc.

Reference in general to surfaces, having one (but only one) system of plane lines of curvature, may be made to Darboux† and to Bianchi‡.

Weingarten Surfaces.

203. One or two incidental references to Weingarten surfaces have already been made; and some special examples have arisen, particularly surfaces having a constant Gaussian measure of curvature and surfaces having a constant mean curvature (including minimal surfaces). We proceed to obtain some properties of these surfaces in general, defining them as *surfaces whose principal curvatures are connected by some functional relation*§

$$F(\alpha, \beta)=0.$$

We refer the surface to its lines of curvature, so that

$$F=0, \quad M=0.$$

The Mainardi-Codazzi relations are

$$L_2=\frac{1}{2}\left(\frac{L}{E}+\frac{N}{G}\right)E_2, \quad N_1=\frac{1}{2}\left(\frac{L}{E}+\frac{N}{G}\right)G_1.$$

* As regards these results, a note by the author, *Messenger of Math.*, vol. xxxviii (1909), pp. 33—44, may be consulted.

† See his treatise, t. iv, pp. 198—266.

‡ *Geometria Differenziale*, t. ii, chap. xxi.

§ They were first discussed in general by Weingarten, *Crelle*, t. lxii (1862), pp. 160—173.

Now

$$\alpha=\frac{G}{N},\quad \beta=\frac{E}{L};$$

hence

$$-\frac{\beta_2}{\beta^2}=\frac{E_2}{2E}\left(\frac{L}{E}+\frac{N}{G}\right)-\frac{L}{E^2}E_2$$

$$=-\frac{E_2}{2E}\left(\frac{L}{E}-\frac{N}{G}\right)$$

$$=-\frac{E_2}{2E}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right),$$

and therefore

$$\frac{E_2}{2E}=\frac{\dfrac{\alpha}{\beta}}{\alpha-\beta}\beta_2,$$

so that

$$E=Pe^{2\int\frac{\alpha}{\beta}\frac{d\beta}{\alpha-\beta}},$$

where P is a function of p only. Taking a new variable p' such that $dp'=P^{\frac{1}{2}}dp$, we effectively make P equal to unity; so we can, without loss of generality, write

$$E=e^{2\int\frac{\alpha}{\beta}\frac{d\beta}{\alpha-\beta}}.$$

Similarly, we can take

$$G=e^{2\int\frac{\beta}{\alpha}\frac{d\alpha}{\beta-\alpha}}.$$

Also

$$L=\frac{E}{\beta},\quad N=\frac{G}{\alpha};$$

thus the fundamental magnitudes of the surface are known, because the integrals in E and G are complete through the relation $F(\alpha,\beta)=0$.

Ex. 1. In the case of a minimal surface, we have

$$\alpha+\beta=0,$$

so that

$$E=\beta,\quad G=\alpha,\quad L=1,\quad N=1.$$

The lines of curvature are the parametric curves; the asymptotic lines are given by

$$dp^2+dq^2=0.$$

Ex. 2. In the case of a surface having its Gaussian measure of curvature equal to a constant, we have

$$\alpha\beta=c,$$

where c is constant; then

$$E=\frac{\beta}{\alpha-\beta},\qquad G=\frac{\alpha}{\beta-\alpha},$$

$$L=\frac{1}{\alpha-\beta},\qquad N=\frac{1}{\beta-\alpha},$$

so that the fundamental magnitudes are known in terms of α and β. As before, the lines of curvature are the parametric curves; the asymptotic lines are given by

$$dp^2-dq^2=0.$$

Ex. 3. In the case of a surface having its mean measure of curvature constant (but not zero), so that

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{1}{c},$$

we have

$$E=\frac{\alpha+\beta}{\alpha-\beta}, \qquad G=\frac{\beta+\alpha}{\beta-\alpha}.$$

As these satisfy the condition that

$$\frac{\partial^2}{\partial p \partial q}\left(\log\frac{E}{G}\right)=0,$$

the lines of curvature form an isometric system*—a known property of surfaces having a constant mean measure of curvature (§ 64).

204. Returning to the general equations for the Weingarten surface, we can express the fundamental magnitudes in terms of a single parameter. Let

$$e^{\int\frac{d\alpha}{\beta-\alpha}}=\frac{1}{\theta},$$

so that

$$\frac{d\alpha}{\beta-\alpha}=-\frac{d\theta}{\theta},$$

and therefore

$$\beta=\alpha-\theta\frac{d\alpha}{d\theta}.$$

Now we easily find

$$E=\beta^2 e^{2\int\frac{d\beta}{\alpha-\beta}}, \quad G=\alpha^2 e^{2\int\frac{d\alpha}{\beta-\alpha}},$$

so that

$$EG=\frac{\alpha^2\beta^2}{(\alpha-\beta)^2};$$

hence

$$G=\frac{\alpha^2}{\theta^2},$$

$$E=\frac{\beta^2}{\left(\frac{d\alpha}{d\theta}\right)^2},$$

$$L=\frac{E}{\beta}=\frac{\alpha-\theta\frac{d\alpha}{d\theta}}{\left(\frac{d\alpha}{d\theta}\right)^2},$$

$$N=\frac{G}{\alpha}=\frac{\alpha}{\theta^2}.$$

Moreover, we have

$$F(\alpha, \beta)=0$$

* For Weingarten surfaces in general, on which the lines of curvature are an isometric system, see a memoir by Demartres, *Ann. de Toulouse*, 2^{me} Sér., t. iv (1902), p. 341. The complete solution of the problem requires the integration of an ordinary non-linear equation of the third order.

in general, say

$$\beta = f(\alpha);$$

then we have

$$\alpha - \theta \frac{d\alpha}{d\theta} = f(\alpha),$$

that is,

$$\frac{d\theta}{\theta} + \frac{d\alpha}{f(\alpha) - \alpha} = 0.$$

The arc-element of the surface is

$$ds^2 = \frac{\beta^2}{\left(\frac{d\alpha}{d\theta}\right)^2} dp^2 + \frac{\alpha^2}{\theta^2} dq^2,$$

where θ is a function of p and q.

But the resolution of the equation $F(\alpha, \beta) = 0$ in the foregoing form $\beta = f(\alpha)$ may not be possible, even when F is a rational function. In that case, the simplest instances would lead to Abelian integrals and algebraic functions in the expressions for E and G; some of them might be worth investigation.

Ex. 1. Let $a = \frac{1}{2}\theta^2$; then

$$\beta = f(a) = a - \theta \frac{da}{d\theta} = -\tfrac{1}{2}\theta^2,$$

so that

$$a + \beta = 0;$$

the surface is minimal. We have

$$E = \tfrac{1}{4}\theta^2, \quad G = \tfrac{1}{4}\theta^2,$$

so that the element of arc is given by the equation

$$ds^2 = \tfrac{1}{4}\theta^2 (dp^2 + dq^2),$$

the lines of curvature being an isometric system. When we substitute in the Gaussian characteristic equation, we find

$$\frac{\partial^2}{\partial p^2}(\log \theta) + \frac{\partial^2}{\partial q^2}(\log \theta) = \frac{1}{\theta^2};$$

or, with the transformation $\theta = e^u$, we have the equation

$$u_{11} + u_{22} = e^{-2u}.$$

Consider the surface of centres of the surface. The arc-elements on the two sheets are (§ 81) given by the relations

$$d\sigma^2 = da^2 + E\left(1 - \frac{a}{\beta}\right)^2 dp^2,$$

$$d\sigma'^2 = d\beta^2 + G\left(1 - \frac{\beta}{a}\right)^2 dq^2,$$

in general for any surface; in the present case, we have

$$d\sigma^2 = \theta^2 (d\theta^2 + dp^2),$$

$$d\sigma'^2 = \theta^2 (d\theta^2 + dq^2),$$

so that, on one sheet, the lines $\theta=$constant and $p=$constant (that is, the lines $a=$constant and $p=$constant), and, on the other sheet, the lines $\theta=$constant and $q=$constant (that is, the lines $\beta=$constant and $q=$constant), are isometric orthogonal systems on the surface of centres.

Ex. 2. Let

$$\theta=\sin\tfrac{1}{2}\omega, \qquad \frac{da}{d\theta}=\cos\tfrac{1}{2}\omega,$$

so that

$$\left(\frac{da}{d\theta}\right)^2=1-\theta^2.$$

Then

$$da=\tfrac{1}{2}\cos^2\tfrac{1}{2}\omega\,d\omega=\tfrac{1}{4}(1+\cos\omega)\,d\omega,$$

and therefore

$$a=\tfrac{1}{4}(\omega+\sin\omega).$$

Similarly

$$\beta=\tfrac{1}{4}(\omega-\sin\omega).$$

Hence the functional equation of the Weingarten surface is

$$2(a-\beta)=\sin 2(a+\beta);$$

and the arc-elements on the sheets of its centro-surface are

$$d\sigma^2=\cos^4\tfrac{1}{2}\omega\,(\tfrac{1}{2}d\omega)^2+\sin^4\tfrac{1}{2}\omega\,(dp)^2,$$

$$d\sigma'^2=\sin^4\tfrac{1}{2}\omega\,(\tfrac{1}{2}d\omega)^2+\cos^4\tfrac{1}{2}\omega\,(dq)^2.$$

Ex. 3. Obtain expressions for the principal radii of curvature in terms of a single parameter for each of the surfaces

$$\frac{1}{a}+\frac{1}{\beta}=\frac{2}{c};$$

$$a\beta=k^2.$$

205. It follows from the functional equation $F(\alpha,\beta)=0$, defining a Weingarten surface, that

$$\alpha_1\beta_2-\alpha_2\beta_1=0.$$

As regards its two-sheeted surface of centres, the Gaussian measure of curvature for one of them is (§ 81)

$$K=-\frac{1}{(\alpha-\beta)^2}\frac{\beta_2}{\alpha_2},$$

and for the other of them is

$$K'=-\frac{1}{(\alpha-\beta)^2}\frac{\alpha_1}{\beta_1};$$

hence for the Weingarten surface the product of the specific curvatures for the sheets of its centro-surface is such that

$$KK'=\frac{1}{(\alpha-\beta)^4}.$$

Again, assuming that the original surface is referred to its lines of

curvature as parametric curves, we have the asymptotic lines on the first sheet given by the equation (§ 81)

$$E\frac{\beta_2}{\beta^2}dp^2 - G\frac{\alpha_2}{\alpha^2}dq^2 = 0,$$

and the asymptotic lines on the second sheet given by the equation

$$-E\frac{\beta_1}{\beta^2}dp^2 + G\frac{\alpha_1}{\alpha^2}dq^2 = 0.$$

When the original surface is any Weingarten surface, these two equations are the same; hence the asymptotic lines on the two sheets of the centro-surface of a Weingarten surface correspond to one another—a result due to Ribaucour (§ 83).

The asymptotic lines on the original surface are given by

$$Ldp^2 + Ndq^2 = 0,$$

that is,

$$\frac{E}{\beta}dp^2 + \frac{G}{\alpha}dq^2 = 0.$$

If the asymptotic lines on the centro-surface correspond to the asymptotic lines on the original surface, their equations must substantially be the same; that is,

$$\frac{\beta_2}{\beta} = -\frac{\alpha_2}{\alpha}, \quad \frac{\beta_1}{\beta} = -\frac{\alpha_1}{\alpha},$$

and therefore

$$\alpha\beta = \text{constant}.$$

Hence the only Weingarten surfaces, such that the asymptotic lines on the centro-surfaces correspond to the asymptotic lines on the original surfaces, are those which have a constant Gaussian measure of curvature.

As another result, also due to Ribaucour (§ 82), we have the theorem that the only surface, such that the lines of curvature on the centro-surface correspond to one another, is a Weingarten surface such that

$$\alpha - \beta = \text{constant};$$

but these lines of curvature are easily seen, from the analysis used (*l. c.*) in establishing the result, not to correspond to the lines of curvature upon the original surface.

206. Consider any elementary arc on either sheet—say the first sheet—of the centro-surface of a Weingarten surface. In general, (§ 81), it is given by

$$d\sigma^2 = E\left(1 - \frac{\alpha}{\beta}\right)^2 dp^2 + d\alpha^2;$$

and therefore, for the Weingarten surfaces $F(\alpha, \beta) = 0$, we have

$$d\sigma^2 = d\alpha^2 + f(\alpha)\, dp^2,$$

where $f(\alpha)$ is a definite function of α, the form of which depends upon the form of F. Also

$$f(\alpha)=\left(1-\frac{\alpha}{\beta}\right)^2 E$$

$$=(\alpha-\beta)^2 e^{2\int\frac{d\beta}{\alpha-\beta}}$$

$$=e^{2\int\frac{d\alpha}{\alpha-\beta}},$$

any additive constant being absorbed into the integral; so that the element of arc on the first sheet of the centro-surface is

$$d\sigma^2=d\alpha^2+e^{2\int\frac{d\alpha}{\alpha-\beta}}dp^2=d\alpha^2+f(\alpha)\,dp^2.$$

Similarly, the element of arc on the second sheet of the centro-surface is

$$d\sigma'^2=d\beta^2+e^{2\int\frac{d\beta}{\beta-\alpha}}dq^2=d\beta^2+g(\beta)\,dq^2.$$

Both these arc-elements are characteristic of a surface of revolution; hence *either sheet of the centro-surface of a Weingarten surface is deformable into a surface of revolution*, a theorem first enunciated* by Weingarten himself. To consider in slight detail the surface of revolution into which we can deform the sheet having

$$d\sigma^2=d\alpha^2+f(\alpha)\,dp^2$$

for its arc-element, let the equation be

$$z=P(r)=P,$$

where the axis of z is the axis of revolution, and r is the distance of a point on the surface from the axis. Its arc-element is

$$ds^2=(1+P'^2)\,dr^2+r^2d\phi^2,$$

and therefore, under deformation, we have

$$(1+P'^2)^{\frac{1}{2}}dr=d\alpha,$$

$$d\phi=dp,$$

$$r^2=f(\alpha).$$

From the last, we have

$$dr=\tfrac{1}{2}f^{-\frac{1}{2}}f'\,d\alpha,$$

and therefore

$$1+P'^2=4\frac{f}{f'^2}.$$

Consequently

$$P=\int P'dr=\int\left(1-\frac{f'^2}{4f}\right)^{\frac{1}{2}}d\alpha;$$

* *Crelle*, t. lix (1861), p. 387.

an equation which, in conjunction with

$$r^2 = f(\alpha),$$

determines the surface of revolution.

Thus, for the minimal surface $\alpha + \beta = 0$, the surface of revolution is given by

$$9z^2 = (x^2 + y^2 - 1)^3;$$

and, for the surface having its Gaussian measure of curvature constant, the surface of revolution is a catenoid.

Ex. Obtain the surface of revolution, to which the centro-surface of the Weingarten surface

$$\alpha + c\beta = k,$$

can be deformed; discussing, in particular, the Ribaucour surface for $c = -1$.

207. A sort of converse of the preceding theorem, also due* to Weingarten, can be enunciated as follows:—

Any surface, that is deformable into a surface of revolution, can be regarded in general as a centro-surface of a Weingarten surface.

The theorem is little more than an interpretation of a different arrangement of the preceding analysis. When the surface of revolution is given, we have

$$d\alpha = (1 + P'^2)^{\frac{1}{2}}\, dr,$$

for the purposes of the theorem; so that, as P' is known, we can regard α as a known function of r, determined by the relation

$$\alpha = \int (1 + P'^2)^{\frac{1}{2}}\, dr.$$

Also

$$r^2 = f(\alpha) = e^{2\int \frac{d\alpha}{\alpha - \beta}};$$

hence

$$\frac{d\alpha}{\alpha - \beta} = \frac{dr}{r},$$

and therefore

$$\alpha - \beta = r(1 + P'^2)^{\frac{1}{2}};$$

that is,

$$\begin{aligned} \beta &= -r(1 + P'^2)^{\frac{1}{2}} + \int (1 + P'^2)^{\frac{1}{2}}\, dr \\ &= -\int rP'P''(1 + P'^2)^{-\frac{1}{2}}\, dr. \end{aligned}$$

Thus α and β are functions of r alone; the surface is a Weingarten surface.

* *Crelle*, t. lxii (1863), p. 160.

208. We must note Lie's theorem* that the lines of curvature on any Weingarten surface can be obtained by quadratures.

The equation of the lines of curvature on any surface can be taken in the form

$$VW = \begin{vmatrix} Edp + Fdq, & Fdp + Gdq \\ Ldp + Mdq, & Mdp + Ndq \end{vmatrix} = 0.$$

We have seen (§ 133) that W is an absolute covariant for all changes of the independent variables; hence, taking the parameters (say u and v) of the lines of curvature as the independent variables, so that

$$F' = 0, \quad M' = 0, \quad L' = \frac{E'}{\beta}, \quad N' = \frac{G'}{\alpha}, \quad V'^2 = E'G',$$

we have

$$W = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) V' du dv$$

$$= \phi du dv,$$

say. Now

$$V'^2 = E'G' = e^{2\int \frac{\alpha}{\beta} \frac{d\beta}{\alpha - \beta} + 2\int \frac{\beta}{\alpha} \frac{d\alpha}{\beta - \alpha}};$$

hence

$$\frac{1}{\phi}\frac{\partial \phi}{\partial u} = \frac{\beta_1 - \alpha_1}{\beta - \alpha} - \frac{\alpha_1}{\alpha} - \frac{\beta_1}{\beta} + \frac{\alpha}{\beta}\frac{\beta_1}{\alpha - \beta} + \frac{\beta}{\alpha}\frac{\alpha_1}{\beta - \alpha}$$

$$= 0,$$

and similarly

$$\frac{1}{\phi}\frac{\partial \phi}{\partial v} = 0.$$

Hence

$$\phi = \text{constant} = 1,$$

say, so that

$$W = du dv.$$

Returning to the original form for W, we have

$$W = \frac{1}{V}\{(EM - FL)\, dp^2 + (EN - GL)\, dp dq + (FN - GM)\, dq^2\}$$

$$= \frac{1}{V}(EM - FL)(dp + \rho dq)(dp + \rho' dq)$$

$$= S(dp + \rho dq)(dp + \rho' dq),$$

where ρ, ρ', S are known quantities. Comparing the two expressions for W, we take

$$du = R(dp + \rho dq), \quad dv = R'(dp + \rho' dq),$$

where

$$RR' = S.$$

* *Darb. Bull.*, 2me Sér., t. iv (1880), p. 300.

As the expressions for du and dv must be perfect differentials, we must have

$$\frac{\partial R}{\partial q}=\rho\frac{\partial R}{\partial p}+R\frac{\partial \rho}{\partial p},\quad \frac{\partial R'}{\partial q}=\rho'\frac{\partial R'}{\partial p}+R'\frac{\partial \rho'}{\partial p}.$$

Hence

$$\begin{aligned}\frac{\partial S}{\partial q}&=R'\frac{\partial R}{\partial q}+R\frac{\partial R'}{\partial q}\\&=\rho R'\frac{\partial R}{\partial p}+\rho' R\frac{\partial R'}{\partial p}+S\left(\frac{\partial \rho}{\partial p}+\frac{\partial \rho'}{\partial p}\right),\end{aligned}$$

that is,

$$\frac{\rho}{R}\frac{\partial R}{\partial p}+\frac{\rho'}{R'}\frac{\partial R'}{\partial p}=\frac{1}{S}\frac{\partial S}{\partial q}-\left(\frac{\partial \rho}{\partial p}+\frac{\partial \rho'}{\partial p}\right).$$

Also

$$\frac{1}{R}\frac{\partial R}{\partial p}+\frac{1}{R'}\frac{\partial R'}{\partial p}=\frac{1}{S}\frac{\partial S}{\partial p};$$

hence $\frac{1}{R}\frac{\partial R}{\partial p}$ and $\frac{1}{R'}\frac{\partial R'}{\partial p}$ are expressible in terms of known quantities. The earlier equations then give $\frac{1}{R}\frac{\partial R}{\partial q}$ and $\frac{1}{R'}\frac{\partial R'}{\partial q}$ in terms of known quantities, so that R and R' are determinable by quadratures. When their values are known, and are substituted in du and dv, then u and v are determinable by quadratures—which is Lie's theorem.

EXAMPLES.

1. A surface has both its systems of lines of curvature plane or spherical. At any point the plane (or sphere) of one system cuts the surface at an angle ω_1, and the plane (or sphere) of the other system cuts the surface at an angle ω_2, while the one plane (or sphere) cuts the other plane (or sphere) at an angle ω_{12}; prove that

$$\cos\omega_{12}=\cos\omega_1\cos\omega_2.$$

2. Shew that the spherical image of a Weingarten surface is given by the equation

$$dS^2=\frac{dp^2}{\gamma^2}+\frac{dq^2}{\phi'^2(\gamma)},$$

where the principal radii of curvature are connected by the relations

$$\beta=\phi(\gamma),\quad \alpha=\phi(\gamma)-\gamma\phi'(\gamma).$$

3. Shew that the asymptotic lines of the centro-surface of a Weingarten surface correspond to nul lines in the spherical representation when

$$\alpha+\beta=\text{constant};$$

and that they correspond to nul lines on the surface when

$$\frac{1}{\alpha}+\frac{1}{\beta}=\text{constant}.$$

4. Shew that, for a Weingarten surface

$$\alpha-\beta=c,$$

the geodesic curvature of the parametric curve p=constant on the first sheet of its centro-surface and the geodesic curvature of the parametric curve q=constant on the second sheet of its centro-surface are equal to one another, the common value being $1/c$.

5. The arc-element on a class of surfaces, referred to lines of curvature as parametric curves, is such that

$$G^{\frac{1}{2}}=\phi'(E^{\frac{1}{2}}),$$

and the Gauss measure of curvature is given by

$$\phi(E^{\frac{1}{2}})\{\phi(E^{\frac{1}{2}})-E^{\frac{1}{2}}G^{\frac{1}{2}}\};$$

prove that the surfaces are of the Weingarten type.

6. Shew that, if a minimal surface has plane curves for one system of lines of curvature, its other system of lines of curvature also is plane. Illustrate the result in connection with Enneper's minimal surface by shewing that every line of curvature is a plane cubic; and verify that its asymptotic lines are skew cubics.

7. A surface has one system of its lines of curvature spherical; prove that the radius of circular curvature of the lines is

$$P+Qg^{-\frac{1}{2}}e_2,$$

where P and Q are functions of the parameter q of the lines, while e and g belong to the spherical image of the surface which is supposed referred to its lines of curvature as parametric curves.

CHAPTER X.

DEFORMATION OF SURFACES.

THE problem of the deformation of surfaces has attracted many investigators. In its most general form, the discussion was initiated in the famous memoir by Gauss, so often quoted; and other investigations of the utmost importance for surfaces in general are due to Bour, Bonnet, Darboux, and Weingarten, among others. References to the original authorities will be found in Darboux's treatise, vol. iii, Book vii, chapters ii—iv. Reference may also be made to chapter vii in vol. i of the treatise by Bianchi; and important analytical developments are contained in the third volume of this treatise, published in 1909.

Among surfaces that are subjected to deformation, special interest attaches to the class of ruled surfaces. This problem seems first to have been considered by Minding*; among later writers, special mention should be made of Bonnet† and Beltrami‡. References are given by Darboux§ and by Bianchi||.

Among the deformations of surfaces which have secured much attention from mathematicians, there is one considerable class, viz. infinitesimal deformations, important alike on account of their intrinsic interest and the variety of results (as well as of methods) connected with them. Moreover, while some of the results are of long standing, much of the body of known doctrine belongs to more recent years. Full references are given by Darboux¶ and by Bianchi**, both of whom have made important contributions to the subject; and no lack of appreciation of the work of Beltrami, Guichard, Kœnigs and Ribaucour is implied, because a special reference is made solely to two memoirs†† by Weingarten. Only the briefest account of the elements of this interesting part of the theory will be given in this chapter; the authorities just quoted should be consulted for fuller discussion.

* *Crelle*, t. xviii (1838), pp. 297—302.
† *Journ. de l'Éc. Poly.*, cah. xlii (1867), pp. 1—151.
‡ *Ann. di Mat.*, t. vii (1865), pp. 105—138.
§ In the chapter occupying pp. 293—315 of vol. iii of his treatise.
|| See his treatise, ch. viii.
¶ *Treatise*, practically throughout, vol. iv.
** *Geometria Differenziale*, vol. ii, pp. 1 *et seq.*, 172 *et seq.*
†† *Crelle*, t. c (1887), pp. 296—310; *Acta Math.*, t. xx (1897), pp. 159—200.

Deformation and Applicability; Initial Conditions.

209. Let a surface be supposed flexible; and let it be changed in any manner and within any limits, provided there is neither stretching nor tearing. Any such change is called a *deformation* of the surface.

Again, let two surfaces be given such that either of them can suffer a deformation into the other, or (what is the same thing) that both of them can suffer deformations into some one and the same third surface. The two given surfaces are said to be *applicable* to one another. The mathematical theory of the applicability of two surfaces is usually (but not always) the same as that of the deformation of one surface; so it will usually be convenient to speak of the latter alone.

Imagine any curve or curves drawn upon a surface; and let the surface be deformed. As there is no stretching or tearing, the distance between any two points on a curve remains unaltered when the distance is measured along the curve; in particular, any infinitesimal arc at a point remains unaltered. The angle between two curves through a point remains unaltered —a property secured by having every infinitesimal arc at a point conserved in length throughout a deformation. If, then, an arc-element on a surface is given by

$$ds^2 = E dp^2 + 2F dp dq + G dq^2,$$

and the deformed arc-element on the surface, however deformed, is given by

$$ds'^2 = E' dp'^2 + 2F' dp' dq' + G' dq'^2,$$

the necessary and sufficient condition is that, for all variations of the parameters, we must have

$$ds = ds'.$$

This comprehensive condition must be translated into more amenable forms.

Manifestly, geodesics (being the shortest distance on a surface between two points) remain geodesics through any deformation; and similarly, the geodesic curvature of any curve (being the arc-rate of deviation of the curve from a geodesic tangent) remains unaltered. In fact, any absolute covariant, which involves only the equation of a curve or curves and the fundamental magnitudes of the first order (being those which occur in the arc-element), remains unaltered in a deformation. In particular, the Gaussian measure of curvature of a surface is expressible in terms of E, F, G and of their derivatives of the first and second orders; accordingly, it remains unaltered during all deformations. The latter property is a condition necessary to secure that one surface can be deformed into another; it is not, however, a generally sufficient condition. But it is sufficient to prevent a sphere or

a hyperboloid of one sheet from being deformed into a developable surface; sufficient, also, to prevent an ellipsoid or a spheroid from being deformed into a sphere. Even a surface of constant curvature cannot be deformed into another surface of constant curvature, unless the constants are the same.

210. As an example of the deformation of one surface into another, tested by the invariability of the arc-element, consider the catenoid of revolution

$$r = (x^2 + y^2)^{\frac{1}{2}} = c \cosh \frac{z}{c}.$$

Taking

$$x = r \cos \phi, \quad y = r \sin \phi,$$

we have

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 + dz^2 \\ &= dr^2 + r^2 d\phi^2 + \frac{c^2}{r^2 - c^2} dr^2 \\ &= \frac{r^2}{r^2 - c^2} dr^2 + r^2 d\phi^2. \end{aligned}$$

Let

$$p = (r^2 - c^2)^{\frac{1}{2}};$$

then

$$ds^2 = dp^2 + (p^2 + c^2)\, d\phi^2.$$

Now consider the helicoid

$$\frac{y}{x} = \tan \frac{z}{a}.$$

Taking

$$y = u \sin v, \quad x = u \cos v, \quad z = av,$$

we have

$$ds'^2 = du^2 + (u^2 + a^2)\, dv^2.$$

Manifestly the arc-elements are the same, for all variations, if

$$u = p, \quad v = \phi, \quad a = c;$$

in other words, the catenoid

$$r = c \cosh \frac{z}{c}$$

can be deformed into the helicoid

$$\frac{y}{x} = \tan \frac{z}{c}.$$

We leave, as an exercise, the verification of the property that the Gaussian measure of curvature is the same for the two surfaces at corresponding points.

211. Owing to a specially individual property of surfaces of constant Gaussian measure of curvature—that such a surface is applicable upon itself in an infinite variety of ways—, we shall discuss them first of all.

Take such a surface; and choose, for its parametric curves, a family of concurrent geodesics and the family of their geodesic parallels. Then the arc-element of the surface is given by

$$ds^2 = dp^2 + D^2 dq^2,$$

where the curves $q =$ constant are the geodesics, and the quantity p is the distance along the geodesic measured from any selected directrix geodesic parallel. Then, if K be the Gaussian measure of curvature, we have (§ 68)

$$K = -\frac{1}{D}\frac{\partial^2 D}{\partial p^2};$$

moreover, when $p = 0$, the general conditions (§ 68) require the limitations

$$\frac{\partial D}{\partial p} = 1, \quad D = 0,$$

for any non-singular part of the surface. When K is constant, there are three typical cases according as K is zero, positive, or negative.

When $K = 0$, we must have

$$\frac{\partial^2 D}{\partial p^2} = 0,$$

so that

$$D = p\phi(q) + \psi(q).$$

The conditions, when $p = 0$, require that $\phi(q) = 1$, $\psi(q) = 0$; thus

$$D = p.$$

Hence the arc-element is given by

$$ds^2 = dp^2 + p^2 dq^2.$$

When K is positive, let its value be $1/a^2$; then

$$\frac{1}{D}\frac{\partial^2 D}{\partial p^2} = -\frac{1}{a^2},$$

so that

$$D = \phi(q)\cos\frac{p}{a} + \psi(q)\sin\frac{p}{a}.$$

The conditions, when $p = 0$, require that

$$\phi(q) = 0, \quad \psi(q) = a.$$

The arc-element is given by

$$ds^2 = dp^2 + a^2 \sin^2\frac{p}{a}\,dq^2,$$

or (what effectively is the same thing) by

$$ds^2 = a^2(dp^2 + \sin^2 p\,dq^2).$$

When K is negative, let its value be $-1/a^2$; then

$$\frac{1}{D}\frac{\partial^2 D}{\partial p^2} = \frac{1}{a^2},$$

so that

$$\sigma = \phi(q)\cosh\frac{p}{a} + \psi(q)\sinh\frac{p}{a}.$$

The conditions, when $p=0$, require that

$$\phi(q)=0, \quad \psi(q)=a.$$

Hence the arc-element is given by

$$ds^2 = dp^2 + a^2\sinh^2\frac{p}{a}\,dq^2,$$

or (what is effectively the same thing) by

$$ds^2 = a^2(dp^2 + \sinh^2 p\,dq^2).$$

Thus (as before, § 155) the surfaces of constant curvature have their arc-element of one or other of the forms

$$ds^2 = dp^2 + p^2dq^2,$$
$$ds^2 = a^2(dp^2 + \sin^2 p\,dq^2),$$
$$ds^2 = a^2(dp^2 + \sinh^2 p\,dq^2).$$

212. Now take two surfaces of the same constant Gaussian curvature. On them choose any two points O and O'; and through each of these points draw a geodesic in any direction, measuring any distance p along the two geodesics. Let the surfaces be referred to the geodesics and geodesic parallels as parametric curves; the elements of arc on the two surfaces are given by

$$ds^2 = dp^2 + f(p)\,dq^2, \quad ds'^2 = dp^2 + f(p)\,dq'^2,$$

for one or other of the three forms of $f(p)$. The arc-elements will be equal, and so the two surfaces will be applicable to one another, if

$$dq^2 = dq'^2,$$

that is, if

$$q - q_0 = q' - q_0',$$

a relation which conserves the angle between corresponding pairs of geodesics through O and O'.

Hence two surfaces of the same constant Gaussian curvature are applicable to one another, by making an arbitrary point on one coincide with an arbitrary point on the other and a second arbitrary point on the first coincide with a second arbitrary point on the second, the geodesic distances between the point-pairs on the two surfaces being the same. Thus two surfaces of the same constant Gaussian curvature are applicable to one another in an infinitude of ways. In particular, a surface of constant Gaussian curvature can be deformed over itself in an infinitude of ways.

213. Among the surfaces of constant Gaussian curvature, which thus are deformable each upon itself, it is convenient to know those that are surfaces of revolution. We refer the surface to its meridians and parallels of

latitude; the former are geodesics which are not necessarily concurrent, and so the analysis in § 211 does not apply; we take the arc-element in the form

$$ds^2 = du^2 + r^2 dv^2,$$

where du is the arc-element of a meridian and r, the distance of a point on the surface from the axis, is a function of u only. Denoting by K the measure of curvature, we have

$$\frac{1}{r}\frac{d^2r}{du^2} = -K;$$

and therefore, for a pseudo-sphere having $-1/a^2$ for its measure of curvature, we have

$$\frac{d^2r}{du^2} = \frac{r}{a^2},$$

so that

$$r = Ae^{\frac{u}{a}} + Be^{-\frac{u}{a}},$$

where A and B are constants. There are three cases:—

(i) when A and B have the same sign; by adding a constant to u (which only means changing the origin of measurement along the meridian), we can make $A = B$, and then the arc-element is

$$ds^2 = du^2 + c^2 \cosh^2 \frac{u}{a} dv^2,$$

while

$$r = c \cosh \frac{u}{a};$$

also

$$du^2 = dr^2 + dz^2,$$

so that

$$z = \int \left(1 - \frac{c^2}{a^2} \sinh^2 \frac{u}{a}\right)^{\frac{1}{2}} du;$$

(ii) when A and B have opposite signs; in the same way as in the first case, we can take $A = -B$, and then the arc-element is

$$ds^2 = du^2 + c^2 \sinh^2 \frac{u}{a} dv^2,$$

while

$$r = c \sinh \frac{u}{a},$$

and

$$z = \int \left(1 - \frac{c^2}{a^2} \cosh^2 \frac{u}{a}\right)^{\frac{1}{2}} du;$$

(iii) when one of the two constants A and B vanishes (both cannot vanish); let $B = 0$, and (as is permissible in the same way as before) take $A = 1$; then the arc-element is

$$ds^2 = du^2 + e^{\frac{2u}{a}} dv^2,$$

while

$$r = e^{\frac{u}{a}} = a \sin \phi,$$

$$z = \int \left(1 - \frac{1}{a^2} e^{2\frac{u}{a}}\right)^{\frac{1}{2}} du$$

$$= a (\log \tan \tfrac{1}{2}\phi + \cos \phi),$$

which is the curve known as a tractrix.

In the first case, we have

$$z = \int \left(\frac{a^2 + c^2 - r^2}{r^2 - c^2}\right)^{\frac{1}{2}} dr,$$

so that r varies between c and $(a^2 + c^2)^{\frac{1}{2}}$; the quantity z is expressible in elliptic functions in the form

$$z = (a^2 + c^2)^{\frac{1}{2}} \{E(\theta) - \theta\},$$

where

$$r = (a^2 + c^2)^{\frac{1}{2}} \operatorname{dn} \theta, \quad k (a^2 + c^2)^{\frac{1}{2}} = a.$$

In the second case, we have

$$z = \int \left(\frac{a^2 - c^2 - r^2}{c^2 + r^2}\right)^{\frac{1}{2}} dr,$$

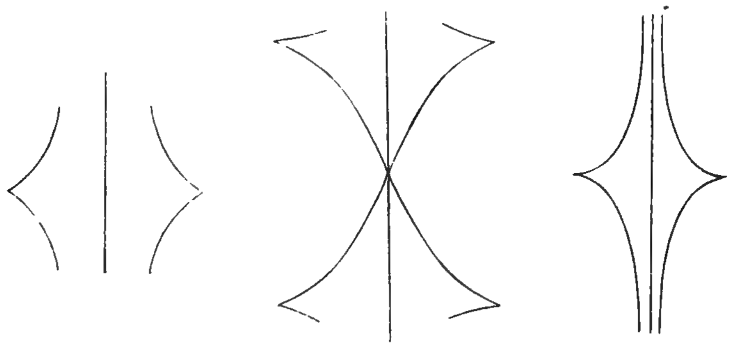
so that r varies between 0 and $(a^2 - c^2)^{\frac{1}{2}}$; the quantity z is expressible in elliptic functions in the form

$$z = a \{E(\theta) - \theta\},$$

where

$$r = (a^2 - c^2)^{\frac{1}{2}} \operatorname{cn} \theta, \quad k = \left(1 - \frac{c^2}{a^2}\right)^{\frac{1}{2}}.$$

In the first case, the pseudo-sphere (said then to be of the hyperbolic type) has its meridian curve as in the figure on the left, the range being there from $-K$ to $+K$; and any part between $(2m-1)K$ and $(2m+1)K$ is the same as the part drawn.



In the second case, the pseudo-sphere (said then to be of the elliptic type) has its meridian curve as in the middle figure, the range between the levels

of the cusps being from 0 to $2K$; and any part between $2mK$ and $(2m+2)K$ is the same as the part drawn.

In the third case, the pseudo-sphere (said then to be of parabolic type) has its meridian curve as in the figure on the right; it has cusps for $\phi = \frac{1}{2}\pi$, and the axis of z is an asymptote, the range for the curve being from $\phi = 0$ to $\phi = \pi$.

214. The formulæ for surfaces of revolution, as regards those deformations which always leave them surfaces of revolution, can be obtained very simply as follows*. Denoting the element of meridian arc by $d\sigma$, and the axial distance by r, we have

$$d\sigma^2 = dr^2 + dz^2,$$
$$ds^2 = d\sigma^2 + r^2 d\phi^2.$$

For the deformed surface, when it remains a surface of revolution, we have

$$ds^2 = d\sigma'^2 + r'^2 d\phi'^2.$$

All arcs are to correspond; hence

$$r'd\phi' = r\,d\phi, \quad d\sigma' = d\sigma.$$

The former is satisfied by

$$r' = kr,$$
$$\phi' = \frac{1}{k}\phi;$$

and the latter gives

$$dz'^2 + dr'^2 = dr^2 + dz^2,$$

that is,

$$dz'^2 = dz^2 + (1 - k^2)\,dr^2.$$

The required surfaces of revolution are given by

$$r' = kr, \quad z' = \int \{dz^2 + (1 - k^2)\,dr^2\}^{\frac{1}{2}}.$$

Thus in the case of a sphere, $z = \sin\theta$, $r = \cos\theta$; so

$$r' = k\cos\theta,$$
$$z' = \int (1 - k^2 \sin^2\theta)^{\frac{1}{2}}\,d\theta,$$

in agreement with the preceding result.

Ex. Obtain the deformations of a hyperboloid of revolution of one sheet, discussing the configuration of the generators.

Deformation; General Equations.

215. From the essential property, that geodesics remain geodesics throughout any deformation of the surface, we can deduce one equation relating to all deformations.

Let the arc-element on the surface be given by

$$ds^2 = du^2 + g^2 dv^2,$$

* Frost, *Solid Geometry*, p. 350.

where g may be regarded as a known function of u and v such that, for small values of u, we have

$$g = u - \tfrac{1}{6}K_0 u^3 + \dots,$$

where K_0 is the measure of curvature at the geodesic pole $u = 0$. In any deformation, let the arc-element be given by

$$ds'^2 = dU^2 + G^2 dV^2;$$

then we must have

$$dU^2 + G^2 dV^2 = du^2 + g^2 dv^2,$$

and the deformations will be given by the knowledge of all values of U, V, G which satisfy this relation. We must have

$$\left(\frac{\partial U}{\partial u}\right)^2 + G^2\left(\frac{\partial V}{\partial u}\right)^2 = 1,$$

$$\frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + G^2\frac{\partial V}{\partial u}\frac{\partial V}{\partial v} = 0,$$

$$\left(\frac{\partial U}{\partial v}\right)^2 + G^2\left(\frac{\partial V}{\partial v}\right)^2 = g^2,$$

three relations involving the three quantities U, V, G. When we eliminate G and the derivatives of V, we find

$$g^2\left(\frac{\partial U}{\partial u}\right)^2 + \left(\frac{\partial U}{\partial v}\right)^2 = g^2,$$

a partial equation of the first order for U containing the function g.

Should, however, g be unobtainable, some other method must be used; accordingly, we shall adopt a more generally effective method.

216. By Bonnet's theorem (§ 37) we know that, when the magnitudes E, F, G, L, M, N are known, the surface is determinate save as to position and orientation. But if only E, F, G are known, so that the arc-element is given, the surface is determinate save as to position, orientation, and deformation. We proceed to indicate the equations for this limited determination of the surface.

We have

$$\begin{aligned} x_{11} - x_1\Gamma - x_2\Delta &= LX, \\ x_{12} - x_1\Gamma' - x_2\Delta' &= MX, \\ x_{22} - x_1\Gamma'' - x_2\Delta'' &= NX, \\ LN - M^2 &= KV^2, \end{aligned}$$

where the quantities Γ, Γ', Γ'', Δ, Δ', Δ''. K, V are known functions of E, F, G and their derivatives. Also

$$\begin{aligned} V^2X^2 &= (y_1 z_2 - y_2 z_1)^2 \\ &= (y_1^2 + z_1^2)(y_2^2 + z_2^2) - (y_1 y_2 + z_1 z_2)^2 \\ &= (E - x_1^2)(G - x_2^2) - (F - x_1 x_2)^2 \\ &= V^2 - (Gx_1^2 - 2Fx_1x_2 + Ex_2^2). \end{aligned}$$

Consequently we have

$$(x_{11} - x_1\Gamma - x_2\Delta)(x_{22} - x_1\Gamma'' - x_2\Delta'') - (x_{12} - x_1\Gamma' - x_2\Delta')^2$$
$$= (LN - M^2) X^2$$
$$= K\{V^2 - (Gx_1^2 - 2Fx_1x_2 + Ex_2^2)\},$$

a partial differential equation, of the second order and the Monge-Ampère type, for the determination of x.

The same partial differential equation is satisfied by y and by z.

Connected with the solution of an equation of this type, and especially with the process of obtaining an integral to satisfy assigned conditions, there is a subsidiary equation (commonly called the equation of characteristics* of the differential equation) which is of fundamental importance. Denote an equation by

$$\Omega = 0;$$

its characteristics are given by

$$\frac{\partial \Omega}{\partial x_{11}} dq^2 - \frac{\partial \Omega}{\partial x_{12}} dp dq + \frac{\partial \Omega}{\partial x_{22}} dp^2 = 0.$$

Thus, for our equation, the characteristics are

$$X(Ndq^2 + 2M dp dq + Ldp^2) = 0,$$

and similarly for the equations satisfied by y and by z; that is, the characteristics are given by

$$Ldp^2 + 2M dp dq + Ndq^2 = 0.$$

Hence *the characteristics of the equation for the determination of the surface are its asymptotic lines*, a result that will be seen to bring these lines into specially significant relation with conditions that may be assigned as governing all deformations of the surface.

217. Some forms of the equation are of special importance; we shall consider three of these forms.

I. Let the equation of the surface be

$$z = f(x, y);$$

and give to p, q, r, s, t their customary significance as the first and the second derivatives of z with respect to x and y. We require the equation of the second order, satisfied by every surface into which the given surface can be deformed; so we take x and y to be the independent variables throughout, and we denote by Z the ordinate of the deformed surface, and by P, Q, R, S, T

* See the author's *Theory of Differential Equations*, vol. vi, chap. xx.

its first and its second derivatives with respect to x and y. Now, for the given surface, we have (p. 60)

$$E = 1 + p^2, \quad F = pq, \quad G = 1 + q^2, \quad V^2 = 1 + p^2 + q^2,$$

$$V^2\Gamma = pr, \quad V^2\Gamma' = ps, \quad V^2\Gamma'' = pt,$$

$$V^2\Delta = qr, \quad V^2\Delta' = qs, \quad V^2\Delta'' = qt,$$

$$K = \frac{rt - s^2}{(1 + p^2 + q^2)^2};$$

hence the equation, which is

$$(R - P\Gamma - Q\Delta)(T - P\Gamma'' - Q\Delta'') - (S - P\Gamma' - Q\Delta')^2$$
$$= K\{V^2 - (GP^2 - 2FPQ + EQ^2)\},$$

becomes

$$(RT - S^2)(1 + p^2 + q^2) - (rT + tR - 2sS)(pP + qQ) + (P^2 + Q^2 - 1)(rt - s^2) = 0.$$

The general value of Z, satisfying this equation, will give the general set of surfaces derivable from

$$z = f(x, y)$$

by deformation. It ought to include (and manifestly it does include) the possibility

$$Z = z.$$

II. Let the surface be referred to nul lines as parametric curves. The arc-element then has the form

$$ds^2 = 4\lambda\, du\, dv;$$

also

$$\Gamma = \frac{\lambda_1}{\lambda}, \quad \Gamma' = 0, \quad \Gamma'' = 0,$$

$$\Delta = 0, \quad \Delta' = 0, \quad \Delta'' = \frac{\lambda_2}{\lambda},$$

$$K = -\frac{1}{2\lambda}\frac{\partial^2 \log\lambda}{\partial u \partial v}.$$

Then, using θ to denote x, y, or z, we have the equation for θ in the form

$$\left(\theta_{11} - \theta_1\frac{\lambda_1}{\lambda}\right)\left(\theta_{22} - \theta_2\frac{\lambda_2}{\lambda}\right) - \theta_{12}^2$$
$$= -\frac{1}{2\lambda}\frac{\partial^2 \log\lambda}{\partial u \partial v}\{-4\lambda^2 + 4\lambda\theta_1\theta_2\}$$
$$= 2(\lambda - \theta_1\theta_2)\frac{\partial^2 \log\lambda}{\partial u \partial v}.$$

To adopt the customary notation for partial differential equations with two independent variables, we write

$$\theta_1 = p, \quad \theta_2 = q, \quad \theta_{11} = r, \quad \theta_{12} = s, \quad \theta_{22} = t;$$

and then, if

$$\frac{\lambda_1}{\lambda} = a, \quad \frac{\lambda_2}{\lambda} = c, \quad \frac{\partial^2 \log \lambda}{\partial u \partial v} = b,$$

our equation for surfaces, deformable into a given surface, is

$$rt - s^2 - cqr - apt = b(\lambda - pq) - acpq,$$

the parametric curves being nul lines on the surfaces. Manifestly we have, among the coefficients a, b, c and the quantity λ, the relations

$$a = \frac{\partial \log \lambda}{\partial u},$$

$$c = \frac{\partial \log \lambda}{\partial v},$$

$$b = \frac{\partial a}{\partial v} = \frac{\partial c}{\partial u} = \frac{\partial^2 \log \lambda}{\partial u \partial v}.$$

The variables u and v are conjugate quantities, known (§ 55) to be derivable by integrating an ordinary equation of the first order; the quantity λ is then a factor, obtainable by merely direct operations.

III. Let the surface be referred to geodesic polar coordinates. The arc-element then has the form

$$ds^2 = du^2 + D^2 dv^2.$$

With corresponding changes of notation, the equation, which determines surfaces that are deformable into a given surface, is

$$r\left(t + pDD_1 - q\frac{D_2}{D}\right) - \left(s - q\frac{D_1}{D}\right)^2 + \frac{D_{11}}{D}(D^2 - D^2p^2 - q^2) = 0,$$

the form being due to Bour. The equation naturally is equivalent to the equation in the preceding type of representation of the surface; but it is dependent upon the determination of the variables u and v, which requires the integration of an ordinary differential equation of the second order.

218. Two other methods of constructing a critical equation—always of the second order, for the surfaces that arise by the deformations of a given surface, should be noted. One of them is due to Bonnet*, the other to Darboux†.

In Bonnet's method, the surface again is referred to its nul lines as parametric curves, and the arc-element is taken in the form

$$ds^2 = 4\lambda^2 du\, dv.$$

All the surfaces, into which it can be deformed, are given by

$$dx^2 + dy^2 + dz^2 = 4\lambda^2 du\, dv,$$

* *Journ. Éc. Polytechn.*, cah. xlii (1867), p. 3.

† See his treatise, vol. iii, p. 253.

so that

$$x_1^2+y_1^2+z_1^2=0,\quad x_2^2+y_2^2+z_2^2=0,$$

$$x_1x_2+y_1y_2+z_1z_2=2\lambda^2.$$

The first two of these equations are satisfied by taking

$$x_1=i\,(m^2+n^2),\quad y_1=m^2-n^2,\quad z_1=2mn,$$

$$x_2=i\,(m'^2+n'^2),\quad y_2=m'^2-n'^2,\quad z_2=2m'n',$$

for any values of m, n, m', n'; and then the third equation is satisfied if

$$mn'-m'n=i\lambda.$$

But we must have

$$\frac{\partial x_1}{\partial v}=\frac{\partial x_2}{\partial u},\quad \frac{\partial y_1}{\partial v}=\frac{\partial y_2}{\partial u},\quad \frac{\partial z_1}{\partial v}=\frac{\partial z_2}{\partial u};$$

and therefore

$$mm_2=m'm_1',$$

$$nn_2=n'n_1',$$

$$nm_2+mn_2=n'm_1'+m'n_1'.$$

Hence

$$\frac{m_2}{m'}=\frac{m_1'}{m}=\frac{n_1'}{n}=\frac{n_2}{n'}$$

$$=\frac{m'^2}{-i\lambda}\frac{\partial}{\partial u}\left(\frac{n'}{m'}\right)$$

$$=\frac{m^2}{i\lambda}\frac{\partial}{\partial v}\left(\frac{n}{m}\right),\ =\theta,$$

say. Also

$$\frac{n'}{m'}=\frac{n}{m}+\frac{i\lambda}{mm'},$$

and therefore

$$-\frac{i\lambda}{m'^2}\theta=\frac{\partial}{\partial u}\left(\frac{n'}{m'}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{n}{m}\right)+\frac{i}{m'}\frac{\partial}{\partial u}\left(\frac{\lambda}{m}\right)-\frac{i\lambda}{m}\frac{m_1'}{m'^2}$$

$$=\frac{\partial}{\partial u}\left(\frac{n}{m}\right)+\frac{i}{m'}\frac{\partial}{\partial u}\left(\frac{\lambda}{m}\right)-\frac{i\lambda}{m'^2}\theta;$$

hence

$$\frac{\partial}{\partial u}\left(\frac{n}{m}\right)=-\frac{i}{m'}\frac{\partial}{\partial u}\left(\frac{\lambda}{m}\right).$$

Similarly

$$\frac{\partial}{\partial v}\left(\frac{n}{m}\right)=\frac{i\lambda}{m^2}\theta=-\frac{i\lambda}{m'}\frac{\partial}{\partial v}\left(\frac{1}{m}\right).$$

Now

$$m^2=\tfrac{1}{2}(y_1-ix_1)=\zeta_1=p,\ \text{say},$$

$$m'^2=\tfrac{1}{2}(y_2-ix_2)=\zeta_2=q,\ \text{say},$$

where p and q have an altered significance, which correspondingly will be associated with r, s, t. Thus

$$\frac{\partial}{\partial u}\left(\frac{n}{m}\right) = -\frac{i}{q^{\frac{1}{2}}}\frac{\partial}{\partial u}\left(\frac{\lambda}{p^{\frac{1}{2}}}\right)$$

$$= -\frac{i}{p^{\frac{1}{2}}q^{\frac{1}{2}}}\lambda_1 - \frac{i\lambda}{q^{\frac{1}{2}}}\frac{\partial}{\partial u}\left(\frac{1}{p^{\frac{1}{2}}}\right),$$

$$\frac{\partial}{\partial v}\left(\frac{n}{m}\right) = -\frac{i\lambda}{q^{\frac{1}{2}}}\frac{\partial}{\partial v}\left(\frac{1}{p^{\frac{1}{2}}}\right);$$

and therefore

$$\frac{\partial}{\partial v}\left(\frac{\lambda_1}{p^{\frac{1}{2}}q^{\frac{1}{2}}}\right) + \frac{\partial}{\partial v}\left(\frac{\lambda}{q^{\frac{1}{2}}}\right)\frac{\partial}{\partial u}\left(\frac{1}{p^{\frac{1}{2}}}\right) = \frac{\partial}{\partial u}\left(\frac{\lambda}{q^{\frac{1}{2}}}\right)\frac{\partial}{\partial v}\left(\frac{1}{p^{\frac{1}{2}}}\right),$$

becoming, on expansion,

$$\lambda\,(rt - s^2) - 2\lambda_2 qr - 2\lambda_1 pt + 4pq\lambda_{12} = 0.$$

This is the required equation; its difference, from the earlier equation in form, is due to the fact that the dependent variable now is $\frac{1}{2}(y - ix)$.

When this equation is integrated so that ζ is known, we know m and m'; and then, by quadrature through the above relations, we find n/m, that is, we know n. The value of n' follows from $mn' - m'n = i\lambda$. Substituting in

$$dx = i\,(m^2 + n^2)\,du + i\,(m'^2 + n'^2)\,dv,$$

$$dy = (m^2 - n^2)\,du + (m'^2 - n'^2)\,dv,$$

$$dz = 2mn\,du + 2m'n'dv,$$

and effecting the quadratures, we have the equations of all surfaces derivable from the given surface by deformation.

219. Next, consider Darboux's method of constructing the critical equation—always of the second order—for surfaces deformable into a given surface. The latter still is referred to its nul lines so that the arc-element is given by

$$ds^2 = 4\lambda\,du\,dv;$$

all the required surfaces are such that

$$dx^2 + dy^2 + dz^2 = 4\lambda\,du\,dv.$$

Thus

$$dx^2 + dy^2 = 4\lambda\,du\,dv - (p\,du + q\,dv)^2$$

$$= -p^2du^2 + 2\,(2\lambda - pq)\,du\,dv - q^2dv^2.$$

The surface, of which the arc-element is given by

$$ds^2 = -p^2du^2 + 2\,(2\lambda - pq)\,du\,dv - q^2dv^2,$$

is thus deformable into a plane; consequently, its Gaussian measure of curvature is zero. We have

$$E = -p^2, \quad F = 2\lambda - pq, \quad G = -q^2, \quad V^2 = 4\lambda\,(pq - \lambda);$$

and therefore

$$\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}) = rt - s^2 - 2\lambda_{12}.$$

Also, with the notation of § 34, we have

$$m = -pr, \qquad m' = -ps, \quad m'' = 2\lambda_2 - pt,$$
$$n = 2\lambda_1 - qr, \quad n' = -qs, \qquad n'' = -qt;$$

and therefore

$$nn'' - n'^2 = -2\lambda_1 qt + q^2(rt - s^2),$$
$$nm'' - 2n'm' + mn'' = 4\lambda_1\lambda_2 - 2\lambda_1 pt - 2\lambda_2 qr + 2pq(rt - s^2),$$
$$mm'' - m'^2 = -2\lambda_2 pr + p^2(rt - s^2).$$

In order that the Gaussian measure of curvature may be zero, we must have

$$\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11})V^2$$
$$= -E(nn'' - n'^2) + F(nm'' - 2n'm' + mn'') - G(mm'' - m'^2),$$

which, when we substitute and reduce, becomes

$$rt - s^2 - \frac{\lambda_2}{\lambda}qr - \frac{\lambda_1}{\lambda}pt = (\lambda - pq)\,2\frac{\partial^2 \log\lambda}{\partial u \partial v} - \frac{\lambda_1\lambda_2}{\lambda^2}pq,$$

the equation in question. Manifestly it is a partial differential equation of the second order, being of the Monge-Ampère type; of course, the dependent variable is not the same as in Bonnet's equation.

Moreover, supposing the value of z known, we have

$$dx^2 + dy^2 = -p^2du^2 + 2(2\lambda - pq)\,du\,dv - q^2dv^2,$$

so that

$$x_1^2 + y_1^2 = -p^2,$$
$$x_1x_2 + y_1y_2 = 2\lambda - pq,$$
$$x_2^2 + y_2^2 = -q^2;$$

and therefore

$$(p^2 + x_1^2)(q^2 + x_2^2) = (2\lambda - pq - x_1x_2)^2,$$

that is,

$$q^2x_1^2 - 2(2\lambda - pq)x_1x_2 + p^2x_2^2 = 4\lambda^2 - 4\lambda pq,$$

an equation of the first order for x.

When x is known from this equation, then

$$y_1^2 = -p^2 - x_1^2, \quad y_1y_2 = 2\lambda - pq - x_1x_2,$$

and so the value of y is derivable by quadrature.

It thus appears that, whatever process be adopted, an essential and critical condition—in the form that leads to surfaces which are deformable into a given surface—is a Monge-Ampère partial differential equation of the second order. The limitations of sufficiency of the equation, in varied possibilities, will be discussed later; we shall now deal with some special examples.

Ex. 1. Consider the surfaces deformable into a plane. The square of the arc-element is equal to dx^2+dy^2, that is,

$$du\,dv,$$

where $u=x+iy$, $v=x-iy$. Thus, for Darboux's equation, $\lambda=\frac{1}{4}$; and so the equation is

$$rt-s^2=0.$$

The intermediate integral is

$$q=f(p);$$

and the primitive is

$$\left.\begin{aligned} z&=au+vf(a)+\phi(a)\\ 0&=\ \ u+vf'(a)+\phi'(a)\end{aligned}\right\},$$

where f and ϕ are arbitrary functions.

The equation for x now is

$$c^2x_1^2-2(2\lambda-ac)x_1x_2+a^2x_2^2=4\lambda^2-4\lambda ac,$$

where $c=f(a)$; and its primitive is

$$\left.\begin{aligned} x&=au+\beta v+\psi(a)\\ 0&=\ \ u+v\frac{d\beta}{da}+\psi'(a)\end{aligned}\right\},$$

where

$$c^2a^2-2(2\lambda-ac)a\beta+a^2\beta^2=4\lambda^2-4\lambda ac.$$

The equations for y are

$$y_1^2=-p^2-x_1^2=-a^2-a^2=\mu^2,$$

$$y_2^2=-q^2-x_2^2=-c^2-\beta^2=\nu^2,$$

and so the most general value is

$$y=\mu u+\nu v+\rho.$$

The equations can be simplified by taking x and y as the independent variables, say $x=u$, $y=v$; then

$$\left.\begin{aligned} z&=ax+yf(a)+\phi(a)\\ 0&=\ \ x+yf'(a)+\phi'(a)\end{aligned}\right\},$$

being a developable surface, as was to be expected.

Ex. 2. Consider the deformations of a sphere, not restricted (as in § 213) to give surfaces of revolution.

When the surface is referred to its nul lines as parametric curves, the arc-element is

$$ds^2=\frac{4}{(1+uv)^2}du\,dv,$$

so that

$$\lambda=\frac{1}{(1+uv)^2},$$

for Darboux's equation. Thus

$$\frac{\lambda_1}{\lambda}=\frac{-2v}{1+uv},\qquad \frac{\lambda_2}{\lambda}=\frac{-2u}{1+uv},$$

$$\frac{\partial^2\log\lambda}{\partial u\,\partial v}=\frac{-2}{(1+uv)^2};$$

and so the differential equation, which governs all the deformations, is

$$rt-s^2+2\frac{uqr+vpt}{1+uv}+4pq\frac{uv-1}{(1+uv)^2}+\frac{4}{(1+uv)^4}=0.$$

It is easy to verify that this equation is satisfied by

$$z=f(uv)=f(w),$$

where

$$f'^2=\frac{4}{(w+1)^4}+A\frac{(w-1)^2}{w(w+1)^4},$$

the sphere itself being given by

$$A=0, \qquad f=\frac{1-w}{1+w}.$$

Ex. 3. Consider the surfaces into which the paraboloid of revolution

$$2z=x^2+y^2$$

can be deformed. To represent the surface, let

$$z=\tfrac{1}{2}\lambda, \quad x=\lambda^{\frac{1}{2}}\cos\theta, \quad y=\lambda^{\frac{1}{2}}\sin\theta;$$

the parameters of the nul lines are

$$2u=\frac{1}{2}\int\frac{1}{\lambda}(1+\lambda)^{\frac{1}{2}}\,d\lambda+i\theta, \qquad 2v=\frac{1}{2}\int\frac{1}{\lambda}(1+\lambda)^{\frac{1}{2}}\,d\lambda-i\theta,$$

and the arc-element is

$$ds^2=4\lambda\,du\,dv.$$

We take

$$\begin{aligned}\xi=u+v&=\frac{1}{2}\int\frac{1}{\lambda}(1+\lambda)^{\frac{1}{2}}\,d\lambda\\&=(\lambda+1)^{\frac{1}{2}}-\tfrac{1}{2}\log\frac{(\lambda+1)^{\frac{1}{2}}+1}{(\lambda+1)^{\frac{1}{2}}-1},\end{aligned}$$

though the integrated form will not be used. Writing $\lambda'=d\lambda/d\xi$, we have

$$\lambda'=\frac{2\lambda}{(1+\lambda)^{\frac{1}{2}}}, \qquad \lambda''=\frac{4\lambda+2\lambda^2}{(1+\lambda)^2};$$

and therefore

$$\frac{\lambda_1}{\lambda}=\frac{\lambda_2}{\lambda}=\frac{2}{(1+\lambda)^{\frac{1}{2}}},$$

$$\frac{\partial^2\log\lambda}{\partial u\partial v}=-\frac{2\lambda}{(1+\lambda)^2},$$

so that the Darboux equation is

$$\begin{aligned}rt-s^2-\frac{2(qr+pt)}{(1+\lambda)^{\frac{1}{2}}}&=-\frac{4\lambda}{(1+\lambda)^2}(\lambda-pq)-\frac{4pq}{1+\lambda}\\&=-\frac{4pq}{(1+\lambda)^2}-\frac{4\lambda^2}{(1+\lambda)^2},\end{aligned}$$

the customary partial differential equation of the second order and the Monge-Ampère type.

Ex. 4. When we retain x and y as the independent variables, and again consider the deformations of the paraboloid of revolution

$$2z=x^2+y^2,$$

so that

$$p=x, \quad q=y, \quad r=1, \quad s=0, \quad t=1,$$

then the critical equation of the deformed surface is

$$(RT-S^2)(1+x^2+y^2)-(R+T)(xP+yQ)+P^2+Q^2-1=0,$$

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p. 371. The two sets of equations in the last three lines should be

$$\left.\begin{aligned} dq + A\,dv - (B + \Delta^{\frac{1}{2}})\,du &= 0 \\ dp - (B - \Delta^{\frac{1}{2}})\,dv + C\,du &= 0 \\ dz - q\,dv - p\,du &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} dq + A\,dv - (B - \Delta^{\frac{1}{2}})\,du &= 0 \\ dp - (B + \Delta^{\frac{1}{2}})\,dv + C\,du &= 0 \\ dz - q\,dv - p\,du &= 0 \end{aligned}\right\}.$$

p. 372. The two sets of equations after the seventh line should be

$$\left.\begin{aligned} \frac{\partial q}{\partial \beta} + A\frac{\partial v}{\partial \beta} - (B + \Delta^{\frac{1}{2}})\frac{\partial u}{\partial \beta} &= 0 \\ \frac{\partial p}{\partial \beta} - (B - \Delta^{\frac{1}{2}})\frac{\partial v}{\partial \beta} + C\frac{\partial u}{\partial \beta} &= 0 \\ \frac{\partial z}{\partial \beta} - q\frac{\partial v}{\partial \beta} - p\frac{\partial u}{\partial \beta} &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} \frac{\partial q}{\partial \alpha} + A\frac{\partial v}{\partial \alpha} - (B - \Delta^{\frac{1}{2}})\frac{\partial u}{\partial \alpha} &= 0 \\ \frac{\partial p}{\partial \alpha} - (B + \Delta^{\frac{1}{2}})\frac{\partial v}{\partial \beta} + C\frac{\partial u}{\partial \alpha} &= 0 \\ \frac{\partial z}{\partial \alpha} - q\frac{\partial v}{\partial \alpha} - p\frac{\partial u}{\partial \alpha} &= 0 \end{aligned}\right\}.$$

say (with an obvious change of notation)

$$rt - s^2 - (r+t)\frac{xp+yq}{1+x^2+y^2} = \frac{1-p^2-q^2}{1+x^2+y^2}.$$

This equation of the second order is of the customary type; and it is equivalent to the equation in the preceding example, regard being paid to the difference of significance in the symbols.

The critical equation of the second order; its integrals.

220. It is clear, from the general theory and from all examples which are not exceedingly special, that the determination of the surfaces into which a given surface can be deformed depends upon the integration of a Monge-Ampère partial differential equation of the second order. Adopting Darboux's initial resolution of the problem, we have the equation in the form

$$rt - s^2 - cqr - apt = b(\lambda - pq) - acpq,$$

where

$$a = \frac{\lambda_1}{\lambda}, \quad c = \frac{\lambda_2}{\lambda}, \quad b = 2a_2 = 2c_1 = 2\frac{\partial^2 \log \lambda}{\partial u \partial v}.$$

The possible methods, at present known, of actually forming the primitive of an equation of this type are set out in treatises on differential equations. In general, no one of the methods is of compelling power; that is to say, a primitive cannot be obtained in finite terms, unless some special form or other characterises the quantity λ. It may at once be said that no intermediate integral (that is, an equivalent partial equation of the first order) of the foregoing equation can be derived by the method of Monge or the equivalent method of Boole; nor can an intermediate integral be derived by the amplification of Darboux's method for proceeding to the primitive of the equation. All that remains therefore is to see how far Ampère's method, which is perfectly general in idea, can prove effectively manipulative in particular cases*.

Stated briefly for the equation

$$rt - s^2 + Ar + 2Bs + Ct = D,$$

where A, B, C, D can be functions of x, y, z, p, q, Ampère's method is as follows. Writing

$$\Delta = B^2 - AC - D,$$

we construct the two systems of subsidiary equations

$$\left.\begin{aligned} dq + Rdv - \Delta^{\frac{1}{2}}du &= 0 \\ dp + \Delta^{\frac{1}{2}}dv + Tdu &= 0 \\ dz - qdv - pdu &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} dq + Rdv + \Delta^{\frac{1}{2}}du &= 0 \\ dp - \Delta^{\frac{1}{2}}dv + Tdu &= 0 \\ dz - qdv - pdu &= 0 \end{aligned}\right\}.$$

* See the author's *Theory of Differential Equations*, vol. vi, ch. xvii; and some remarks in a lecture before the Rome congress of mathematicians (1908) in vol. i of the *Atti*, as well as in a presidential address to the London Mathematical Society in 1906.

We attempt to frame some integrable combination of the first set; let it be

$$f(p, q, z, u, v, \lambda) = \alpha,$$

where α is an arbitrary parameter of integration. We attempt also to frame some integrable combination of the second set; let it be

$$g(p, q, z, u, v, \lambda) = \beta,$$

where β is another arbitrary parameter of integration. The quantities α and β are then made the independent variables; and we form the equations

$$\left.\begin{aligned}\frac{\partial q}{\partial\beta} + R\frac{\partial v}{\partial\beta} - \Delta^{\frac{1}{2}}\frac{\partial u}{\partial\beta} = 0\\ \frac{\partial p}{\partial\beta} + \Delta^{\frac{1}{2}}\frac{\partial v}{\partial\beta} + T\frac{\partial u}{\partial\beta} = 0\\ \frac{\partial z}{\partial\beta} - q\frac{\partial v}{\partial\beta} - p\frac{\partial u}{\partial\beta} = 0\end{aligned}\right\}, \qquad \left.\begin{aligned}\frac{\partial q}{\partial\alpha} + R\frac{\partial v}{\partial\alpha} + \Delta^{\frac{1}{2}}\frac{\partial u}{\partial\alpha} = 0\\ \frac{\partial p}{\partial\alpha} - \Delta^{\frac{1}{2}}\frac{\partial v}{\partial\alpha} + T\frac{\partial u}{\partial\alpha} = 0\\ \frac{\partial z}{\partial\alpha} - q\frac{\partial v}{\partial\alpha} - p\frac{\partial u}{\partial\alpha} = 0\end{aligned}\right\}.$$

These equations are to be integrated; the integration has the same range of practical possibility as the construction of the preceding quantities f and g. Arbitrary parameters of partial integration for each set are made arbitrary functions of the latent variable; and so the primitive may be obtained. But, as already remarked, it happens only too often in individual cases that, while the theory is complete, the practicability of a primitive in finite terms is out of the question*.

And manifestly, at this stage, the analysis is much more concerned with the integration of partial differential equations than with the properties of deformation of which they merely are the expression.

Ex. 1. Taking the equation for the deformation of a paraboloid of revolution as given in § 219, Ex. 3, shew that an integrable combination of the first set of subsidiary equations in the preceding statement of process is given by

$$p - q + 2\left\{(1+\lambda)\left(1 - \frac{pq}{\lambda}\right)\right\}^{\frac{1}{2}} = \alpha,$$

and that an integrable combination of the second set is given by

$$p - q - 2\left\{(1+\lambda)\left(1 - \frac{pq}{\lambda}\right)\right\}^{\frac{1}{2}} = \beta.$$

Complete the primitive†.

Ex. 2. Shew that the surfaces of revolution into which the paraboloid

$$2z = x^2 + y^2 = r^2$$

can be deformed are given by

$$z = \tfrac{1}{2}r(r^2 + 2c)^{\frac{1}{2}} + c\log\{r + (r^2 + 2c)^{\frac{1}{2}}\},$$

where c is a parametric constant.

* For a detailed construction of the results stated, see the author's *Theory of Differential Equations*, vol. vi, ch. xvii.

† See a memoir by Calapso, *Rend. Circ. Mat. di Palermo*, t. xv (1901), pp. 1—32.

221. But, though there is the customary impossibility of integrating the critical equation in finite terms, there exists the important theorem due to Cauchy which governs the existence of integrals of partial differential equations. When it is applied to an equation of the second order, the theorem affirms the existence of a uniform integral, which is uniquely determined by the properties that z and one of its derivatives—say p—acquire assigned values along any given curve that is not a characteristic. The properties can be stated in another form. Let the curve be

$$\phi(u, v) = 0,$$

so that

$$\frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0.$$

Now

$$dz = p\,du + q\,dv,$$

while z and p are given along the curve; hence q also is known along the curve, and therefore the properties can be regarded as giving the values of p, q, z along the curve. When u and v are x and y, as in the equation in § 217 (I.), we can say that a surface exists as an integral of the equation, uniquely determined by the requirements of passing through an assigned curve and touching a given developable surface along the curve. But the assigned curve must not be a characteristic; in the present case, therefore, it must not belong to either of the families of asymptotic lines.

In the most general form, the quantities x, y, z satisfy one and the same equation of the second order. Suppose that, when $u = \alpha$, where α is a parameter and u is not the variable of an asymptotic line, the values of x, y, z are required to be $A(v)$, $B(v)$, $C(v)$, and the values of x_1, y_1, z_1 are required to be $a(v)$, $b(v)$, $c(v)$. Then

$$\left.\begin{aligned} a^2 + b^2 + c^2 &= E \\ aA' + bB' + cC' &= F \\ A'^2 + B'^2 + C'^2 &= G \end{aligned}\right\},$$

where E, F, G are given quantities; thus, in the least restricted assignment of conditions, three arbitrary elements appear to survive. The two independent variables are unassigned and arbitrary, yet they would have to disappear when the surface is represented by a single equation; so they may be considered as absorbing two out of three arbitrary elements. It follows therefore that one arbitrary element certainly survives, when all requirements are met and the surface is given by a single equation; and so a question arises as to the use which can be made of this disposable arbitrary element. Some instances are given in the propositions which follow.

Some illustrations as to deformation.

222. For instance, can a surface S be deformed, while some curve C upon it is kept rigid?

On the surface, take a set of orthogonal curves with parameters u and v, where

$$v = 0$$

is the equation of the curve C; and let Σ denote the deformed surface. The correspondence between the points of the surfaces S and Σ is birational; hence, by Tissot's theorem (§ 156), an orthogonal system on S has an orthogonal system on Σ as its homologue, and there is only one such system. By hypothesis, C is conserved through the deformations; hence the system of curves given by the parameters u and v on Σ is orthogonal.

Again, as C remains rigid, its circular curvature is unaltered; and the geodesic curvature is not changed under deformation. Hence, with the usual notation (§ 104), the quantity $\sin \varpi$ is unchanged along C; and therefore the normal to Σ along C coincides with the normal to S along C.

Denote by x', y', z' the coordinates for Σ, and by X', Y', Z' the direction-cosines of the normal. Then, along C, we have

$$\left.\begin{array}{lll} x' = x, & y' = y, & z' = z \\ x_1' = x_1, & y_1' = y_1, & z_1' = z_1 \\ x_2' = x_2, & y_2' = y_2, & z_2' = z_2 \\ X' = X, & Y' = Y, & Z' = Z \end{array}\right\},$$

that is, when $v = 0$. From the second and the third lines we have

$$x_{11}' = x_{11}, \quad y_{11}' = y_{11}, \quad z_{11}' = z_{11},$$

$$x_{12}' = x_{12}, \quad y_{12}' = y_{12}, \quad z_{12}' = z_{12},$$

when $v = 0$. But

$$x_{11} = x_1\Gamma + x_2\Delta + LX, \qquad x_{12} = x_1\Gamma' + x_2\Delta' + MX,$$

$$x_{11}' = x_1'\Gamma + x_2'\Delta + L'X', \quad x_{12}' = x_1'\Gamma' + x_2'\Delta' + M'X',$$

and so for the other coordinates, the quantities Γ, Δ, Γ', Δ' depending only upon E, F, G which remain unaltered. Hence

$$L' = L, \quad M' = M,$$

when $v = 0$. The Gaussian measure of curvature is unaltered by the deformation, so that

$$L'N' - M'^2 = LN - M^2,$$

that is,

$$L'N' = LN;$$

and therefore $N' = N$ when $v = 0$, unless $L' = L = 0$. Thus, if $N' = N$, we have

$$x_{22}' = x_{22}, \quad y_{22}' = y_{22}, \quad z_{22}' = z_{22},$$

when $v = 0$.

Thus, except in the excluded case, all the second derivatives of x', y', z' agree with those of x, y, z, when $v = 0$. The same holds of all the derivatives of all orders when $v = 0$; hence, taking the Taylor expansions, we have

$$x' = x, \quad y' = y, \quad z' = z,$$

everywhere, that is, S and Σ coincide. There has been no deformation.

In the excluded case, $L = 0$; we cannot infer that $N' = N$, or that $x_{22}' = x_{22}$, $y_{22}' = y_{22}$, $z_{22}' = z_{22}$. When $L = 0$, the asymptotic lines are

$$2M\,du\,dv + N\,dv^2 = 0,$$

that is, the curve C is an asymptotic line. We cannot now infer that

$$x' = x, \quad y' = y, \quad z' = z,$$

everywhere, so that S and Σ do not coincide. There has been deformation. Hence a surface can be deformed while a curve upon it remains rigid, only if the curve is an asymptotic line*.

The simplest example is that of a hyperboloid of one sheet. The hyperboloid can be deformed† so that its generators continue generators, like a netted flexible frame of straight rigid rods.

223. Next, consider those deformations (if any) of a surface such that one given curve C traced on the surface is deformed into another given curve C'.

Denoting the circular curvature of C' by $1/\rho'$, and the angle between the normal to the deformed surface at any point of C' and the principal normal of C' by ϖ', we have

$$\frac{\sin \varpi'}{\rho'} = \frac{1}{\gamma},$$

owing to the persistence in value of the geodesic curvature of any curve on the surface; hence in all cases

$$\frac{1}{\rho'} \geqslant \frac{1}{\gamma},$$

as regards magnitude. Consequently the circular curvature of the final curve C' must be at least as great as the initial (and unchanged) geodesic curvature of C at the point. There are two cases to consider.

* On account of this property, asymptotic lines are sometimes called *lines of folding*.

† For details, see Cayley, *Coll. Math. Papers*, vol. xi, p. 66.

I. Let

$$\frac{1}{\rho'} > \frac{1}{\gamma}.$$

On the deformed surface, take an orthogonal system of curves determined by parameters u and v, such that the final curve C' is given by $v=0$; as any function of u can be substituted for u, let the magnitude of u be chosen so that, along $v=0$, the arc of C' measured from some fixed point is equal to u. Then the arc-element on the deformed surface is given by

$$ds^2 = E\,du^2 + G\,dv^2,$$

where $E=1$ when $v=0$. Also, du is the arc-element of C on the undeformed surface; and E and G are known throughout.

The principal trihedron of C' is known at every point. Hence the values of x_1, y_1, z_1 are known when $v=0$, for they are the direction-cosines of the tangent to C'. The values of $\rho' x_{11}, \rho' y_{11}, \rho' z_{11}$ are known when $v=0$, for they are the direction-cosines (say $\cos\xi, \cos\eta, \cos\zeta$) of the principal normal to C'; thus, as ρ' is known, the values of x_{11}, y_{11}, z_{11} are known when $v=0$. And the direction-cosines (say $\cos\lambda, \cos\mu, \cos\nu$) of the binormal to C' are known, that is, when $v=0$.

As ϖ' is the angle between the normal to the deformed surface and the principal normal to C', it follows that $\frac{1}{2}\pi - \varpi'$ is the angle between the line whose direction-cosines are $G^{-\frac{1}{2}}x_2, G^{-\frac{1}{2}}y_2, G^{-\frac{1}{2}}z_2$; hence

$$G^{-\frac{1}{2}}x_2 = \sin\varpi' \cos\xi - \cos\varpi' \cos\lambda,$$

$$G^{-\frac{1}{2}}y_2 = \sin\varpi' \cos\eta - \cos\varpi' \cos\mu,$$

$$G^{-\frac{1}{2}}z_2 = \sin\varpi' \cos\zeta - \cos\varpi' \cos\nu.$$

Thus the values of x_2, y_2, z_2 are known when $v=0$; and so also the values of x_{12}, y_{12}, z_{12} are known when $v=0$. Moreover, the variable x satisfies the equation

$$\left(x_{11} - \frac{E_1}{2E}x_1 + \frac{E_2}{2G}x_2\right)\left(x_{22} + \frac{G_1}{2E}x_1 - \frac{G_2}{2G}x_2\right) - \left(x_{12} - \frac{E_2}{2E}x_1 - \frac{G_1}{2G}x_2\right)^2$$
$$= K\{EG - (Gx_1^2 + Ex_2^2)\},$$

while y and z satisfy the same equation; hence the values of x_{22}, y_{22}, z_{22} are known when $v=0$. Thus, partly from the data and partly from the nature of the case, all the first and second derivatives of x, y, z are known along the curve C'. And similarly for all the higher derivatives; *e.g.*, the values of $x_{111}, x_{112}, x_{122}$, when $v=0$, are the u-derivatives of x_{11}, x_{12}, x_{22}, and therefore are known, while the value of x_{222} is obtained by differentiating the critical equation with respect to v and then inserting in the derived equation the values of the other quantities which are known.

Now consider deformations of the surface in general. For the purpose, we require integrals of the foregoing critical equation which are such that, when $v=0$,

$$\left.\begin{array}{ll} x = x', & x_2 = x_2' \\ y = y', & y_2 = y_2' \\ z = z', & z_2 = z_2' \end{array}\right\}.$$

They exist, and they are uniquely determined by these relations. Thus the curve on the deformed surface, given by $v=0$, coincides with C'; in other words, the deformation of the original surface is possible. A curve, originally given by $v=0$, is deformed so as to coincide with C'.

Moreover, in the present case, we have

$$\frac{\sin \varpi'}{\rho'} = \frac{1}{\gamma}:$$

so that there are two (supplementary) values of ϖ'. Thus we have the result:—

It is possible (in two different ways) to deform a surface, so that a given curve C traced upon it can be deformed into a given curve C', provided the circular curvature of C' is greater than the geodesic curvature of C on the original surface.

II. The other case arises when

$$\frac{1}{\rho'} = \frac{1}{\gamma}.$$

We must then have $\varpi' = \frac{1}{2}\pi$. Also, when $v=0$, we have

$$\frac{L}{E} = \frac{\cos \varpi'}{\rho'} = 0,$$

and $E=1$, when $v=0$; consequently

$$L=0,$$

and therefore the curve C' is an asymptotic line. Hence, as in the preceding case, we are led to the theorem:—

It is possible to deform a surface so that a given curve C should become an asymptotic line C' in the deformed surface, provided the geodesic curvature of C is equal to the circular curvature of C'.

These results will suffice to indicate the manner, in which the external arbitrary data in Cauchy's existence-theorem concerning integrals of the critical equation can be used to obtain some conditioned deformations of a given surface. Further developments will be found in Darboux's discussion of the subject*.

* See his treatise, vol. iii, especially pp. 253—292.

Deformation of Scrolls.

224. Among the surfaces which can be subjected to deformation, a special interest attaches to scrolls, thereby meaning surfaces which are ruled and are such that their rectilinear generators of any one system do not meet, the surface not being developable. Hitherto, all the deformations under consideration have related to the conservation of the arc-element, so that only the fundamental magnitudes of the first order have been involved; we now proceed to deformations limited, more or less, by fundamental magnitudes of the second order.

Some hints have been given that a ruled surface can be deformed, while each single generator is kept rigidly straight, though these generators are not kept rigidly connected with one another. Accordingly, we first consider those deformations, which allow a ruled surface to be changed into a ruled surface.

Suppose that, if possible, such a deformation exists under which the generators of one surface (being, of course, asymptotic lines of one system, and also geodesics on the surface) do not deform into the generators of the other surface (being also, of course, asymptotic lines of one system, and also geodesics on the surface). Let p be the parametric variable of the unconserved generators on the first surface, and q the parametric variable of the unconserved generators on the second surface; and take p and q as the parametric variables of reference for both surfaces. As the arc-elements of the two surfaces are the same, we have

$$Edp^2 + 2Fdpdq + Gdq^2 = E'dp^2 + 2F'dpdq + G'dq^2,$$

for all variations of p and q; hence

$$E = E', \quad F = F', \quad G = G'.$$

The asymptotic lines of the first surface are, as to one set, given by $p =$ constant, and the general equation is

$$Ldp^2 + 2Mdpdq + Ndq^2 = 0.$$

Hence we have

$$N = 0.$$

The asymptotic lines of the second surface are, as to one set (not being the set of the first surface), given by $q =$ constant, and the general equation is

$$L'dp^2 + 2M'dpdq + N'dq^2 = 0.$$

Hence we have

$$L' = 0.$$

The measure of curvature for the two surfaces is the same; hence

$$L'N' - M'^2 = LN - M^2,$$

that is,

$$M' = \pm M.$$

Next, $p=$ constant is a geodesic on the first surface, because the asymptotic lines of the system are generators; hence

$$\Gamma'' = 0.$$

Again, $q=$ constant is a geodesic on the second surface, for the same reason; hence

$$\Delta = 0.$$

For the first surface, we have $N=0$, $\Gamma''=0$, $\Delta=0$; hence its Mainardi-Codazzi relations are

$$L_2 + \Gamma M = M_1 + \Gamma' L + \Delta' M, \quad \Delta'' M = M_2 + \Gamma' M.$$

For the second surface, we have $L'=0$, $\Gamma''=0$, $\Delta=0$; hence its Mainardi-Codazzi relations are

$$\Gamma M' = M_1' + \Delta' M', \quad N_1' + \Delta'' M' = M_2' + \Gamma' M' + \Delta' N'.$$

When the condition $M' = \pm M$ is used, the first two of these relations give

$$L_2 = \Gamma' L,$$

and the other two give

$$N_1' = \Delta' N'.$$

The quantity Γ' is the same for both surfaces, and the final value of L after deformation (being L') vanishes; hence the former inference leads to

$$L = 0.$$

The quantity Δ' is the same for both surfaces, and the initial value of N' before deformation (being N) vanishes; hence the latter inference leads to

$$N' = 0.$$

Thus there is a third surface which has its asymptotic lines (being geodesics) given by $p=$ constant, $q=$ constant; that is, the surface is a ruled quadric. Each of the two surfaces can be deformed into this quadric, on the hypothesis that the generators of the first do not deform into the generators of the second; and a quadric is the only proper surface with two systems of linear generators, for the intersection of a ruled surface of order n by its tangent plane is composed of a generator and a proper curve of order $n-1$. Hence we have the theorem* :—

When two ruled surfaces are deformable into one another, then either :—

(i) *the system of generators of one of them deforms into the system of generators of the other*; *or*

* It is due to Bonnet, *Journ. de l'Éc. Poly.*, cah. xlii (1867), p. 44.

(ii) *each of them can be deformed into a ruled quadric, the generators of one surface deforming into one set of quadric generators, and those of the other surface deforming into the other set of quadric generators.*

225. We now proceed to consider the more general deformation of ruled surfaces; for this purpose, it is sufficient to obtain the surfaces which have the same arc-element as a ruled surface. The discussion will be restricted to real surfaces.

On a given ruled surface, let a curve C (to be called the directrix) be drawn so as to meet all the generators. The position of a point on the surface is uniquely determined by

(i) the arc of the directrix measured from a fixed point on the curve, say v;

(ii) the direction-cosines (say a, b, c) of the generator of the surface that passes through the point of C;

(iii) the distance (say u) along the generator from the point where it meets C;

and the coordinates of the point on the surface then are

$$x = p + au, \quad y = q + bu, \quad z = r + cu,$$

where p, q, r are the coordinates of the point on C through which the generator passes. In these expressions, the quantities p, q, r, a, b, c are functions of v only. As p', q', r' are the direction-cosines of the tangent to C, we have

$$a^2 + b^2 + c^2 = 1,$$

$$p'^2 + q'^2 + r'^2 = 1,$$

$$ap' + bq' + cr' = \cos\theta = D,$$

where θ is the angle between the tangent to C and the generator; and then, when we write

$$a'^2 + b'^2 + c'^2 = A,$$

$$a'p' + b'q' + c'r' = B,$$

where A, B, D are functions of v alone, the arc-element on the ruled surface is given by

$$ds^2 = du^2 + 2D\,du\,dv + (Au^2 + 2Bu + 1)\,dv^2.$$

Accordingly, all the surfaces into which the ruled surface can be deformed must have this arc-element; and so, for the general equations relating to all surfaces, we have

$$E = 1, \quad F = D, \quad G = Au^2 + 2Bu + 1,$$

$$V^2 = Au^2 + 2Bu + 1 - D^2.$$

Also

$$x_1 = a, \qquad y_1 = b, \qquad z_1 = c,$$

$$x_2 = p' + ua', \quad y_2 = q' + ub', \quad z_2 = r' + uc';$$

and therefore

$$VX = br' - cq' + u(bc' - cb'),$$

$$VY = cp' - ar' + u(ca' - ac'),$$

$$VZ = aq' - bp' + u(ab' - ba').$$

The quantities Γ, Γ', Γ'', Δ, Δ', Δ'', are such that

$$V^2\Gamma = 0,$$

$$V^2\Gamma' = -(Au + B)D,$$

$$V^2\Gamma'' = (D' - Au - B)(Au^2 + 2Bu + 1) - (u^2A' + 2uB')D,$$

$$V^2\Delta = 0,$$

$$V^2\Delta' = Au + B,$$

$$V^2\Delta'' = u^2A' + 2uB' - (D' - Au - B)D.$$

Further, we have

$$x_{11} = 0, \qquad y_{11} = 0, \qquad z_{11} = 0,$$

$$x_{12} = a', \qquad y_{12} = b', \qquad z_{12} = c',$$

$$x_{22} = p'' + ua'', \quad y_{22} = q'' + ub'', \quad z_{22} = r'' + uc'';$$

and therefore

$$L = Xx_{11} + Yy_{11} + Zz_{11} = 0,$$

$$VM = VXx_{12} + VYy_{12} + VZz_{12}$$

$$= \begin{vmatrix} a, & b, & c \\ p', & q', & r' \\ a', & b', & c' \end{vmatrix},$$

$$VN = VXx_{22} + VYy_{22} + VZz_{22}$$

$$= \begin{vmatrix} a \quad , & b \quad , & c \\ p' + ua', & q' + ub', & r' + uc' \\ p'' + ua'', & q'' + ub'', & r'' + uc'' \end{vmatrix}$$

$$= \xi u^2 + 2\eta u + \zeta,$$

where ξ, η, ζ are functions of v alone. Squaring the determinant which gives M, we have

$$V^2M^2 = \begin{vmatrix} 1, & D, & 0 \\ D, & 1, & B \\ 0, & B, & A \end{vmatrix} = A - AD^2 - B^2;$$

and therefore the Gaussian measure of curvature is

$$K = -\frac{M^2}{V^2} = -\frac{1}{V^4}(A - AD^2 - B^2),$$

a result also obtainable from Gauss's characteristic equation by inserting the values

$$m = 0, \quad m' = 0, \qquad m'' = D' - Au - B,$$

$$n = 0, \quad n' = Au + B, \quad n'' = u^2A' + 2uB',$$

$$E_{22} = 0, \quad F_{12} = 0, \qquad G_{11} = 2A.$$

As the general equation of asymptotic lines is

$$Ldu^2 + 2Mdudv + Ndv^2 = 0,$$

and as $L = 0$ for our ruled surface, one system of the asymptotic lines is given by

$$v = \text{constant},$$

that is, by the generators, as is to be expected. The other system of asymptotic lines is given by the equation

$$2Mdu + Ndv = 0,$$

that is, by

$$\frac{du}{dv} = -\frac{N}{2M} = -\frac{V}{2}\frac{\xi u^2 + 2\eta u + \zeta}{(A - AD^2 - B^2)^{\frac{1}{2}}}.$$

As the coefficients of the powers of u on the right-hand side are functions of v alone, the equation is of the Riccati type; its primitive is of the form

$$u = \frac{\lambda\alpha + \beta}{\lambda\gamma + \delta},$$

where λ is an arbitrary constant, and α, β, γ, δ are known functions of v. Accordingly, this is the integral equation of the non-generator family of asymptotic lines; and the members of the family are given by the varying values of the parameter λ.

Take four values λ_1, λ_2, λ_3, λ_4 of λ, thus choosing four non-generator asymptotic lines; and let u_1, u_2, u_3, u_4 be the corresponding values of u. Then we verify at once that

$$\frac{(u_1 - u_2)(u_3 - u_4)}{(u_1 - u_3)(u_2 - u_4)} = \frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)},$$

a constant for the same four lines. Now u is the distance along a generator; and therefore the anharmonic ratio of the four points, where any generator intersects four given non-generator asymptotic lines, is constant.

Line of Striction.

226. Take any generator given by

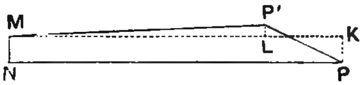
$$x = p + au, \quad y = q + bu, \quad z = r + cu,$$

and a consecutive generator for a consecutive value of v, so that its equations are

$$x = p + p'dv + U(a + a'dv),$$
$$y = q + q'dv + U(b + b'dv),$$
$$z = r + r'dv + U(c + c'dv).$$

Draw the shortest distance between the two. Let u be the distance along the former generator from the directrix curve to the foot of this shortest distance, and $u + du$ the distance along the latter generator for that curve to the other foot of the shortest distance. The direction-cosines λ, μ, ν of the shortest distance are such that

$$\lambda a + \mu b + \nu c = 0,$$
$$\lambda a' + \mu b' + \nu c' = 0,$$



and therefore

$$\frac{\lambda}{b'c - bc'} = \frac{\mu}{c'a - ca'} = \frac{\nu}{a'b - ab'} = \frac{1}{A^{\frac{1}{2}}}.$$

The length $d\sigma$ of this shortest distance is

$$d\sigma = \lambda p'dv + \mu q'dv + \nu r'dv$$
$$= \frac{VM}{A^{\frac{1}{2}}} dv.$$

Let $d\phi$ be the angle between the consecutive generators PN and $P'M$, and let MN be the shortest distance between them. Then, in the figure, we have

$$PP' = dv, \quad MN = KP = d\sigma, \quad ud\phi = P'L, \quad du = LK = -PP'\cos\theta = -dv\cos\theta,$$

and therefore

$$u^2 d\phi^2 + d\sigma^2 + dv^2 \cos^2\theta = dv^2,$$

so that

$$u^2 \left(\frac{d\phi}{dv}\right)^2 = \sin^2\theta - \frac{V^2M^2}{A}$$
$$= 1 - D^2 - \left(1 - D^2 - \frac{B^2}{A}\right) = \frac{B^2}{A}.$$

But, as usual,

$$d\phi^2 = da^2 + db^2 + dc^2$$
$$= Adv^2;$$

hence

$$u^2 = \frac{B^2}{A^2},$$

and therefore

$$u = \pm \frac{B}{A}.$$

The sign must be determined. When we take the spherical image of the generators, the quantities a', b', c' are proportional to the direction-cosines of the spherical arc which makes an obtuse angle with PP', and the direction-cosines of PP' are p', q', r'; hence B, which is equal to $a'p' + b'q' + c'r'$, is negative. The quantity A is necessarily positive; and u is taken positive along the direction a, b, c. Hence*

$$u = -\frac{B}{A}.$$

The limiting position of the foot of the shortest distance between two consecutive generators is called the *centre* of the generator (sometimes, also, the centre of greatest density). The locus of this centre is called the *line of striction* of the ruled surface. Obviously the shortest distance between two consecutive generators is not itself part of the line of striction.

The equation of the line of striction in terms of the parameters of the surface is

$$Au + B = 0.$$

As A is not zero for real surfaces, the line is determinate; and it is the directrix curve, if $B = 0$.

227. We know (§ 105) that the geodesic curvature of any curve $\phi(u, v) = 0$ on a general surface is given by

$$\frac{V}{\gamma} = \frac{\partial}{\partial u}\left(\frac{F\phi_2 - G\phi_1}{\Theta}\right) + \frac{\partial}{\partial v}\left(\frac{F\phi_1 - E\phi_2}{\Theta}\right),$$

where

$$\Theta = (E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2)^{\frac{1}{2}}.$$

Now the directrix curve is given by $u = 0$, so that $\phi_1 = 1$ and $\phi_2 = 0$; hence its geodesic curvature is given by

$$\frac{V}{\gamma} = -\frac{\partial}{\partial u}(G^{\frac{1}{2}}) + \frac{\partial}{\partial v}\left(\frac{F}{G^{\frac{1}{2}}}\right).$$

* The result is obtained, without reference to any figure, by Darboux (t. iii, p. 299) as follows. The shortest distance between two given generators will be obtained by making ds^2, where

$$ds^2 = du^2 + 2D\,du\,dv + (Au^2 + 2Bu + 1)\,dv^2,$$

a minimum when u and du are regarded as variables, while v and dv are fixed. Hence

$$Au + B = 0, \quad du + D\,dv = 0.$$

The former is the required result; the latter is the incidental relation

$$du = -dv \cos\theta.$$

For the ruled surface in general, we have

$$E=1, \quad F=\cos\theta, \quad G=Au^2+2Bu+1,$$
$$V^2=Au^2+2Bu+\sin^2\theta;$$

and therefore

$$\frac{\partial}{\partial u}(G^{\frac{1}{2}})=\frac{Au+B}{G^{\frac{1}{2}}},$$
$$=B,$$

for the directrix curve, and

$$\frac{\partial}{\partial v}\left(\frac{F}{G^{\frac{1}{2}}}\right)=-\frac{\sin\theta}{G^{\frac{1}{2}}}\frac{d\theta}{dv}-\frac{F}{2G^{\frac{3}{2}}}(A'u^2+2B'u)$$
$$=-\sin\theta\frac{d\theta}{dv},$$

for the directrix curve. Hence the geodesic curvature of the directrix curve is given by

$$\frac{1}{\gamma}=-\frac{B}{\sin\theta}-\frac{d\theta}{dv}.$$

Also the curve is chosen arbitrarily, subject to the condition that it intersects the generators; and so we have the theorem:—

When a curve is drawn upon a ruled surface so as to intersect all the generators, and when it has any two of the three properties:—

(i) *that it is a geodesic;*

(ii) *that it is a line of striction;*

(iii) *that it cuts the generators at a constant angle;*

it has the third property also.

228. Next, consider the orthogonal trajectories of the generators. On the surface, the family of generators is given by

$$\delta v=0.$$

The orthogonal trajectory of this family upon any surface in general is given by the equation (§ 26)

$$(Edu+Fdv)\,\delta u=0;$$

and therefore on the ruled surface it is given by

$$du+\cos\theta\, dv=0,$$

that is, as θ is a function of v only, by

$$u+\int\cos\theta\, dv=\text{constant}.$$

Now suppose that one of these orthogonal trajectories is chosen as the directrix curve; we then have

$$\theta=\tfrac{1}{2}\pi,$$

so that

$$F=0, \quad \phi_1=1, \quad \phi_2=0, \quad V^2=Au^2+2Bu+1=G,$$

and so the geodesic curvature of the curve is

$$\frac{1}{\gamma}=-\frac{Au+B}{Au^2+2Bu+1}.$$

Thus the line of striction is the locus of points on the ruled surface where the geodesic curvature of the orthogonal trajectories of the generators vanishes.

229. When an orthogonal trajectory of the generators is chosen as the directrix curve, $\theta=\frac{1}{2}\pi$; the arc-element of the surface is then given by

$$ds^2=du^2+\left(u^2+2\frac{B}{A}u+\frac{1}{A}\right)Adv^2.$$

Let

$$Adv^2=dt^2, \quad -\frac{B}{A}=\alpha, \quad \frac{1}{A}(A-B^2)^{\frac{1}{2}}=\beta;$$

then

$$ds^2=du^2+\{(u-\alpha)^2+\beta^2\}\,dt^2.$$

The centre of the generator is given by $u=\alpha$. Let the normal to the surface at any point on the generator make an angle Ω with the normal to the surface at the centre of the generator. At any current point, we have

$$\left.\begin{aligned} VX&=br'-cq'+u(bc'-cb')\\ VY&=cp'-ar'+u(ca'-ac')\\ VZ&=aq'-bp'+u(ab'-ba')\end{aligned}\right\},$$

while the corresponding quantities at the centre are given by

$$\left.\begin{aligned} V_0X_0&=br'-cq'+\alpha(bc'-cb')\\ V_0Y_0&=cp'-ar'+\alpha(ca'-ac')\\ V_0Z_0&=aq'-bp'+\alpha(ab'-ba')\end{aligned}\right\},$$

where

$$V^2=Au^2+2Bu+1=\frac{(u-\alpha)^2+\beta^2}{\alpha^2+\beta^2},$$

$$V_0=A\alpha^2+2B\alpha+1=\frac{\beta^2}{\alpha^2+\beta^2}.$$

Now

$$\begin{aligned} VV_0\cos\Omega&=XX_0+YY_0+ZZ_0\\ &=\Sigma(br'-cq')^2+(u+\alpha)\Sigma(br'-cq')(bc'-cb')+u\alpha\Sigma(bc'-cb')^2\\ &=1+(u+\alpha)B+u\alpha A\\ &=\frac{\beta^2}{\alpha^2+\beta^2};\end{aligned}$$

hence

$$\cos\Omega = \frac{\beta}{\{(u-\alpha)^2+\beta^2\}^{\frac{1}{2}}},$$

and therefore

$$u - \alpha = \beta \tan\Omega,$$

giving the angle between the normals at the points u and α on the same generator.

Let the tangent plane at the centre of a generator be turned round the generator through an angle ϕ. In its displaced position, it is the tangent plane at a point on the generator given by

$$u_1 - \alpha = \beta \tan\phi;$$

and it is the normal plane at a point on the generator given by

$$u_2 - \alpha = -\beta\cot\phi;$$

thus

$$(u_1-\alpha)(u_2-\alpha)+\beta^2 = 0.$$

Hence any plane through a generator is a tangent plane at some point on the generator and is a normal plane at some other point on the generator; and, for different planes, the two points generate an involution having its centre on the line of striction.

Beltrami's theorem on ruled surfaces.

230. The preceding general properties are necessary to facilitate the discussion of our main question as to how far a ruled surface is determined by a given arc-element.

When the element is given, the quantities A, B, D are known. We then have

$$\left.\begin{aligned} a^2+b^2+c^2 &= 1 \\ a'^2+b'^2+c'^2 &= A \end{aligned}\right\},$$

$$\left.\begin{aligned} p'^2+q'^2+r'^2 &= 1 \\ ap'+bq'+cr' &= D \\ a'p'+b'q'+c'r' &= B \end{aligned}\right\},$$

five equations in all, consistent with one another and satisfied by six functions of v. Hence one of the six functions can be taken arbitrarily, or an arbitrary relation among them can be postulated; hence there is an infinitude of ruled surfaces, which possess an assigned arc-element of the form, proper to a given ruled surface.

Accordingly, let a relation

$$f(a,\ b,\ c) = 0$$

be chosen arbitrarily. This relation, together with the first of the preceding equations, determines b and c as functions of a; when these values are substituted in the second equation, the determination of a as a function of v is a matter of quadrature. Thus we can regard a, b, c as known functions of v.

For the determination of p, q, r, we have

$$\begin{aligned} ap' + bq' + cr' &= D, \\ a'p' + b'q' + c'r' &= B, \\ (bc' - b'c)p' + (ca' - c'a)q' + (ab' - a'b)r' &= -VM \\ &= -(A - AD^2 - B^2)^{\frac{1}{2}} = J, \end{aligned}$$

say, where J has either sign of the radical. When these three equations, linear in p', q', r', are solved, we find

$$p' = Da + \frac{B}{A}a' + \frac{J}{A}(bc' - b'c),$$

$$q' = Db + \frac{B}{A}b' + \frac{J}{A}(ca' - c'a),$$

$$r' = Dc + \frac{B}{A}c' + \frac{J}{A}(ab' - a'b),$$

while the equation

$$p'^2 + q'^2 + r'^2 = 1$$

is satisfied in virtue of the value of J^2. The determination of p, q, r is then, again, a matter of quadrature.

As $a^2 + b^2 + c^2 = 1$, the values of a, b, c give a spherical image of the generators, through the radii of the sphere which are parallel to them; the aggregate of these radii forms a cone which is called the *director cone*. Hence as the equation

$$f(a, b, c) = 0$$

was taken arbitrarily, and as no condition was subsequently imposed on the equation, it follows that the director cone of a ruled surface possessing an assigned arc-element of the proper form can be taken arbitrarily.

Thus there remains a disposable element through which some added external condition can be satisfied.

231. One property of the preceding solution, first rendered significant by Beltrami*, is to be noted. When a, b, c are regarded as known, there are three linear equations for p', q', r'; but the quantity J in those equations can have either sign. Hence, *given a ruled surface, there is another (and different) ruled surface applicable in such a way that the corresponding generators are parallel and in the same sense.*

* *Ann. di Mat.*, t. vii (1865), pp. 139—150.

As an illustration, consider the paraboloid

$$2z = \frac{x^2}{l} - \frac{y^2}{m},$$

(where l and m have the same sign), so as to obtain the associated ruled surface.

The generators can be taken in the form

$$\left.\begin{aligned} x &= \tfrac{1}{2}(l^2 + lm)^{\frac{1}{2}} \tan\theta + u\cos\theta\cos\alpha \\ y &= -\tfrac{1}{2}(m^2 + lm)^{\frac{1}{2}} \tan\theta + u\cos\theta\sin\alpha \\ z &= u\sin\theta \end{aligned}\right\},$$

where

$$\tan\alpha = \left(\frac{m}{l}\right)^{\frac{1}{2}};$$

thus

$$a = \cos\theta\cos\alpha, \quad b = \cos\theta\sin\alpha, \quad c = \sin\theta,$$

$$p = \tfrac{1}{2}(l^2 + lm)^{\frac{1}{2}} \tan\theta, \quad q = -\tfrac{1}{2}(m^2 + lm)^{\frac{1}{2}} \tan\theta, \quad r = 0.$$

As

$$p'^2 + q'^2 + r'^2 = 1,$$

we have

$$\theta' = \frac{2}{l+m}\cos^2\theta,$$

so that

$$p' = \left(\frac{l}{l+m}\right)^{\frac{1}{2}}, \quad q' = -\left(\frac{m}{l+m}\right)^{\frac{1}{2}}, \quad r' = 0.$$

Thus

$$A = \frac{4}{(l+m)^2}\cos^4\theta,$$

$$B = -\frac{2(l-m)}{(l+m)^2}\cos^2\theta\sin\theta,$$

$$D = \frac{l-m}{l+m}\cos\theta;$$

and therefore

$$\frac{B}{A} = -\tfrac{1}{2}(l-m)\frac{\sin\theta}{\cos^2\theta}, \quad \frac{J}{A} = \pm\frac{(lm)^{\frac{1}{2}}}{\cos^2\theta}.$$

Substituting these values, we find

$$p' = \left(\frac{l}{l+m}\right)^{\frac{1}{2}}, \quad q' = -\left(\frac{m}{l+m}\right)^{\frac{1}{2}}, \quad r' = 0,$$

so that

$$p = \tfrac{1}{2}(l^2 + lm)^{\frac{1}{2}} \tan\theta, \quad q = -\tfrac{1}{2}(m^2 + lm)^{\frac{1}{2}} \tan\theta, \quad r = 0,$$

leading to the original surface; or else

$$p' = \left(\frac{l}{l+m}\right)^{\frac{1}{2}} \frac{l-3m}{l+m}, \quad q' = \left(\frac{m}{l+m}\right)^{\frac{1}{2}} \frac{3l-m}{l+m}, \quad r' = 0,$$

so that

$$p=\frac{1}{2}\left(\frac{l}{l+m}\right)^{\frac{1}{2}}(l-3m)\tan\theta,\quad q=\frac{1}{2}\left(\frac{m}{l+m}\right)^{\frac{1}{2}}(3l-m)\tan\theta,\quad r=0.$$

Thus the associated ruled surface is given by

$$\left.\begin{aligned} x&=\frac{1}{2}\left(\frac{l}{l+m}\right)^{\frac{1}{2}}(l-3m)\tan\theta+u\cos\theta\cos\alpha\\ y&=\frac{1}{2}\left(\frac{m}{l+m}\right)^{\frac{1}{2}}(3l-m)\tan\theta+u\cos\theta\sin\alpha\\ z&=u\sin\theta\end{aligned}\right\};$$

its Cartesian equation is

$$z=xy(lm)^{-\frac{1}{2}}-\frac{1}{2}\frac{x^2}{l}\frac{3l-m}{l+m}-\frac{1}{2}\frac{y^2}{m}\frac{l-3m}{l+m},$$

and therefore it is another ruled paraboloid.

Ex. 1. Shew that the ruled surface, which can be applied to the hyperboloid of revolution

$$\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=1,$$

so that corresponding generators are parallel, is the helicoid defined by the equations

$$\left.\begin{aligned}\frac{x}{a}&=\frac{u}{(a^2+c^2)^{\frac{1}{2}}}\cos\frac{v}{a}+\frac{a^2-c^2}{a^2+c^2}\sin\frac{v}{a}\\ \frac{y}{a}&=\frac{u}{(a^2+c^2)^{\frac{1}{2}}}\sin\frac{v}{a}-\frac{a^2-c^2}{a^2+c^2}\cos\frac{v}{a}\\ \frac{z}{c}&=\frac{u}{(a^2+c^2)^{\frac{1}{2}}}+2\frac{a}{(a^2+c^2)^{\frac{1}{2}}}v\end{aligned}\right\}.$$

Ex. 2. Obtain equations of the generators of the cubic scroll

$$x^2z=y^2(y-1),$$

in the form

$$p=v,\qquad q=1,\qquad r=0,$$
$$a=\frac{v^3}{\theta},\qquad b=\frac{v^2}{\theta},\qquad c=\frac{1}{\theta},$$

where

$$\theta=(1+v^4+v^6)^{\frac{1}{2}};$$

and shew that the ruled surface associated with the cubic scroll, so that corresponding generators are parallel, depends upon the equations

$$\frac{p'}{v^6+9v^2-4}=\frac{q'}{12v}=\frac{r'}{-4v^3}=\frac{1}{v^6+9v^2+4}.$$

Ex. 3. Discuss similarly the cubic scrolls*

$$x^2z+y^2w=0,\quad x(yw+xz)+y^3=0,$$

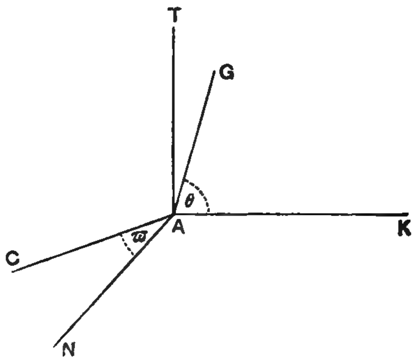
where w is a linear function of x, y, z.

* These equations give the two distinct kinds of cubic scrolls; see Cayley, *Coll. Math. Papers*, vol. v, pp. 211—213. It may be added that there are no undegenerate cubic scrolls of revolution.

Beltrami's method.

232. The preceding method is due to Minding; it leaves an arbitrary element in the solution without giving any clear indication of the way in which the element might be used to satisfy some imposed condition. Another method has been devised by Beltrami*; it pays special regard to the changes in the directrix curve during the deformation of the surface.

In the deformations of the ruled surface which leave it a ruled surface, the arc-element persists unaltered; hence the quantities D, A, B, u are unchanged. The directrix curve will be deformed, though its geodesic curvature is unaltered; let P, Q, R be the point corresponding to p, q, r. In the figure, let AK be the tangent at A (the point P, Q, R) to the directrix curve; KAT the tangent plane, and AG the generator through A making the unaltered angle $\theta \,(= \cos^{-1} D)$ with AK; AN the normal to the surface, and AC the principal normal to the curve, lying in the normal plane NAT.



Let l, m, n be the direction-cosines of AG, while v is the arc measured along the directrix curve from some fixed point; let ρ be the radius of curvature of the directrix curve, and let its direction make an angle ϖ with the principal normal to the surface. Then

$$\rho P''l + \rho Q''m + \rho R''n = \cos\psi,$$

where ψ is the angle between AC and AG; the direction-cosines of AG are $\cos\theta$, 0, $\sin\theta$, while those of AC are 0, $\cos\varpi$, $\sin\varpi$, so that

$$\cos\psi = \sin\theta \sin\varpi.$$

Thus

$$lP'' + mQ'' + nR'' = \sin\theta \frac{\sin\varpi}{\rho}.$$

But

$$lP' + mQ' + nR' = \cos\theta,$$

and therefore

$$\begin{aligned} lP'' + mQ'' + nR'' &= -\theta' \sin\theta - (l'P' + m'Q' + n'R') \\ &= -\theta' \sin\theta - B; \end{aligned}$$

hence

$$\frac{\sin\varpi}{\rho} = -\theta' - \frac{B}{\sin\theta},$$

* References to Minding and to Beltrami have already been given: see p. 354.

in accordance with the result of § 227. The equation expresses the property that the geodesic curvature of the directrix curve remains unaltered.

Let the direction-cosines of AK be $\cos\alpha$, $\cos\beta$, $\cos\gamma$; those of AC be $\cos\xi$, $\cos\eta$, $\cos\zeta$; and those of the binormal to the directrix curve be $\cos\lambda$, $\cos\mu$, $\cos\nu$. As the generator AG makes an angle θ with AK, an angle $\cos^{-1}(\sin\theta\sin\varpi)$ with AC, and an angle $\cos^{-1}(\sin\theta\cos\varpi)$ with the binormal, we have

$$\left.\begin{aligned} l &= \cos\theta\cos\alpha + \sin\theta\sin\varpi\cos\xi + \sin\theta\cos\varpi\cos\lambda \\ m &= \cos\theta\cos\beta + \sin\theta\sin\varpi\cos\eta + \sin\theta\cos\varpi\cos\mu \\ n &= \cos\theta\cos\gamma + \sin\theta\sin\varpi\cos\zeta + \sin\theta\cos\varpi\cos\nu \end{aligned}\right\}.$$

For the directrix curve, the Serret-Frenet formulæ are

$$\begin{aligned} \frac{d\cos\alpha}{dv} &= \frac{\cos\xi}{\rho}, \\ \frac{d\cos\xi}{dv} &= -\frac{\cos\alpha}{\rho} + \frac{\cos\lambda}{\sigma}, \\ \frac{d\cos\lambda}{dv} &= \qquad\quad -\frac{\cos\xi}{\sigma}, \end{aligned}$$

where $1/\sigma$ is the torsion of the directrix curve. We thus have l', m', n'.

Now the quantities A and B are unaltered throughout the deformations, and they are known functions of v. Also θ is a known function of v. But

$$\begin{aligned} B &= l'P' + m'Q' + n'R' = l'\cos\alpha + m'\cos\beta + n'\cos\gamma, \\ A &= l'^2 + m'^2 + n'^2. \end{aligned}$$

When we substitute in the former, we have

$$\frac{\sin\varpi}{\rho} = -\theta' - \frac{B}{\sin\theta},$$

a result already obtained. When we substitute in the latter, and reduce slightly, we find

$$A - B^2 = \left\{\frac{\cos\theta}{\rho} - \frac{\sin\theta\cos\varpi}{\sigma} + \frac{d}{dv}(\sin\theta\sin\varpi)\right\}^2 + \left\{\frac{d}{dv}(\sin\theta\cos\varpi) + \frac{\sin\theta\sin\varpi}{\sigma}\right\}^2.$$

Thus there are two equations involving ϖ, ρ, σ, the other quantities being known functions of v; when ϖ is eliminated between the two equations, a single relation of a form

$$\Psi\left(\frac{d\rho}{dv}, \rho, \sigma, v\right) = 0$$

results. This is an ordinary differential equation, of the first order and of the second degree, which governs any assigned shape for the deformed directrix curve.

When it is satisfied, we can regard ρ and σ (and then ϖ from the first equation) as known functions of v. The Serret-Frenet formulæ would then be satisfied, as the curve is given, so that the direction-cosines of the principal lines are known; and then the values of l, m, n are known.

In that case, the surface can be deformed.

Ex. 1. Consider the possibility of deforming a ruled surface, so that it shall remain ruled and that any curve drawn upon it becomes an asymptotic line.

The curve is, of course, taken as the directrix curve. In its deformed shape, it is an asymptotic line. The osculating plane of the latter is the tangent plane to the surface, so that

$$\varpi = \tfrac{1}{2}\pi ;$$

hence

$$\frac{1}{\rho} = -\theta' - \frac{B}{\sin\theta}.$$

Also

$$A - B^2 = \left(\frac{\cos\theta}{\rho} + \theta'\cos\theta\right)^2 + \frac{\sin^2\theta}{\sigma^2}$$

$$= B^2\cot^2\theta + \frac{\sin^2\theta}{\sigma^2},$$

and therefore

$$\frac{1}{\sigma} = \frac{(A\sin^2\theta - B^2)^{\frac{1}{2}}}{\sin^2\theta}.$$

Thus the ruled surface can be deformed, remaining ruled, so that a given curve upon it can be deformed into an asymptotic line, provided the curvatures of the asymptotic line are given (and therefore the asymptotic line is defined) by the preceding equations, where the quantities θ, A, B are defined in connection with the original given curve on the undeformed surface.

Ex. 2. When the directrix curve is chosen so as to be a geodesic, in all deformations of the surface it remains a geodesic. Hence

$$\frac{\sin\varpi}{\rho} = 0,$$

for its geodesic curvature is zero; hence $\varpi = 0$ for all deformed shapes unless one of them is a straight line when $\frac{1}{\rho} = 0$, while ϖ is not settled by this equation.

In general, therefore, we have

$$\theta' + \frac{B}{\sin\theta} = 0,$$

so that

$$B = D'.$$

Also, in general (so that the deformed geodesic is not a straight line), the final shape of the geodesic is such that

$$A - B^2 = \left(\frac{\cos\theta}{\rho} - \frac{\sin\theta}{\sigma}\right)^2 + \theta'^2\cos^2\theta,$$

that is,

$$\left(\frac{\cos\theta}{\rho} - \frac{\sin\theta}{\sigma}\right)^2 = A - \theta'^2.$$

Ex. 3. Shew that, when a ruled surface is deformed, remaining ruled, while some assigned geodesic becomes a straight line, the equations of the surface in its final form are

$$x = u \sin\theta \cos\psi, \quad y = u \sin\theta \sin\psi, \quad z = v + u\cos\theta,$$

where

$$\psi = \int \frac{(A - \theta'^2)^{\frac{1}{2}}}{\sin\theta} dv.$$

Ex. 4. Determine the condition that must be satisfied by the quantities A, B, θ, in order that a given curve on a ruled surface may, in the deformations which leave it ruled, assume the form of a plane line of curvature.

Other illustrations will be found in Beltrami's memoir and in Darboux's treatise (vol. iii).

Infinitesimal Deformations.

233. When we come to consider the whole aggregate of deformations of surfaces, there are two modes of approaching the subject.

In the first of them, we can have the coordinates of any point on the surfaces expressed in a form

$$x = f(p, q, t), \quad y = g(p, q, t), \quad z = h(p, q, t),$$

where p and q are current variables on a surface, and t is a parameter varying from one surface to another. Should t survive in the eliminant which results from eliminating p and q between the three equations, that eliminant represents a family of surfaces which, for continuous values of t, can change into one another in continuous succession. If the fundamental quantities E, F, G, defined as usual by the relations

$$E = x_1^2 + y_1^2 + z_1^2, \quad F = x_1x_2 + y_1y_2 + z_1z_2, \quad G = x_2^2 + y_2^2 + z_2^2,$$

are independent of t, then any two of the surfaces are applicable to one another; they are usually (but not always*) deformable into one another. In particular, when two surfaces arise from values of t that differ only infinitesimally from one another, each of the surfaces is regarded as an infinitesimal deformation of the other.

In the second of the methods of discussion, the actual infinitesimal displacement of a point on the surface (subject, of course, to the persisting conditions of deformation of surfaces) is considered, rather than the whole body of the surface. We take

$$x' = x + \epsilon X, \quad y' = y + \epsilon Y, \quad z' = z + \epsilon Z,$$

where ϵ is a small constant of negligible square, and X, Y, Z are functions of the current superficial variables; and then any arc-element of the surface is to remain unaltered, either actually, or subject to changes of small quantities

* An exception occurs in the case of Beltrami's associated ruled surfaces which, owing to the difference in the sign of J for the two surfaces (§ 230), are not deformable into one another.

that are of the second or higher orders. Such an instance of infinitesimal deformation is provided by a small rotation of the surface given by

$$x' = x + \epsilon y, \quad y' = y - \epsilon x, \quad z' = z.$$

The question, as to how far the two modes are equivalent, belongs mainly to the theory of continuous groups of transformation. As we are concerned with continuous deformations, rather than with the applicability of surfaces (whether deformed or not), we shall deal with the second mode.

In either case, the essential and sufficient condition of applicability of two surfaces is that the relation

$$dx'^2 + dy'^2 + dz'^2 = dx^2 + dy^2 + dz^2$$

should be satisfied; when the deformations to be considered are infinitesimal, this relation is also essential and sufficient. When the values

$$x' = x + \epsilon X, \quad y' = y + \epsilon Y, \quad z' = z + \epsilon Z$$

are substituted, and when the terms multiplied by ϵ^2 are neglected because the deformation is infinitesimal, then (on the removal of a factor ϵ) we have the equation

$$dxdX + dydY + dzdZ = 0,$$

which is critical for our purpose. It is the resolution of this equation which contains the solution of the problem; and there are various ways of resolving it.

Before proceeding to two of the ways, we may note an interpretation of the equation which shews that our problem is analytically tantamount to another of an apparently quite different kind. The quantities X, Y, Z, being functions of the two parameters in x, y, z, are the coordinates of a point on an (unknown) surface; and dX, dY, dZ determine an arc-element on the unknown surface, just as dx, dy, dz determine an arc-element on the given surface. The critical equation

$$dxdX + dydY + dzdZ = 0$$

expresses the condition that the two arc-elements on the two surfaces are always perpendicular to one another: and so the problem of infinitesimal deformation is analytically equivalent to the problem of determining a surface that is associated with a given surface by means of orthogonal arc-elements*.

* It may be added that a corresponding result applies in the case of deformations which are not infinitesimal. Let two surfaces be deformable into one another, so that

$$dx^2 + dy^2 + dz^2 = dx'^2 + dy'^2 + dz'^2;$$

then, writing

$$X = x + x', \quad Y = y + y', \quad Z = z + z',$$
$$X' = x - x', \quad Y' = y - y', \quad Z' = z - z',$$

the two new surfaces are such that

$$dXdX' + dYdY' + dZdZ' = 0,$$

and therefore they correspond by orthogonal arc-elements. The transformation is the basis of one of Weingarten's methods; see § 237.

234. We have seen that, when a surface

$$z = f(x, y)$$

is referred to x and y as the parametric variables, and when the ordinate of the surface (however deformed) is denoted by Z, the equation for the determination of Z is (§ 217)

$$(RT - S^2)(1 + p^2 + q^2) - (rT + tR - 2sS)(pP + qQ) + (P^2 + Q^2 - 1)(rt - s^2) = 0.$$

Now consider an infinitesimal deviation, represented by

$$Z = z + \epsilon Z',$$

where the square of ϵ can be neglected; denoting the derivatives of Z' by P', Q', R', S', T', we have

$$\begin{aligned} RT - S^2 &= rt - s^2 + \epsilon(rT' + tR' - 2sS'), \\ rT + tR - 2sS &= 2(rt - s^2) + \epsilon(rT' + tR' - 2sS'), \\ pP + qQ &= p^2 + q^2 + \epsilon(pP' + qQ'), \\ P^2 + Q^2 - 1 &= p^2 + q^2 - 1 + 2\epsilon(pP' + qQ'). \end{aligned}$$

The critical equation, after substitution, rejection of cancelling terms, and division by ϵ, gives

$$rT' + tR' - 2sS' = 0.$$

235. The preceding infinitesimal deformation gives the variation of an ordinate alone. Consider a more general infinitesimal deformation, represented by

$$Z = z + \epsilon Z'', \quad X = x + \epsilon X', \quad Y = y + \epsilon Y',$$

governed by the critical relation

$$dx dX' + dy dY' + dz dZ'' = 0.$$

The quantity Z'' can be taken the same function of X and Y, as Z' is of x and y; as Z'' is multiplied by ϵ, while X and Y differ from x and from y by small quantities, we can substitute Z' for Z'' in the expression for Z. Thus Z' is determined by the equation

$$rT' + tR' - 2sS' = 0;$$

the infinitesimal deformation is represented by

$$Z = z + \epsilon Z', \quad X = x + \epsilon X', \quad Y = y + \epsilon Y',$$

while we have

$$dx dX' + dy dY' + dz dZ' = 0.$$

The latter gives

$$dx\left(\frac{\partial X'}{\partial x} dx + \frac{\partial X'}{\partial y} dy\right) + dy\left(\frac{\partial Y'}{\partial x} dx + \frac{\partial Y'}{\partial y} dy\right) + (p dx + q dy)(P' dx + Q' dy) = 0,$$

for all variations of x and y; hence

$$\frac{\partial X'}{\partial x} + pP' = 0,$$

$$\frac{\partial X'}{\partial y} + \frac{\partial Y'}{\partial x} + (pQ' + qP') = 0,$$

$$\frac{\partial Y'}{\partial y} + qQ' = 0,$$

three equations to be satisfied by X' and Y', when Z' is known. Let

$$\frac{\partial X'}{\partial y} + pQ' = U, \quad \frac{\partial Y'}{\partial x} + qP' = -U.$$

Then

$$\frac{\partial U}{\partial x} = \frac{\partial^2 X'}{\partial x \partial y} + pS' + rQ',$$

while, from the first,

$$0 = \frac{\partial^2 X'}{\partial x \partial y} + pS' + sP',$$

so that

$$\frac{\partial U}{\partial x} = rQ' - sP'.$$

Again,

$$-\frac{\partial U}{\partial y} = \frac{\partial^2 Y'}{\partial x \partial y} + qS' + tP',$$

while, from the third,

$$0 = \frac{\partial^2 Y'}{\partial x \partial y} + qS' + sQ',$$

so that

$$\frac{\partial U}{\partial y} = sQ' - tP'.$$

Hence

$$\frac{\partial}{\partial y}(rQ' - sP') = \frac{\partial}{\partial x}(sQ' - tP'),$$

leading to

$$rT' - 2sS' + tR' = 0,$$

the equation* which is satisfied by Z'. Consequently, when Z' is known, we determine X' (save as to an additive function of y) and Y' (save as to an additive function of x) from the equations

$$\frac{\partial X'}{\partial x} + pP' = 0, \quad \frac{\partial Y'}{\partial y} + qQ' = 0;$$

and then these arbitrary functions must be such as to satisfy the equations

$$\begin{aligned} \frac{\partial X'}{\partial y} + pQ' &= -\left(\frac{\partial Y'}{\partial x} + qP'\right) \\ &= U \\ &= \int \{Q'(r\,dx + s\,dy) - P'(s\,dx + t\,dy)\} \\ &= \int (Q'dp - P'dq). \end{aligned}$$

* This analysis is another establishment of the equation for Z' deduced from the equation $dx\,dX + dy\,dY + dz\,dZ = 0$.

Ex. 1. Consider the infinitesimal deformations of the paraboloid of revolution

$$2z=x^2+y^2.$$

We have

$$p=x,\quad q=y,\quad r=1,\quad s=0,\quad t=1.$$

The equation for Z' is

$$R'+T'=0,$$

so that

$$Z'=\phi'(x+iy)+\psi'(x-iy),$$

where ϕ and ψ are arbitrary functions of their arguments, and their derivatives are taken in the expression for Z' to avoid transformations needed to effect subsequent quadratures. Then

$$\begin{aligned}-X'&=x\phi'(x+iy)-\phi(x+iy)+x\psi'(x-iy)-\psi(x-iy)+A(y),\\ -Y'&=y\phi'(x+iy)+i\phi(x+iy)+y\psi'(x-iy)-i\psi(x-iy)+B(x),\\ U&=i\phi'(x+iy)-i\psi'(x-iy),\end{aligned}$$

where $A(y)$ and $B(x)$ are arbitrary functions of their arguments. When we substitute, we find

$$A'(y)=0,\quad B'(x)=0,$$

so that A is a constant, as is B. These quantities occurring in X' and Y' merely give an infinitesimal uniform displacement of the surface perpendicular to the axis of revolution. Neglecting this displacement, we have the infinitesimal deformation given by the equations

$$\begin{aligned}-X'&=x\{\phi'(x+iy)+\psi'(x-iy)\}-\{\phi(x+iy)+\psi(x-iy)\},\\ -Y'&=y\{\phi'(x+iy)+\psi'(x-iy)\}+i\{\phi(x+iy)-\psi(x-iy)\},\\ Z'&=\phi'(x+iy)+\psi'(x-iy),\end{aligned}$$

where ϕ and ψ are arbitrary functions of their arguments.

Ex. 2. Shew that the infinitesimal deformations of the paraboloid

$$z=xy$$

are given by

$$\begin{aligned}Z&=z+\epsilon(\xi'+\eta'),\\ X&=x+\epsilon\{2\eta-y(\xi'+\eta')\},\\ Y&=y+\epsilon\{2\xi-x(\xi'+\eta')\},\end{aligned}$$

where ξ is any function of x, and η is any function of y.

236. In various investigations, we have seen that it can be convenient to refer a surface to its nul lines as parametric curves, the arc-element being given by the relation

$$d\sigma^2=4\lambda\, du\, dv.$$

Then, denoting the derivatives of z with respect to u and v by p, q, r, s, t, we obtained the equation characteristic of any deformed z in the shape (§ 217)

$$rt-s^2-\frac{\lambda_2}{\lambda}qr-\frac{\lambda_1}{\lambda}pt=(\lambda-pq)\,2\frac{\partial^2\log\lambda}{\partial u\,\partial v}-\frac{\lambda_1\lambda_2}{\lambda^2}pq.$$

We proceed, as in § 234, to obtain the equations for the infinitesimal deformation of the surface. Writing

$$x=x_0+\epsilon X',\quad y=y_0+\epsilon Y',\quad z=z_0+\epsilon Z',$$

where x_0, y_0, z_0 are the coordinates of the point on the undeformed surface and the square of ϵ is to be neglected, we denote the derivatives of Z' with respect to u and v by P', Q', R', S', T', and similarly for the derivatives of z_0

Substituting, omitting the terms which cancel, and removing a factor ϵ, we have the equation for Z' in the form

$$t_0 R' - 2s_0 S' + r_0 T' - \frac{\lambda_2}{\lambda}(q_0 R' + r_0 Q') - \frac{\lambda_1}{\lambda}(p_0 T' + t_0 P') + (q_0 P' + p_0 Q')\left(2\frac{\partial^2 \log \lambda}{\partial u \partial v} + \frac{\lambda_1 \lambda_2}{\lambda^2}\right) = 0.$$

The governing equation

$$dx_0 dX' + dy_0 dY' + dz_0 dZ' = 0,$$

on the supposition that Z' is known, thus gives the (consistent) equations for X' and Y' in the form

$$\left.\begin{aligned} \frac{\partial x_0}{\partial u}\frac{\partial X'}{\partial u} + \frac{\partial y_0}{\partial u}\frac{\partial Y'}{\partial u} + p_0 P' &= 0 \\ \frac{\partial x_0}{\partial v}\frac{\partial X'}{\partial u} + \frac{\partial y_0}{\partial v}\frac{\partial Y'}{\partial u} + q_0 P' &= U \\ \frac{\partial x_0}{\partial u}\frac{\partial X'}{\partial v} + \frac{\partial y_0}{\partial u}\frac{\partial Y'}{\partial v} + p_0 Q' &= -U \\ \frac{\partial x_0}{\partial v}\frac{\partial X'}{\partial v} + \frac{\partial y_0}{\partial v}\frac{\partial Y'}{\partial v} + q_0 Q' &= 0 \end{aligned}\right\}.$$

These are the general equations for the infinitesimal deformation of the surface. The result depends primarily on the integration of the Monge equation of the second order, whatever be the surface.

As an illustration, take the case of the general minimal surface; we have (§ 174)

$$\lambda = \tfrac{1}{4}(1 + uv)^2 f''' g''',$$

where f is any function of u only, while g is any function of v only. Also

$$z_0 = uf'' - f' + vg'' - g',$$

so that

$$p_0 = uf''', \quad q_0 = vg''',$$
$$r_0 = uf'''' + f''', \quad s_0 = 0, \quad t_0 = vg'''' + g''',$$
$$\frac{\lambda_1}{\lambda} = \frac{2v}{1 + uv} + \frac{f''''}{f'''}, \quad \frac{\lambda_2}{\lambda} = \frac{2u}{1 + uv} + \frac{g''''}{g'''}.$$

When these values are substituted and reduction takes place, the equations for the infinitesimal deformation become

$$g''' R' + f''' T' + g'''\left(\frac{2v}{1 - uv} - \frac{f''''}{f'''}\right) P' + f'''\left(\frac{2u}{1 - uv} - \frac{g''''}{g'''}\right) Q' = 0,$$

$$\left.\begin{aligned} (1 - u^2)\frac{\partial X'}{\partial u} + i(1 + u^2)\frac{\partial Y'}{\partial u} + 2uP' &= 0 \\ (1 - v^2)\frac{\partial X'}{\partial u} - i(1 + v^2)\frac{\partial Y'}{\partial u} + 2vP' &= \frac{2U}{g'''} \\ (1 - u^2)\frac{\partial X'}{\partial v} + i(1 + u^2)\frac{\partial Y'}{\partial v} + 2uQ' &= -\frac{2U}{f'''} \\ (1 - v^2)\frac{\partial X'}{\partial v} - i(1 + v^2)\frac{\partial Y'}{\partial v} + 2vQ' &= 0 \end{aligned}\right\},$$

the quantity U ultimately dropping out of any particular solution, when it has been completed.

Hence, in order to obtain the full expression of the infinitesimal deformation of a minimal surface in general, it is necessary to solve the foregoing partial differential equation which is of the Monge type in the second order*, as well as equations of the first order ultimately integrable by quadratures alone.

Ex. Thus, for Enneper's surface (§ 177, Ex. 1), we have

$$f=u^3, \quad g=v^3;$$

and so the equation for Z' is

$$R'+T'+2\frac{vP'+uQ'}{1-uv}=0,$$

of which a particular solution is

$$Z'=\kappa(1-uv)^3,$$

where κ is a constant. When we introduce new variables u' and v', such that

$$u'=u+iv, \quad v'=u-iv,$$

the differential equation becomes a Laplace equation, with equal invariants†; it can be expressed in the form

$$\frac{\partial^2 Z''}{\partial u'\partial v'}=\frac{8u'v'}{\{4-i(v'^2-u'^2)\}^2}Z'',$$

where

$$Z'=\{4-i(v'^2-u'^2)\}Z''.$$

Weingarten's Method.

237. We now come to Weingarten's method‡. He discusses, not merely the surfaces into which a given surface can be deformed but also two other surfaces which, at each stage of the deformation, can be associated with the surface.

Suppose that S' and S'' are two surfaces in space, such that S' can be deformed continuously into S''. Let x', y', z' be any point on S'; and let x'', y'', z'' be the corresponding point on S''. The necessary and sufficient condition for the deformational correspondence of the surfaces is that the relation

$$dx'^2+dy'^2+dz'^2=dx''^2+dy''^2+dz''^2$$

shall be satisfied everywhere for all variations along the surfaces.

* See Darboux, vol. iv, §§ 913—915.

† This is a special illustration of the general theory: see Darboux, vol. iv, ch. ii.

‡ Only an elementary sketch will be given here. His chief memoirs have already been mentioned (p. 354). References have already (*l.c.*) been given to the accounts and developments of his investigations which are to be found in the treatises by Darboux and by Bianchi.

Now let

$$x=\tfrac{1}{2}(x'+x''),\qquad y=\tfrac{1}{2}(y'+y''),\qquad z=\tfrac{1}{2}(z'+z''),$$
$$u=\tfrac{1}{2}(-x'+x''),\quad v=\tfrac{1}{2}(-y'+y''),\quad w=\tfrac{1}{2}(-z'+z'');$$

the point x, y, z describes a surface S, which can be regarded as a middle between S' and S''. This surface S is made the clue to the whole development. Clearly

$$dxdu+dydv+dzdw=0;$$

and so, with p and q as the parametric variables, we have

$$x_1u_1+y_1v_1+z_1w_1=0,$$
$$x_1u_2+x_2u_1+y_1v_2+y_2v_1+z_1w_2+z_2w_1=0,$$
$$x_2u_2+y_2v_2+z_2w_2=0.$$

We use the customary notation for the magnitudes of the first order and the second order connected with the surface S; and, for the purpose of dealing with these equations, we introduce a central function ϕ under the definition

$$2V\phi=x_1u_2+y_1v_2+z_1w_2-(x_2u_1+y_2v_1+z_2w_1).$$

Then the foregoing equations are

$$\left.\begin{aligned} x_1u_1+y_1v_1+z_1w_1&=0\\ x_1u_2+y_1v_2+z_1w_2&=V\phi\\ x_2u_1+y_2v_1+z_2w_1&=-V\phi\\ x_2u_2+y_2v_2+z_2w_2&=0\end{aligned}\right\}.$$

From the first of these, we have

$$\Sigma x_1u_{12}+\Sigma x_{12}u_1=0;$$

and, from the second, we have

$$\Sigma x_1u_{12}+\Sigma x_{11}u_2=\frac{\partial}{\partial p}(V\phi),$$

so that

$$\frac{\partial}{\partial p}(V\phi)=\Sigma u_2x_{11}-\Sigma u_1x_{12}.$$

Similarly, from the third and the fourth,

$$\frac{\partial}{\partial q}(V\phi)=\Sigma u_2x_{12}-\Sigma u_1x_{22}.$$

Also, as usual (§ 34), we have

$$x_{11}=x_1\Gamma\ \ +x_2\Delta\ \ +LX,$$
$$x_{12}=x_1\Gamma'\ +x_2\Delta'\ +MX,$$
$$x_{22}=x_1\Gamma''+x_2\Delta''+NX,$$

and similarly for the derivatives of y and of z; all the coefficients concerned belonging to the middle surface S. Then substituting, we find

$$\frac{\partial}{\partial p}(V\phi)=V\phi\Gamma+L\Sigma Xu_2+V\phi\Delta'-M\Sigma Xu_1.$$

But we have

$$\frac{V_1}{V} = \Gamma + \Delta';$$

and therefore

$$\phi_1 = \frac{1}{V}(L\Sigma X u_2 - M\Sigma X u_1).$$

Similarly

$$\phi_2 = \frac{1}{V}(M\Sigma X u_2 - N\Sigma X u_1).$$

238. Two cases occur, according as the Gaussian measure of curvature of our middle surface S is zero or is not zero.

When the Gaussian measure of curvature is zero everywhere, then

$$LN - M^2 = 0;$$

and so we have

$$N\phi_1 - M\phi_2 = 0, \quad -M\phi_1 + L\phi_2 = 0,$$

equivalent to only a single equation. *The central function ϕ satisfies a partial differential equation of the first order.*

When the Gaussian measure of curvature is not zero everywhere, the two equations can be resolved; they give

$$\Sigma X u_1 = -\frac{1}{KV}(L\phi_2 - M\phi_1),$$

$$\Sigma X u_2 = -\frac{1}{KV}(M\phi_2 - N\phi_1).$$

Hence

$$\frac{\partial}{\partial p}\left(\frac{N\phi_1 - M\phi_2}{KV}\right) + \frac{\partial}{\partial q}\left(\frac{L\phi_2 - M\phi_1}{KV}\right) = -\Sigma X_2 u_1 + \Sigma X_1 u_2.$$

Now (§ 29)

$$V^2 X_1 = (FM - GL)\ x_1 + (FL - EM)\, x_2,$$

$$V^2 X_2 = (FN - GM)\, x_1 + (FM - EN)\, x_2,$$

and similarly for the derivatives of Y and of Z; hence

$$V^2 \Sigma X_2 u_1 = -(FM - EN)\, V\phi,$$

$$V^2 \Sigma X_1 u_2 = \quad (FM - GL)\, V\phi,$$

and therefore

$$\frac{\partial}{\partial p}\left(\frac{N\phi_1 - M\phi_2}{KV}\right) + \frac{\partial}{\partial q}\left(\frac{L\phi_2 - M\phi_1}{KV}\right) + \frac{EN - 2FM + GL}{V}\phi = 0.$$

The central function ϕ satisfies a partial differential equation of the second order and the Monge type.

As regards the quantities u, v, w, we have

$$x_1 u_1 + y_1 v_1 + z_1 w_1 = 0,$$

$$x_2 u_1 + y_2 v_1 + z_2 w_1 = -\phi V,$$

$$X u_1 + Y v_1 + Z w_1 = -\frac{1}{KV}(L\phi_2 - M\phi_1),$$

which can be solved for u_1, v_1, w_1. There is a similar set for u_2, v_2, w_2. The resulting values are easily found to be

$$\left.\begin{aligned}
u_1 &= \frac{1}{KV}\{-L(X\phi_2 - \phi X_2) + M(X\phi_1 - \phi X_1)\}\\
v_1 &= \frac{1}{KV}\{-L(Y\phi_2 - \phi Y_2) + M(Y\phi_1 - \phi Y_1)\}\\
w_1 &= \frac{1}{KV}\{-L(Z\phi_2 - \phi Z_2) + M(Z\phi_1 - \phi Z_1)\}\\
u_2 &= \frac{1}{KV}\{-M(X\phi_2 - \phi X_2) + N(X\phi_1 - \phi X_1)\}\\
v_2 &= \frac{1}{KV}\{-M(Y\phi_2 - \phi Y_2) + N(Y\phi_1 - \phi Y_1)\}\\
w_2 &= \frac{1}{KV}\{-M(Z\phi_2 - \phi Z_2) + N(Z\phi_1 - \phi Z_1)\}
\end{aligned}\right\}.$$

Thus, when ϕ is known, the determination of the quantities u, v, w is effected by a process of quadratures.

We therefore have the following result:—

Take any surface S and, in connection with it, determine a function ϕ, by the appropriate partial equation of the first order when S is developable, and by the appropriate partial equation of the second order when S is not developable; and construct the quantities u, v, w. Then there are two surfaces S' and S'', given by the equations

$$x'' = x + u, \quad y'' = y + v, \quad z'' = z + w,$$

$$x' = x - u, \quad y' = y - v, \quad z' = z - w,$$

such that each can be deformed into the other.

Ex. 1. Let the surface be referred to its asymptotic lines as parametric curves, so that

$$L=0, \quad N=0.$$

We do not then have $M=0$, so that the function ϕ satisfies a partial equation of the second order. This equation is easily found to be

$$\phi_{12} - \frac{1}{4}\left(\frac{K_1}{K}\phi_2 + \frac{K_2}{K}\phi_1\right) + FK\phi = 0,$$

a Laplace equation with equal invariants which, on the transformation

$$\phi = \Phi K^{\frac{1}{4}},$$

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acquires the canonical form

$$\Phi_{12}+\Theta\Phi=0,$$

where

$$\Theta=\frac{K_{12}}{4K}-\frac{5}{16}\frac{K_1K_2}{K^2}+FK.$$

This is the simplest form of the partial equation of the second order which, ultimately in some form or other, must be solved before the central function ϕ can be determined.

Ex. 2. Shew that, if the parametric curves on the surface S are such that

$$L=N, \quad M=0,$$

the equation for the function ϕ can be transformed to

$$\Phi_{11}+\Phi_{22}=\Theta'\Phi,$$

where

$$\phi=\Phi K^{\frac{1}{4}},$$

and Θ' depends solely upon the surface.

239. Consider, under the new analysis, a question already (§ 222) discussed; can a surface be deformed while some curve upon it remains rigidly fixed?

As the parametric variables are at our disposal, again choose them so that one of them is constant along the rigid curve, say $p=a$. Now along this curve, we have

$$x''=x', \quad y''=y', \quad z''=z',$$

that is,

$$u=0, \quad v=0, \quad w=0,$$

when $p=a$; and therefore also

$$u_2=0, \quad v_2=0, \quad w_2=0,$$

when $p=a$. We shall assume that we are dealing with real surfaces throughout; the assumption, that V does not vanish, then is no limitation.

We have

$$x_1u_2+y_1v_2+z_1w_2=V\phi,$$

in general; hence we have

$$\phi=0, \quad \phi_2=0, \quad \phi_{22}=0, \ \ldots,$$

when $p=a$. Again, we have

$$\Sigma Xu_2=-\frac{1}{KV}(M\phi_2-N\phi_1),$$

in general; and therefore, if the middle surface S is not developable, we have

$$N\phi_1=0,$$

when $p=a$.

Suppose that N does not vanish (the influence of the alternative to each of the suppositions can be considered later); so we have

$$\phi_1=0,$$

when $p=a$, and therefore

$$\phi_{12}=0, \ \ldots,$$

when $p=a$.

The differential equation satisfied by ϕ is

$$\frac{\partial}{\partial p}\left(\frac{N\phi_1 - M\phi_2}{V}\right) + \frac{\partial}{\partial q}\left(\frac{L\phi_2 - M\phi_1}{V}\right) + \frac{EN - 2FM + GL}{V}\phi = 0$$

in general; hence

$$\phi_{11}=0,$$

when $p=a$, and so all the q-derivatives of ϕ_{11} vanish when $p=a$. Differentiate the general characteristic differential equation with respect to p, and then make $p=a$; we have

$$\phi_{111}=0,$$

when $p=a$, and so all the q-derivatives of ϕ_{111} vanish when $p=a$.

Similarly for all the derivatives of ϕ when $p=a$—each of them vanishes. Taking a Taylor expansion of ϕ in any non-singular region round the rigid curve, we see that ϕ vanishes everywhere in such a region. Hence, when regard is paid to the expressions for the derivatives for u, v, w, and to the fact that u, v, w vanish when $p=a$, it is clear that

$$u=0, \quad v=0, \quad w=0,$$

everywhere in the region. Consequently

$$x''=x', \quad y''=y', \quad z''=z',$$

everywhere; and so there is no deformation between S' and S''. It therefore follows that *usually a surface cannot be deformed while a curve upon it is kept rigid.*

240. The negative conditions, under which the preceding result has been obtained, are three. It was assumed that the surface S is real—the quantity V was supposed not to vanish; the surface S was assumed not to be developable; and the magnitude N was supposed not to vanish.

We shall maintain the first condition. If $V=0$, the two nul lines through a point coincide; so that not merely would the surface be imaginary, but it would belong to a very special class of imaginary surfaces.

Consider the possibility that N should vanish; in that case, we should have (along $p=a$)

$$Xx_{22} + Yy_{22} + Zz_{22} = 0.$$

And we always have

$$Xx_2 + Yy_2 + Zz_2 = 0;$$

hence

$$X : Y : Z = y_2 z_{22} - z_2 y_{22} : z_2 x_{22} - x_2 z_{22} : x_2 y_{22} - y_2 x_{22}$$
$$= y'z'' - z'y'' : z'x'' - x'z'' : x'y'' - y'x'',$$

where dashes denote differentiation along the curve $p=a$. When this curve is not a straight line, it has a definite direction for its binormal; hence our

assumption implies that the normal to the surface coincides with the binormal to the rigid curve along the whole length of the curve. Now when N is zero, we cannot infer that $\phi_1 = 0$ along the curve, and so we should have a function ϕ satisfying the equation of the second order, vanishing along the rigid curve, but not vanishing everywhere on the surface. Thus the analytical condition $N = 0$ is necessary for the conclusion.

When the surface S is developable, the function ϕ satisfies a partial equation of the first order

$$N\phi_1 - M\phi_2 = 0.$$

An argument similar to the earlier argument shews that, if N is not zero, the function ϕ, which satisfies this equation and vanishes along the rigid curve, vanishes everywhere; and then there is no deformation. Thus the condition is unnecessary, when N is not zero; and the latter condition has just been retained.

Summing up, we have Weingarten's theorems*:—

When two surfaces deformable into one another coincide along a curve, which is not a straight line and the points of which are self-congruent in any deformation, the whole surfaces coincide unless the normals to the middle surface of the two surfaces constitute the binormals of the common curve.

When the measure of curvature of a real surface is everywhere positive, N cannot vanish; and so the exception cannot arise. Hence:—

Surfaces of a positive measure of curvature cannot be deformed, if a curve or part of a curve (not being a straight line) on the surface is kept rigid.

Also:—

Surfaces of a negative measure of curvature cannot be deformed, if a curve or part of a curve other than an asymptotic line on the surface is kept rigid.

If the curve, which is to be kept rigid, is an asymptotic line on the surface, we can have $N = 0$; we may have $\phi = 0$ along the line, where ϕ satisfies the equation of the second order, and yet we may have ϕ different from zero elsewhere. The surface may be deformable. The definite establishment of a theorem, that it is deformable, would require the derivation, from the equations, of the set of deformable surfaces which have their Gaussian measure of curvature everywhere negative and which possess one asymptotic line in common.

For further developments of the subject, reference should be made to the memoirs by Weingarten.

* *Crelle*, t. c (1887), p. 307; the earliest (but only partial) establishment of the second of them was made by Jellett.

EXAMPLES.

1. Shew that it is possible to deform a surface so that a given curve becomes a line of curvature on the deformed surface. Are there any conditions to be satisfied?

2. Shew that a surface cannot be deformed so that a whole system of asymptotic lines remains asymptotic, unless it is a ruled surface of which the asymptotic lines are generators.

3. Prove that the Gaussian measure of curvature of a ruled surface is greater at the line of striction than elsewhere along a generator.

4. Spheres are drawn according to any law, which makes the centres lie upon a surface and their radii a function of the position of the centre upon the surface; and their envelope is formed. Shew that the normals to the spheres at the points of contact with their envelope remain rigidly connected with the surface on which their centres lie when this surface is deformed.

5. Shew that a scroll can always be deformed into another scroll so as to make the generators of the first become the principal normals of any one of their orthogonal trajectories.

6. Shew that, if a scroll can be deformed into another scroll so that its generators become the principal normals of two of their orthogonal trajectories, the equation

$$\alpha^2+\beta^2+2a\alpha+2b\beta+c=0,$$

(where a, b, c are constants, and α, β are the magnitudes of § 229), must be satisfied.

7. Shew that when a hyperboloid of revolution of one sheet is deformed, while its generators remain rectilinear, its principal circular section becomes a (Bertrand) curve such that

$$\frac{1}{\rho}=\lambda\left(\frac{1}{k}-\frac{1}{\sigma}\right),$$

where λ and k are constants.

8. Prove that the only real ruled surfaces, which can be deformed into surfaces of revolution, are the one-sheeted hyperboloid of revolution and the minimal helicoid.

9. Prove that a pseudo-sphere can be deformed in an unlimited number of ways so as to leave an asymptotic line rigid and to conserve the principal radii of curvature along the line; and that it can be deformed in one way so that any two lines through a point on the surface become asymptotic lines for the deformed surface.

10. A given surface can be deformed into a ruled surface, a family of geodesics becoming the generators. At the points where this family meets an asymptotic line, the rectilinear tangents to the geodesic are drawn; prove that they generate a ruled surface into which the given surface can be deformed.

CHAPTER XI.

Triply Orthogonal Systems of Surfaces.

The present chapter is devoted to triply orthogonal systems in ordinary space. No account will be taken of multiply orthogonal systems in space of more than three dimensions.

The first important theorem—that the intersections of three triply orthogonal surfaces are lines of curvature on the surfaces—was obtained by Dupin in 1813. Later, the subject attracted the attention of a multitude of mathematicians, among whom particular mention of Lamé should be made; the theory of curvilinear coordinates in space, and a large body of developed results, owe their origin to him. Later, in 1846, it was pointed out by Bouquet that any arbitrarily chosen surface cannot belong to a triply orthogonal system. In 1862, Bonnet had shewn that the determination of such a system must depend upon a partial differential equation of the third order; and this equation of the third order was first obtained by Cayley in 1872. Soon there followed the researches of Darboux, on orthogonal systems, as on so many parts of differential geometry; and many workers, among whom Bianchi may be specially named, have laboured in the field.

It is unnecessary to set out detailed references to the many memoirs that are concerned with the subject. The reader, who wishes to obtain a comprehensive grasp of the theory, must refer to Darboux's treatise *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*, published (in its completed form) in 1910. He will there find a systematic exposition of the theory, which deals with all the important matters and includes many of the latest developments. In that work, ample references to the original memoirs are given.

Curvilinear coordinates in space; fundamental magnitudes.

241. Just as a point on a surface can be determined by two variables which are taken as the parameters of two families of curves on the surface, so a point in ordinary space can be determined by three variables which are the parameters of three families of surfaces in the space. We shall assume that, in any region which will be considered, the surfaces are uniform, regular, and free from singularities; hence through any ordinary point of space there will pass three surfaces, one (and only one) belonging to each of the three families.

We shall denote* by u, v, w the parameters of the three surfaces, so that these are given by

$$u(x, y, z) = u, \quad v(x, y, z) = v, \quad w(x, y, z) = w.$$

The Jacobian of u, v, w with respect to x, y, z is not identically zero; and we do not consider regions of space where, at points or along lines connected with any of the surfaces, the Jacobian might happen to vanish (though not identically) or to become infinite. The surface w = constant will contain two families of curves, given as its intersections with the family of surfaces u = constant and the family v = constant; thus u and v can be taken as the parametric variables for the representation of points on a w-surface. And so for a u-surface and a v-surface.

When we suppose the variables x, y, z expressed in terms of u, v, w, we take them in a form

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w).$$

Naturally we have the same excluding suppositions about the Jacobian of x, y, z with respect to u, v, w as about the former Jacobian, the product of the two being unity.

We require derivatives with respect to u, v, w, and also derivatives with respect to x, y, z. We write

$$\frac{\partial u}{\partial x} = u_1, \quad \frac{\partial u}{\partial y} = u_2, \quad \frac{\partial u}{\partial z} = u_3, \quad \frac{\partial^2 u}{\partial x^2} = u_{11}, \ \ldots,$$

and so on, with a similar notation for the derivatives of v and of w; and we also write (no confusion need be caused by the identity of the suffixes)

$$\frac{\partial x}{\partial u} = x_1, \quad \frac{\partial x}{\partial v} = x_2, \quad \frac{\partial x}{\partial w} = x_3, \quad \frac{\partial^2 x}{\partial u^2} = x_{11}, \ \ldots,$$

and so on, with a similar notation for the derivatives of y and of z.

Three quantities h_1, h_2, h_3 are introduced under the definitions

$$\left.\begin{aligned} u_1^2 + u_2^2 + u_3^2 &= h_1^2 \\ v_1^2 + v_2^2 + v_3^2 &= h_2^2 \\ w_1^2 + w_2^2 + w_3^2 &= h_3^2 \end{aligned}\right\}.$$

Moreover, it is customary to assume that the three families of surfaces are everywhere orthogonal to one another; and so we have

$$\begin{aligned} u_1 v_1 + u_2 v_2 + u_3 v_3 &= 0, \\ v_1 w_1 + v_2 w_2 + v_3 w_3 &= 0, \\ w_1 u_1 + w_2 u_2 + w_3 u_3 &= 0. \end{aligned}$$

* The notations are very varied. In addition to u, v, w, the quantities ρ, ρ_1, ρ_2 are used (by Lamé, and Darboux); others (*e.g.* Bianchi) use ρ_1, ρ_2, ρ_3; Cayley and Salmon have used p, q, r; and not a few writers use α, β, γ.

Now, as u, v, w are independent functions of three independent variables x, y, z, we have

$$u_1 x_1 + u_2 y_1 + u_3 z_1 = 1,$$

$$v_1 x_1 + v_2 y_1 + v_3 z_1 = 0,$$

$$w_1 x_1 + w_2 y_1 + w_3 z_1 = 0;$$

and therefore, from the second and third of these, we have

$$\frac{x_1}{v_2 w_3 - v_3 w_2} = \frac{y_1}{v_3 w_1 - v_1 w_3} = \frac{z_1}{v_1 w_2 - v_2 w_1}.$$

But

$$v_1 u_1 + v_2 u_2 + v_3 u_3 = 0, \quad w_1 u_1 + w_2 u_2 + w_3 u_3 = 0,$$

so that

$$\frac{u_1}{v_2 w_3 - v_3 w_2} = \frac{u_2}{v_3 w_1 - v_1 w_3} = \frac{u_3}{v_1 w_2 - v_2 w_1};$$

thus

$$\frac{x_1}{u_1} = \frac{y_1}{u_2} = \frac{z_1}{u_3} = \theta,$$

say. Substituting in

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 1,$$

we have

$$h_1^2 \theta = 1;$$

hence

$$\frac{x_1}{u_1} = \frac{y_1}{u_2} = \frac{z_1}{u_3} = \frac{1}{h_1^2}.$$

Similarly

$$\frac{x_2}{v_1} = \frac{y_2}{v_2} = \frac{z_2}{v_3} = \frac{1}{h_2^2},$$

$$\frac{x_3}{w_1} = \frac{y_3}{w_2} = \frac{z_3}{w_3} = \frac{1}{h_3^2}.$$

Hence

$$\left.\begin{aligned} x_1 x_2 + y_1 y_2 + z_1 z_2 = 0 \\ x_2 x_3 + y_2 y_3 + z_2 z_3 = 0 \\ x_3 x_1 + y_3 y_1 + z_3 z_1 = 0 \end{aligned}\right\}.$$

We also introduce three quantities H_1, H_2, H_3, under the definitions

$$\left.\begin{aligned} x_1^2 + y_1^2 + z_1^2 = H_1^2 \\ x_2^2 + y_2^2 + z_2^2 = H_2^2 \\ x_3^2 + y_3^2 + z_3^2 = H_3^2 \end{aligned}\right\}.$$

Manifestly

$$h_1^2 H_1^2 = 1, \quad h_2^2 H_2^2 = 1, \quad h_3^2 H_3^2 = 1;$$

or, if we give positive values to h_1, h_2, h_3, H_1, H_2, H_3, then

$$h_1 H_1 = 1, \quad h_2 H_2 = 1, \quad h_3 H_3 = 1.$$

The following relations can easily be established, and may be useful hereafter:—

$$\left.\begin{aligned} v_2w_3 - v_3w_2 &= \frac{h_2h_3}{h_1}u_1 \\ v_3w_1 - v_1w_3 &= \frac{h_2h_3}{h_1}u_2 \\ v_1w_2 - v_2w_1 &= \frac{h_2h_3}{h_1}u_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} y_2z_3 - y_3z_2 &= \frac{H_2H_3}{H_1}x_1 \\ z_2x_3 - z_3x_2 &= \frac{H_2H_3}{H_1}y_1 \\ x_2y_3 - x_3y_2 &= \frac{H_2H_3}{H_1}z_1 \end{aligned}\right\},$$

$$\left.\begin{aligned} w_2u_3 - w_3u_2 &= \frac{h_3h_1}{h_2}v_1 \\ w_3u_1 - w_1u_3 &= \frac{h_3h_1}{h_2}v_2 \\ w_1u_2 - w_2u_1 &= \frac{h_3h_1}{h_2}v_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} y_3z_1 - y_1z_3 &= \frac{H_3H_1}{H_2}x_2 \\ z_3x_1 - z_1x_3 &= \frac{H_3H_1}{H_2}y_2 \\ x_3y_1 - x_1y_3 &= \frac{H_3H_1}{H_2}z_2 \end{aligned}\right\},$$

$$\left.\begin{aligned} u_2v_3 - u_3v_2 &= \frac{h_1h_2}{h_3}w_1 \\ u_3v_1 - u_1v_3 &= \frac{h_1h_2}{h_3}w_2 \\ u_1v_2 - u_2v_1 &= \frac{h_1h_2}{h_3}w_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} y_1z_2 - y_2z_1 &= \frac{H_1H_2}{H_3}x_3 \\ z_1x_2 - z_2x_1 &= \frac{H_1H_2}{H_3}y_3 \\ x_1y_2 - x_2y_1 &= \frac{H_1H_2}{H_3}z_3 \end{aligned}\right\}.$$

Also

$$J\left(\frac{u,\ v,\ w}{x,\ y,\ z}\right) = \begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_3, & v_3, & w_3 \end{vmatrix} = h_1h_2h_3,$$

$$J\left(\frac{x,\ y,\ z}{u,\ v,\ w}\right) = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = H_1H_2H_3.$$

Ex. In § 195, the equations of a Dupin cyclide were obtained in the form

$$\left.\begin{aligned} x &= \mu\frac{a}{c} + \frac{b^2}{c}\,\frac{c\cos\theta - \mu}{a - c\cos\theta\cos\psi} \\ y &= \frac{b\,(a - \mu\cos\psi)}{a - c\cos\theta\cos\psi}\sin\theta \\ z &= \frac{b\,(c\cos\theta - \mu)}{a - c\cos\theta\cos\psi}\sin\psi \end{aligned}\right\},$$

where $c^2 = a^2 - b^2$. Thus three families of surfaces are given by regarding μ, θ, ψ as the family parameters; and the equations of the three families are easily proved to be

$$\left.\begin{aligned} (x^2+y^2+z^2-\mu^2+b^2)^2 &= 4\,(ax-c\mu)^2 + 4b^2y^2 \\ (x^2+y^2+z^2-\mu^2-b^2)^2 &= 4\,(cx-a\mu)^2 - 4b^2z^2 \end{aligned}\right\},$$

(these two equations of the Dupin cyclide being equivalent),

$$x^2+y^2+z^2+b^2-2\frac{by}{\sin\theta}=\frac{1}{c^2}(ax-by\cot\theta)^2,$$

$$x^2+y^2+z^2+b^2-2bz\cot\psi=\frac{1}{a^2}(cx-bz\operatorname{cosec}\psi)^2.$$

It is not difficult to verify by direct substitution that

$$\frac{\partial x}{\partial\mu}\frac{\partial x}{\partial\theta}+\frac{\partial y}{\partial\mu}\frac{\partial y}{\partial\theta}+\frac{\partial z}{\partial\mu}\frac{\partial z}{\partial\theta}=0,$$

$$\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\psi}+\frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\psi}+\frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\psi}=0,$$

$$\frac{\partial x}{\partial\psi}\frac{\partial x}{\partial\mu}+\frac{\partial y}{\partial\psi}\frac{\partial y}{\partial\mu}+\frac{\partial z}{\partial\psi}\frac{\partial z}{\partial\mu}=0;$$

hence the surfaces are orthogonal to one another. Thus the three families of surfaces (one of them being a family of Dupin cyclides) are a triply orthogonal system*.

242. The construction of the fundamental magnitudes of the second order for the surfaces requires the derivatives of x, y, z of the second order with respect to u, v, w.

When we differentiate $x_1x_2+y_1y_2+z_1z_2=0$ with respect to w, and similarly for the other two corresponding relations symmetric with it, we have

$$\left.\begin{array}{l} 0=x_1x_{23}+x_2x_{13}+y_1y_{23}+y_2y_{13}+z_1z_{23}+z_2z_{13} \\ 0=x_2x_{13}+x_3x_{12}+y_2y_{13}+y_3y_{12}+z_2z_{13}+z_3z_{12} \\ 0=x_1x_{23}+x_3x_{12}+y_1y_{23}+y_3y_{12}+z_1z_{23}+z_3z_{12} \end{array}\right\};$$

which are easily seen to be equivalent to

$$\left.\begin{array}{l} x_1x_{23}+y_1y_{23}+z_1z_{23}=0 \\ x_2x_{31}+y_2y_{31}+z_2z_{31}=0 \\ x_3x_{12}+y_3y_{12}+z_3z_{12}=0 \end{array}\right\}.$$

Further,

$$\left.\begin{array}{l} x_1x_{11}+y_1y_{11}+z_1z_{11}=H_1\dfrac{\partial H_1}{\partial u} \\ x_1x_{12}+y_1y_{12}+z_1z_{12}=H_1\dfrac{\partial H_1}{\partial v} \\ x_2x_{11}+y_2y_{11}+z_2z_{11}=-H_1\dfrac{\partial H_1}{\partial v} \\ x_1x_{13}+y_1y_{13}+z_1z_{13}=H_1\dfrac{\partial H_1}{\partial w} \\ x_3x_{11}+y_3y_{11}+z_3z_{11}=-H_1\dfrac{\partial H_1}{\partial w} \end{array}\right\},$$

* In regard to the systems which include Dupin's cyclides, two memoirs by Darboux, *Mém. de l'Acad.*, t. li (1908), n$^{\text{s}}$ 1, n° 2, as well as Note iii at the end of his treatise on orthogonal systems, should be consulted.

$$\left.\begin{aligned}
x_2x_{12}+y_2y_{12}+z_2z_{12}&=H_2\frac{\partial H_2}{\partial u}\\
x_1x_{22}+y_1y_{22}+z_1z_{22}&=-H_2\frac{\partial H_2}{\partial u}\\
x_2x_{22}+y_2y_{22}+z_2z_{22}&=H_2\frac{\partial H_2}{\partial v}\\
x_2x_{23}+y_2y_{23}+z_2z_{23}&=H_2\frac{\partial H_2}{\partial w}\\
x_3x_{22}+y_3y_{22}+z_3z_{22}&=-H_2\frac{\partial H_2}{\partial w}
\end{aligned}\right\},\qquad
\left.\begin{aligned}
x_3x_{13}+y_3y_{13}+z_3z_{13}&=H_3\frac{\partial H_3}{\partial u}\\
x_1x_{33}+y_1y_{33}+z_1z_{33}&=-H_3\frac{\partial H_3}{\partial u}\\
x_3x_{23}+y_3y_{23}+z_3z_{23}&=H_3\frac{\partial H_3}{\partial v}\\
x_2x_{33}+y_2y_{33}+z_2z_{33}&=-H_3\frac{\partial H_3}{\partial v}\\
x_3x_{33}+y_3y_{33}+z_3z_{33}&=H_3\frac{\partial H_3}{\partial w}
\end{aligned}\right\}.$$

The direction-cosines of the normals to the three surfaces are given by

$$\left.\begin{aligned}
\alpha_1&=\frac{u_1}{h_1}=\frac{x_1}{H_1}, & \beta_1&=\frac{u_2}{h_1}=\frac{y_1}{H_1}, & \gamma_1&=\frac{u_3}{h_1}=\frac{z_1}{H_1}\\
\alpha_2&=\frac{v_1}{h_2}=\frac{x_2}{H_2}, & \beta_2&=\frac{v_2}{h_2}=\frac{y_2}{H_2}, & \gamma_2&=\frac{v_3}{h_2}=\frac{z_2}{H_2}\\
\alpha_3&=\frac{w_1}{h_3}=\frac{x_3}{H_3}, & \beta_3&=\frac{w_2}{h_3}=\frac{y_3}{H_3}, & \gamma_3&=\frac{w_3}{h_3}=\frac{z_3}{H_3}
\end{aligned}\right\}.$$

The parametric variables on the w-surface are u and v; hence the fundamental magnitudes L, M, N for that surface are

$$\begin{aligned}
L&=\alpha_3x_{11}+\beta_3y_{11}+\gamma_3z_{11}=-\frac{H_1}{H_3}\frac{\partial H_1}{\partial w},\\
M&=\alpha_3x_{12}+\beta_3y_{12}+\gamma_3z_{12}=0,\\
N&=\alpha_3x_{22}+\beta_3y_{22}+\gamma_3z_{22}=-\frac{H_2}{H_3}\frac{\partial H_2}{\partial w}.
\end{aligned}$$

Similar results are deducible for the u-surface and the v-surface. The whole table of the six fundamental magnitudes for each of the three surfaces is as follows:—

Surface	Superficial parameters	E	F	G	L	M	N
$u=a$	$v,\ w$	H_2^2	0	H_3^2	$-\frac{H_2}{H_1}\frac{\partial H_2}{\partial u}$	0	$-\frac{H_3}{H_1}\frac{\partial H_3}{\partial u}$
$v=b$	$w,\ u$	H_3^2	0	H_1^2	$-\frac{H_3}{H_2}\frac{\partial H_3}{\partial v}$	0	$-\frac{H_1}{H_2}\frac{\partial H_1}{\partial v}$
$w=c$	$u,\ v$	H_1^2	0	H_2^2	$-\frac{H_1}{H_3}\frac{\partial H_1}{\partial w}$	0	$-\frac{H_2}{H_3}\frac{\partial H_2}{\partial w}$

243. It will be noticed that $F=0$, $M=0$ for each surface; so we have Dupin's theorem:—

When three surfaces cut orthogonally, the curves of intersection are lines of curvature on each surface.

The last theorem can be associated with a theorem of Joachimsthal's already (§ 128) proved—that, if two surfaces cut one another along a curve at a constant angle, and if the curve be a line of curvature for either surface, it is a line of curvature upon the other also.

When the constant angle is a right angle, the theorem can be established very simply by the following method which is an adaptation of the method of Puiseux to be used hereafter (§ 259, *post*) for triply orthogonal systems. Let the surfaces be transferred to any point current along the line of intersection; take the tangent plane to one of them at the point as the plane $z=0$, and the tangent plane to the other as the plane $y=0$; then, in the immediate vicinity of the origin, the equations of the two surfaces have the form

$$0 = z + ax^2 + by^2 + 2Cxy + \dots,$$
$$0 = y + a'x^2 + c'z^2 + 2Bxz + \dots,$$

where, in each case, the unexpressed terms are of order smaller than the retained terms near the origin. The curve of intersection of the surfaces at the origin (which is a current point on the curve) is

$$z=0, \quad y=0;$$

on the former surface, it is a line of curvature if $C=0$, and on the latter if $B=0$. The condition of orthogonality everywhere is

$$\begin{aligned}(2ax + 2Cy + \dots)&(2a'x + 2Bz + \dots)\\ &+ (1 + \dots)(2by + 2Cx + \dots)\\ &+ (1 + \dots)(2c'z + 2Bx + \dots) = 0;\end{aligned}$$

along the curve of intersection, we have

$$-z = ax^2 + \text{higher powers of } x, \quad -y = a'x^2 + \text{higher powers of } x;$$

hence we have

$$B + C = 0,$$

so that the vanishing of B or C means the vanishing of the other.

244. The expressions for all the second derivatives of x, y, z with regard to u, v, w are derivable from the foregoing equations. We have

$$x_1x_{11} + y_1y_{11} + z_1z_{11} = H_1\frac{\partial H_1}{\partial u},$$

$$x_2x_{11} + y_2y_{11} + z_2z_{11} = -H_1\frac{\partial H_1}{\partial v},$$

$$x_3x_{11} + y_3y_{11} + z_3z_{11} = -H_1\frac{\partial H_1}{\partial w};$$

and therefore

$$\left.\begin{aligned} x_{11} &= x_1 \frac{1}{H_1}\frac{\partial H_1}{\partial u} + x_2\left(-\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}\right) + x_3\left(-\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right)\\ y_{11} &= y_1 \frac{1}{H_1}\frac{\partial H_1}{\partial u} + y_2\left(-\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}\right) + y_3\left(-\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right)\\ z_{11} &= z_1 \frac{1}{H_1}\frac{\partial H_1}{\partial u} + z_2\left(-\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}\right) + z_3\left(-\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right)\end{aligned}\right\}.$$

Again,

$$x_1x_{23} + y_1y_{23} + z_1z_{23} = 0,$$

$$x_2x_{23} + y_2y_{23} + z_2z_{23} = H_2\frac{\partial H_2}{\partial w},$$

$$x_3x_{23} + y_3y_{23} + z_3z_{23} = H_3\frac{\partial H_3}{\partial v};$$

and therefore

$$\left.\begin{aligned} x_{23} &= x_2\frac{1}{H_2}\frac{\partial H_2}{\partial w} + x_3\frac{1}{H_3}\frac{\partial H_3}{\partial v}\\ y_{23} &= y_2\frac{1}{H_2}\frac{\partial H_2}{\partial w} + y_3\frac{1}{H_3}\frac{\partial H_3}{\partial v}\\ z_{23} &= z_2\frac{1}{H_2}\frac{\partial H_2}{\partial w} + z_3\frac{1}{H_3}\frac{\partial H_3}{\partial v}\end{aligned}\right\}.$$

It may be noted that the last three equations are the forms of equations of conjugate systems on the u-surface; as they are also orthogonal, they are necessarily lines of curvature. Similarly for the other surfaces.

The corresponding results for the other derivatives can be obtained by the cyclical interchange of variables. The remaining formulæ which, with those already given, constitute the full aggregate of second derivatives, are as follows:—

$$\left.\begin{aligned} x_{22} &= x_1\left(-\frac{H_2}{H_1^2}\frac{\partial H_2}{\partial u}\right) + x_2\left(\frac{1}{H_2}\frac{\partial H_2}{\partial v}\right) + x_3\left(-\frac{H_2}{H_3^2}\frac{\partial H_2}{\partial w}\right)\\ y_{22} &= y_1\left(-\frac{H_2}{H_1^2}\frac{\partial H_2}{\partial u}\right) + y_2\left(\frac{1}{H_2}\frac{\partial H_2}{\partial v}\right) + y_3\left(-\frac{H_2}{H_3^2}\frac{\partial H_2}{\partial w}\right)\\ z_{22} &= z_1\left(-\frac{H_2}{H_1^2}\frac{\partial H_2}{\partial u}\right) + z_2\left(\frac{1}{H_2}\frac{\partial H_2}{\partial v}\right) + z_3\left(-\frac{H_2}{H_3^2}\frac{\partial H_2}{\partial w}\right)\end{aligned}\right\};$$

$$\left.\begin{aligned} x_{31} &= x_3\frac{1}{H_3}\frac{\partial H_3}{\partial u} + x_1\frac{1}{H_1}\frac{\partial H_1}{\partial w}\\ y_{31} &= y_3\frac{1}{H_3}\frac{\partial H_3}{\partial u} + y_1\frac{1}{H_1}\frac{\partial H_1}{\partial w}\\ z_{31} &= z_3\frac{1}{H_3}\frac{\partial H_3}{\partial u} + z_1\frac{1}{H_1}\frac{\partial H_1}{\partial w}\end{aligned}\right\};$$

$$\left.\begin{aligned}
x_{33} &= x_1\left(-\frac{H_3}{H_1^2}\frac{\partial H_3}{\partial u}\right) + x_2\left(-\frac{H_3}{H_2^2}\frac{\partial H_2}{\partial v}\right) + x_3\left(\frac{1}{H_3}\frac{\partial H_3}{\partial w}\right)\\
y_{33} &= y_1\left(-\frac{H_3}{H_1^2}\frac{\partial H_3}{\partial u}\right) + y_2\left(-\frac{H_3}{H_2^2}\frac{\partial H_2}{\partial v}\right) + y_3\left(\frac{1}{H_3}\frac{\partial H_3}{\partial w}\right)\\
z_{33} &= z_1\left(-\frac{H_3}{H_1^2}\frac{\partial H_3}{\partial u}\right) + z_2\left(-\frac{H_3}{H_2^2}\frac{\partial H_2}{\partial v}\right) + z_3\left(\frac{1}{H_3}\frac{\partial H_3}{\partial w}\right)
\end{aligned}\right\};$$

and

$$\left.\begin{aligned}
x_{12} &= x_1\frac{1}{H_1}\frac{\partial H_1}{\partial v} + x_2\frac{1}{H_2}\frac{\partial H_2}{\partial u}\\
y_{12} &= y_1\frac{1}{H_1}\frac{\partial H_1}{\partial v} + y_2\frac{1}{H_2}\frac{\partial H_2}{\partial u}\\
z_{12} &= z_1\frac{1}{H_1}\frac{\partial H_1}{\partial v} + z_2\frac{1}{H_2}\frac{\partial H_2}{\partial u}
\end{aligned}\right\}.$$

From these, we have

$$x_{11}^2 + y_{11}^2 + z_{11}^2 = \left(\frac{\partial H_1}{\partial u}\right)^2 + \frac{H_1^2}{H_2^2}\left(\frac{\partial H_1}{\partial v}\right)^2 + \frac{H_1^2}{H_3^2}\left(\frac{\partial H_1}{\partial w}\right)^2,$$

$$x_{11}x_{12} + y_{11}y_{12} + z_{11}z_{12} = \frac{\partial H_1}{\partial u}\frac{\partial H_1}{\partial v} - \frac{H_1}{H_2}\frac{\partial H_2}{\partial u}\frac{\partial H_1}{\partial v},$$

$$x_{11}x_{23} + y_{11}y_{23} + z_{11}z_{23} = -\frac{H_1}{H_2}\frac{\partial H_1}{\partial v}\frac{\partial H_2}{\partial w} - \frac{H_1}{H_3}\frac{\partial H_1}{\partial w}\frac{\partial H_3}{\partial v}.$$

$$x_{12}x_{13} + y_{12}y_{13} + z_{12}z_{13} = \frac{\partial H_1}{\partial v}\frac{\partial H_1}{\partial w},$$

$$x_{23}^2 + y_{23}^2 + z_{23}^2 = \left(\frac{\partial H_2}{\partial w}\right)^2 + \left(\frac{\partial H_3}{\partial v}\right)^2;$$

together with others derivable by circular interchange of the variables u, v, w and of the suffixes.

Some of these relations for the second derivatives of x, y, z can be expressed in another form which will be used later. As

$$\alpha_1 = \frac{1}{H_1}x_1, \quad \alpha_2 = \frac{1}{H_2}x_2, \quad \alpha_3 = \frac{1}{H_3}x_3,$$

we have

$$\frac{\partial \alpha_1}{\partial u} = \frac{1}{H_1}x_{11} - \frac{x_1}{H_1^2}\frac{\partial H_1}{\partial u} = -\frac{1}{H_2}\frac{\partial H_1}{\partial v}\alpha_2 - \frac{1}{H_3}\frac{\partial H_1}{\partial w}\alpha_3,$$

on using the value of x_{11}; and so for the others. The tale of the results is:—

$$\left.\begin{aligned}
\frac{\partial \alpha_1}{\partial u} &= -\frac{1}{H_2}\frac{\partial H_1}{\partial v}\alpha_2 - \frac{1}{H_3}\frac{\partial H_1}{\partial w}\alpha_3\\
\frac{\partial \alpha_1}{\partial v} &= \frac{1}{H_1}\frac{\partial H_2}{\partial u}\alpha_2\\
\frac{\partial \alpha_1}{\partial w} &= \phantom{-\frac{1}{H_2}\frac{\partial H_1}{\partial v}\alpha_2 -{}} \frac{1}{H_1}\frac{\partial H_3}{\partial u}\alpha_3
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\frac{\partial \alpha_2}{\partial u} &= \frac{1}{H_2}\frac{\partial H_1}{\partial v}\alpha_1 \\
\frac{\partial \alpha_2}{\partial v} &= -\frac{1}{H_3}\frac{\partial H_2}{\partial w}\alpha_3 - \frac{1}{H_1}\frac{\partial H_2}{\partial u}\alpha_1 \\
\frac{\partial \alpha_2}{\partial w} &= \frac{1}{H_2}\frac{\partial H_3}{\partial v}\alpha_3
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\frac{\partial \alpha_3}{\partial u} &= \frac{1}{H_3}\frac{\partial H_1}{\partial w}\alpha_1 \\
\frac{\partial \alpha_3}{\partial v} &= \frac{1}{H_3}\frac{\partial H_2}{\partial w}\alpha_2 \\
\frac{\partial \alpha_3}{\partial w} &= -\frac{1}{H_1}\frac{\partial H_3}{\partial u}\alpha_1 - \frac{1}{H_2}\frac{\partial H_3}{\partial v}\alpha_2
\end{aligned}\right\},$$

together with corresponding results for β_1, β_2, β_3 and γ_1, γ_2, γ_3.

Further, it is important (especially for some of the equations of triply orthogonal systems) to associate a magnitude

$$\theta = x^2 + y^2 + z^2$$

with x, y, z, and to have the corresponding equations; these are

$$\left.\begin{aligned}
\theta_{11} - 2H_1^2 &= \theta_1 \frac{1}{H_1}\frac{\partial H_1}{\partial u} + \theta_2\left(-\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}\right) + \theta_3\left(-\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right) \\
\theta_{22} - 2H_2^2 &= \theta_1\left(-\frac{H_2}{H_1^2}\frac{\partial H_2}{\partial u}\right) + \theta_2 \frac{1}{H_2}\frac{\partial H_2}{\partial v} + \theta_3\left(-\frac{H_2}{H_3^2}\frac{\partial H_2}{\partial w}\right) \\
\theta_{33} - 2H_3^2 &= \theta_1\left(-\frac{H_3}{H_1^2}\frac{\partial H_3}{\partial u}\right) + \theta_2\left(-\frac{H_3}{H_2^2}\frac{\partial H_3}{\partial v}\right) + \theta_3 \frac{1}{H_3}\frac{\partial H_3}{\partial w}
\end{aligned}\right\},$$

and

$$\left.\begin{aligned}
\theta_{23} &= \theta_2 \frac{1}{H_2}\frac{\partial H_2}{\partial w} + \theta_3 \frac{1}{H_3}\frac{\partial H_3}{\partial v} \\
\theta_{31} &= \theta_3 \frac{1}{H_3}\frac{\partial H_3}{\partial u} + \theta_1 \frac{1}{H_1}\frac{\partial H_1}{\partial w} \\
\theta_{12} &= \theta_1 \frac{1}{H_1}\frac{\partial H_1}{\partial v} + \theta_2 \frac{1}{H_2}\frac{\partial H_2}{\partial u}
\end{aligned}\right\}.$$

It will be noticed that the forms of the last three equations are the same as the forms of the corresponding equations for x, y, z—a property to be compared with the corresponding property, noted in § 77, for surfaces when they are referred to lines of curvature as parametric curves.

245. As the fundamental magnitudes of the three surfaces are known, and as the parametric curves are lines of curvature on each of the surfaces, the principal curvatures can be written down at once.

For the surface $u = a$, the principal curvature along $v = b$ is N/G, that is,

$$-\frac{1}{H_1 H_3}\frac{\partial H_3}{\partial u};$$

and along $w=c$ it is L/E, that is,

$$-\frac{1}{H_1H_2}\frac{\partial H_2}{\partial u}.$$

For the surface $v=b$, the principal curvature along $w=c$ is

$$-\frac{1}{H_2H_1}\frac{\partial H_1}{\partial v},$$

and along $u=a$ is

$$-\frac{1}{H_2H_3}\frac{\partial H_3}{\partial v}.$$

For the surface $w=c$, the principal curvature along $u=a$ is

$$-\frac{1}{H_3H_2}\frac{\partial H_2}{\partial w},$$

and along $v=b$ is

$$-\frac{1}{H_3H_1}\frac{\partial H_1}{\partial w}.$$

Ex. It is known that a triply orthogonal system is constituted by the complete set of confocal quadrics

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=1,$$

for various values of λ. Taking u, v, w to be the parametric values of λ for a point in space, shew that

$$\left.\begin{aligned} 4H_1^2&=\frac{(u-v)(u-w)}{(a+u)(b+u)(c+u)}\\ 4H_2^2&=\frac{(v-w)(v-u)}{(a+v)(b+v)(c+v)}\\ 4H_3^2&=\frac{(w-u)(w-v)}{(a+w)(b+w)(c+w)}\end{aligned}\right\};$$

and obtain all the fundamental magnitudes of the second order.

Lamé relations satisfied by H_1, H_2, H_3.

246. Although there are only three functions H_1, H_2, H_3 of the three independent variables u, v, w, yet it appears that they satisfy a set of differential relations, which can be obtained in the same way as the Mainardi-Codazzi relations for the fundamental magnitudes of a surface. The space-relations are six in number. One set of three is made up of the equations

$$\left.\begin{aligned} \frac{\partial}{\partial u}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial v}\right)+\frac{1}{H_3^2}\frac{\partial H_1}{\partial w}\frac{\partial H_2}{\partial w}&=0\\ \frac{\partial}{\partial v}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial w}\right)+\frac{1}{H_1^2}\frac{\partial H_2}{\partial u}\frac{\partial H_3}{\partial u}&=0\\ \frac{\partial}{\partial w}\left(\frac{1}{H_3}\frac{\partial H_1}{\partial w}\right)+\frac{\partial}{\partial u}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u}\right)+\frac{1}{H_2^2}\frac{\partial H_3}{\partial v}\frac{\partial H_1}{\partial v}&=0\end{aligned}\right\};$$

they sometimes are called Gauss relations, more often Lamé relations, and may be briefly written in the form

$$[u, v]=0, \quad [v, w]=0, \quad [w, u]=0.$$

The other set of three is made up of the equations

$$\left.\begin{aligned}
\frac{\partial^2 H_1}{\partial v \partial w}-\frac{1}{H_2}\frac{\partial H_2}{\partial w}\frac{\partial H_1}{\partial v}-\frac{1}{H_3}\frac{\partial H_3}{\partial v}\frac{\partial H_1}{\partial w}=0\\
\frac{\partial^2 H_2}{\partial w \partial u}-\frac{1}{H_3}\frac{\partial H_3}{\partial u}\frac{\partial H_2}{\partial w}-\frac{1}{H_1}\frac{\partial H_1}{\partial w}\frac{\partial H_2}{\partial u}=0\\
\frac{\partial^2 H_3}{\partial u \partial v}-\frac{1}{H_1}\frac{\partial H_1}{\partial v}\frac{\partial H_3}{\partial u}-\frac{1}{H_2}\frac{\partial H_2}{\partial u}\frac{\partial H_3}{\partial v}=0
\end{aligned}\right\};$$

they sometimes are called Mainardi relations, more often Lamé relations, and may be briefly written in the form

$$\{v, w\}=0, \quad \{w, u\}=0, \quad \{u, v\}=0.$$

All the expressions for the derivatives of x, y, z of the second order have been obtained; these derivatives of x are linear in x_1, x_2, x_3, and similarly for the second derivatives of y and of z. Now we must have

$$\frac{\partial x_{11}}{\partial v}=\frac{\partial x_{12}}{\partial u}, \quad \frac{\partial y_{11}}{\partial v}=\frac{\partial y_{12}}{\partial u}, \quad \frac{\partial z_{11}}{\partial v}=\frac{\partial z_{12}}{\partial u}.$$

When we substitute the values of x_{11} and x_{12}, y_{11} and y_{12}, z_{11} and z_{12}, effect the differential operations, substitute again for the second derivatives which are introduced by these operations, and reduce, we find

$$0=[u, v]\,x_2+\{v, w\}\,x_3,$$
$$0=[u, v]\,y_2+\{v, w\}\,y_3,$$
$$0=[u, v]\,z_2+\{v, w\}\,z_3;$$

and therefore

$$[u, v]=0, \quad \{v, w\}=0.$$

Similarly from the necessary relations

$$\frac{\partial x_{11}}{\partial w}=\frac{\partial x_{13}}{\partial u}, \quad \frac{\partial y_{11}}{\partial w}=\frac{\partial y_{13}}{\partial u}, \quad \frac{\partial z_{11}}{\partial w}=\frac{\partial z_{13}}{\partial u},$$

we find

$$[w, u]=0, \quad \{v, w\}=0;$$

from the necessary relations

$$\frac{\partial x_{12}}{\partial v}=\frac{\partial x_{22}}{\partial u}, \quad \frac{\partial y_{12}}{\partial v}=\frac{\partial y_{22}}{\partial u}, \quad \frac{\partial z_{12}}{\partial v}=\frac{\partial z_{22}}{\partial u},$$

we find

$$[u, v]=0, \quad \{w, u\}=0;$$

from the necessary relations

$$\frac{\partial x_{12}}{\partial w}=\frac{\partial x_{23}}{\partial u}, \quad \frac{\partial y_{12}}{\partial w}=\frac{\partial y_{23}}{\partial u}, \quad \frac{\partial z_{12}}{\partial w}=\frac{\partial z_{23}}{\partial u},$$

we find

$$[w, u] = 0, \quad \{w, u\} = 0;$$

from the necessary relations

$$\frac{\partial x_{12}}{\partial w} = \frac{\partial x_{13}}{\partial v}, \quad \frac{\partial y_{12}}{\partial w} = \frac{\partial y_{13}}{\partial v}, \quad \frac{\partial z_{12}}{\partial w} = \frac{\partial z_{13}}{\partial v},$$

we find

$$[v, w] = 0, \quad \{v, w\} = 0;$$

from the necessary relations

$$\frac{\partial x_{13}}{\partial v} = \frac{\partial x_{23}}{\partial u}, \quad \frac{\partial y_{13}}{\partial v} = \frac{\partial y_{23}}{\partial u}, \quad \frac{\partial z_{13}}{\partial v} = \frac{\partial z_{23}}{\partial u},$$

we find

$$[u, v] = 0, \quad \{u, v\} = 0;$$

from the necessary relations

$$\frac{\partial x_{13}}{\partial w} = \frac{\partial x_{33}}{\partial u}, \quad \frac{\partial y_{13}}{\partial w} = \frac{\partial y_{33}}{\partial u}, \quad \frac{\partial z_{13}}{\partial w} = \frac{\partial z_{33}}{\partial u},$$

we find

$$[w, u] = 0, \quad \{u, v\} = 0;$$

from the necessary relations

$$\frac{\partial x_{22}}{\partial w} = \frac{\partial x_{23}}{\partial v}, \quad \frac{\partial y_{22}}{\partial w} = \frac{\partial y_{23}}{\partial v}, \quad \frac{\partial z_{22}}{\partial w} = \frac{\partial z_{23}}{\partial v},$$

we find

$$[v, w] = 0, \quad \{w, u\} = 0;$$

and from the necessary relations

$$\frac{\partial x_{23}}{\partial w} = \frac{\partial x_{33}}{\partial v}, \quad \frac{\partial y_{23}}{\partial w} = \frac{\partial y_{33}}{\partial v}, \quad \frac{\partial z_{23}}{\partial w} = \frac{\partial z_{33}}{\partial v},$$

we find

$$[v, w] = 0, \quad \{u, v\} = 0.$$

Hence there are six relations in all, in the two sets as arranged.

247. It thus appears that the three quantities H_1, H_2, H_3, which arise from a triply orthogonal system, must satisfy six partial differential equations of the second order; the independent variables do not explicitly occur in these equations. Two questions at once propound themselves.

The first can be put into the form:—supposing that three quantities H_1, H_2, H_3 are known, or are found, as satisfying these six equations, what is their significance for the construction of a triply orthogonal system?

The second arises out of an obvious doubt. We are to have six partial differential equations satisfied by three quantities, regarded as dependent variables; and the independent variables do not occur. It is manifest that, if the equations were quite general in form, common solutions would not exist.

But the equations are special in form, and have (in the two sets) a circular symmetry in the variables; and so they may possess common solutions. Thus a question arises as to whether any conditions must be satisfied in order that the equations may coexist. Our concern, however, is rather with systems of surfaces than with systems of equations; and so the question will rather relate to the conditions that may be required of families which can belong to triply orthogonal systems.

The two questions will be taken in turn.

Extension of Bonnet's theorem (§ 37) *to space.*

248. Accordingly, we proceed to investigate how far a triply orthogonal system is determined by three quantities H_1, H_2, H_3, which are given in value and satisfy the six characteristic partial equations of the second order.

The equations to be satisfied by the coordinates of a point in space, being the intersection of particular members of the three families (if they exist), are three sets for each coordinate. One of these sets is

$$\left.\begin{array}{l} x_{11} - x_1 \dfrac{1}{H_1}\dfrac{\partial H_1}{\partial u} - x_2\left(-\dfrac{H_1}{H_2^2}\dfrac{\partial H_1}{\partial v}\right) - x_3\left(-\dfrac{H_1}{H_3^2}\dfrac{\partial H_1}{\partial w}\right) = 0 \\ x_{12} - x_1 \dfrac{1}{H_1}\dfrac{\partial H_1}{\partial v} - x_2 \dfrac{1}{H_2}\dfrac{\partial H_2}{\partial u} = 0 \\ x_{13} - x_1 \dfrac{1}{H_1}\dfrac{\partial H_1}{\partial w} - x_3 \dfrac{1}{H_3}\dfrac{\partial H_3}{\partial u} = 0 \end{array}\right\},$$

being a set linear in x_1, x_2, x_3, and in their first derivatives with regard to u. The other two sets for x also are linear in x_1, x_2, x_3, and in their first derivatives with regard to v and to w respectively. The three sets coexist, after the earlier analysis of § 246, provided the six characteristic equations are satisfied by H_1, H_2, H_3.

As the foregoing set of equations is linear in x_1, x_2, x_3, the primitive is of the form

$$\left.\begin{array}{l} x_1 = \xi a_1 + \eta b_1 + \zeta c_1 \\ x_2 = \xi a_2 + \eta b_2 + \zeta c_2 \\ x_3 = \xi a_3 + \eta b_3 + \zeta c_3 \end{array}\right\},$$

where ξ, η, ζ are arbitrary functions of v and w, and where three linearly independent sets of particular solutions are given by

$$\begin{aligned} x_1,\ x_2,\ x_3 &= a_1,\ a_2,\ a_3; \\ &= b_1,\ b_2,\ b_3; \\ &= c_1,\ c_2,\ c_3. \end{aligned}$$

Also, no limitations upon ξ, η, ζ are imposed by the above set of differential equations.

249. The second set of equations in the derivatives of x is

$$\left.\begin{aligned} &x_{21} - x_1 \frac{1}{H_1}\frac{\partial H_1}{\partial v} - x_2 \frac{1}{H_2}\frac{\partial H_2}{\partial u} = 0 \\ &x_{22} - x_1\left(-\frac{H_2}{H_1^2}\frac{\partial H_2}{\partial u}\right) - x_2 \frac{1}{H_2}\frac{\partial H_2}{\partial v} - x_3\left(-\frac{H_2}{H_3^2}\frac{\partial H_2}{\partial w}\right) = 0 \\ &x_{23} - x_2 \frac{1}{H_2}\frac{\partial H_2}{\partial w} - x_3 \frac{1}{H_3}\frac{\partial H_3}{\partial v} = 0 \end{aligned}\right\};$$

if the triply orthogonal system exists, this set must be satisfied by the primitive of the first set. When

$$x_1,\ x_2,\ x_3 = a_1,\ a_2,\ a_3$$

are substituted in the left-hand sides of these equations, let the latter become A_1, A_2, A_3 respectively*; and similarly let them become B_1, B_2, B_3 when

$$x_1,\ x_2,\ x_3 = b_1,\ b_2,\ b_3$$

are substituted, and become C_1, C_2, C_3 when

$$x_1,\ x_2,\ x_3 = c_1,\ c_2,\ c_3$$

are substituted. Then, in order that the foregoing set may be satisfied by the primitive of the first set, it is necessary and sufficient that the equations

$$\left.\begin{aligned} a_1\frac{\partial \xi}{\partial v} + b_1\frac{\partial \eta}{\partial v} + c_1\frac{\partial \zeta}{\partial v} &= -(\xi A_1 + \eta B_1 + \zeta C_1) \\ a_2\frac{\partial \xi}{\partial v} + b_2\frac{\partial \eta}{\partial v} + c_2\frac{\partial \zeta}{\partial v} &= -(\xi A_2 + \eta B_2 + \zeta C_2) \\ a_3\frac{\partial \xi}{\partial v} + b_3\frac{\partial \eta}{\partial v} + c_3\frac{\partial \zeta}{\partial v} &= -(\xi A_3 + \eta B_3 + \zeta C_3) \end{aligned}\right\}$$

should be satisfied by values of ξ, η, ζ which are independent of u. This last requirement will be satisfied if, at the same time, the equations

$$\left.\begin{aligned} \frac{\partial a_1}{\partial u}\frac{\partial \xi}{\partial v} + \frac{\partial b_1}{\partial u}\frac{\partial \eta}{\partial v} + \frac{\partial c_1}{\partial u}\frac{\partial \zeta}{\partial v} &= -\left(\xi\frac{\partial A_1}{\partial u} + \eta\frac{\partial B_1}{\partial u} + \zeta\frac{\partial C_1}{\partial u}\right) \\ \frac{\partial a_2}{\partial u}\frac{\partial \xi}{\partial v} + \frac{\partial b_2}{\partial u}\frac{\partial \eta}{\partial v} + \frac{\partial c_2}{\partial u}\frac{\partial \zeta}{\partial v} &= -\left(\xi\frac{\partial A_2}{\partial u} + \eta\frac{\partial B_2}{\partial u} + \zeta\frac{\partial C_2}{\partial u}\right) \\ \frac{\partial a_3}{\partial u}\frac{\partial \xi}{\partial v} + \frac{\partial b_3}{\partial u}\frac{\partial \eta}{\partial v} + \frac{\partial c_3}{\partial u}\frac{\partial \zeta}{\partial v} &= -\left(\xi\frac{\partial A_3}{\partial u} + \eta\frac{\partial B_3}{\partial u} + \zeta\frac{\partial C_3}{\partial u}\right) \end{aligned}\right\}$$

also are satisfied. That they are so satisfied, in connection with the equations which now define ξ, η, ζ, can be established as follows.

Take the set of equations now defining ξ, η, ζ, and multiply them by

$$\frac{1}{H_1}\frac{\partial H_1}{\partial u},\qquad -\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v},\qquad -\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w};$$

then adding, and remembering that $x_1, x_2, x_3 = a_1, a_2, a_3$ constitute a particular

* The quantity A_1 formally is zero: it is retained for the sake of symmetry.

system of solutions for the first set of three equations in x, and similarly for $x_1, x_2, x_3 = b_1, b_2, b_3$, and for $x_1, x_2, x_3 = c_1, c_2, c_3$, we have

$$\begin{aligned}\frac{\partial a_1}{\partial u}\frac{\partial \xi}{\partial v}+\frac{\partial b_1}{\partial u}\frac{\partial \eta}{\partial v}+\frac{\partial c_1}{\partial u}\frac{\partial \zeta}{\partial v} &\\ = -\xi\left(A_1\frac{1}{H_1}\frac{\partial H_1}{\partial u}-A_2\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}-A_3\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right)&\\ -\eta\left(B_1\frac{1}{H_1}\frac{\partial H_1}{\partial u}-B_2\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}-B_3\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right)&\\ -\zeta\left(C_1\frac{1}{H_1}\frac{\partial H_1}{\partial u}-C_2\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}-C_3\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right)&.\end{aligned}$$

Similarly for the second and the third of the set of equations for ξ, η, ζ. Accordingly, they all will be satisfied, if only nine relations of the type

$$\frac{\partial A_1}{\partial u}=A_1\frac{1}{H_1}\frac{\partial H_1}{\partial u}-A_2\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}-A_3\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}$$

are satisfied. Now

$$A_1=a_{21}-a_1\frac{1}{H_1}\frac{\partial H_1}{\partial v}-a_2\frac{1}{H_2}\frac{\partial H_2}{\partial u},$$

so that

$$\frac{\partial A_1}{\partial u}=\frac{\partial a_{21}}{\partial u}-a_{11}\frac{1}{H_1}\frac{\partial H_1}{\partial v}-a_{12}\frac{1}{H_2}\frac{\partial H_2}{\partial u}-a_1\frac{\partial}{\partial u}\left(\frac{1}{H_1}\frac{\partial H_1}{\partial v}\right)-a_2\frac{\partial}{\partial u}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial u}\right).$$

But

$$\begin{aligned}\frac{\partial a_{21}}{\partial u}=\frac{\partial a_{11}}{\partial v}&\\ =a_{12}\frac{1}{H_1}\frac{\partial H_1}{\partial u}+a_{22}\left(-\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}\right)+a_{23}\left(-\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right)&\\ +a_1\frac{\partial}{\partial v}\left(\frac{1}{H_1}\frac{\partial H_1}{\partial u}\right)-a_2\frac{\partial}{\partial v}\left(\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}\right)-a_3\frac{\partial}{\partial v}\left(\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}\right)&.\end{aligned}$$

In the expression for $\frac{\partial A_1}{\partial u}$, insert this value for $\frac{\partial a_{21}}{\partial u}$; substitute for a_{11}, a_{12} from the original equations which they satisfy; gather together the terms in a_{22}, a_{23}, and substitute for a_{22}, a_{23} from A_2, A_3: then, after reductions which are merely laborious, we find

$$\frac{\partial A_1}{\partial u}=-A_2\frac{H_1}{H_2^2}\frac{\partial H_1}{\partial v}-A_3\frac{H_1}{H_3^2}\frac{\partial H_1}{\partial w}.$$

Also $A_1=0$. Hence the first of the nine relations is satisfied.

In precisely the same way, the other eight of those relations can be proved to be satisfied.

Consequently the three equations defining ξ, η, ζ possess a primitive in which ξ, η, ζ occur as functions that do not involve u. As those equations are homogeneous and linear in ξ, η, ζ and their derivatives with regard to v, the primitive is of the form

$$\left.\begin{aligned}\xi &= \rho\xi' + \sigma\xi'' + \tau\xi'''\\ \eta &= \rho\eta' + \sigma\eta'' + \tau\eta'''\\ \zeta &= \rho\zeta' + \sigma\zeta'' + \tau\zeta'''\end{aligned}\right\},$$

where ρ, σ, τ are arbitrary constants so far as derivation with respect to v is concerned; and where ξ', η', ζ' are one special set of solutions, ξ'', η'', ζ'' are another special set, and ξ''', η''', ζ''' are a third special set, these three sets being linearly independent of one another. When these values of ξ, η, ζ are substituted in x_1, x_2, x_3, we have

$$\left.\begin{aligned}x_1 &= \rho\alpha_1 + \sigma\beta_1 + \tau\gamma_1\\ x_2 &= \rho\alpha_2 + \sigma\beta_2 + \tau\gamma_2\\ x_3 &= \rho\alpha_3 + \sigma\beta_3 + \tau\gamma_3\end{aligned}\right\}$$

as the simultaneous primitive of the first two of the three sets of equations satisfied by x; and ρ, σ, τ are (so far as concerns these two sets) arbitrary functions of w. Also

$$\begin{aligned}x_1,\ x_2,\ x_3 &= \alpha_1,\ \alpha_2,\ \alpha_3;\\ &= \beta_1,\ \beta_2,\ \beta_3;\\ &= \gamma_1,\ \gamma_2,\ \gamma_3;\end{aligned}$$

are special simultaneous sets of solutions of the two sets of equations.

250. The third set of equations for x, viz.

$$\left.\begin{aligned}x_{13} - x_1\frac{1}{H_1}\frac{\partial H_1}{\partial w} \qquad\qquad\qquad\qquad - x_3\frac{1}{H_3}\frac{\partial H_3}{\partial u} &= 0\\ x_{23} \qquad\qquad - x_2\frac{1}{H_2}\frac{\partial H_2}{\partial w} - x_3\frac{1}{H_3}\frac{\partial H_3}{\partial v} &= 0\\ x_{33} + x_1\frac{H_3}{H_1^2}\frac{\partial H_3}{\partial u} + x_2\frac{H_3}{H_2^2}\frac{\partial H_3}{\partial v} - x_3\frac{1}{H_3}\frac{\partial H_3}{\partial w} &= 0\end{aligned}\right\},$$

must be satisfied. All that remains at our disposal for satisfying them are three arbitrary (and so disposable) functions ρ, σ, τ, of w. The procedure is similar to the procedure for the preceding set; the difference lies in the fact that, when the linear equations of the first order for ρ, σ, τ are formed, two subsidiary systems have to be satisfied instead of one alone. All the necessary tests are satisfied in virtue of the six relations between H_1, H_2, H_3; and the result is that the most general values of ρ, σ, τ are

$$\left.\begin{aligned}\rho &= \lambda\rho_1 + \mu\rho_2 + \nu\rho_3\\ \sigma &= \lambda\sigma_1 + \mu\sigma_2 + \nu\sigma_3\\ \tau &= \lambda\tau_1 + \mu\tau_2 + \nu\tau_3\end{aligned}\right\};$$

where λ, μ, ν are arbitrary constants; and where the other quantities are functions of w only, forming linearly independent sets of solutions of the equations for ρ, σ, τ.

When these values are substituted in the expressions for x_1, x_2, x_3, we have (as the ultimate primitive of the nine equations satisfied by x) the expressions

$$\left.\begin{aligned} x_1 &= \lambda X_1 + \mu Y_1 + \nu Z_1 \\ x_2 &= \lambda X_2 + \mu Y_2 + \nu Z_2 \\ x_3 &= \lambda X_3 + \mu Y_3 + \nu Z_3 \end{aligned}\right\}.$$

The quantities λ, μ, ν are arbitrary constants; the other quantities are functions of u, v, w, which combine to form special sets of simultaneous solutions of the equations.

The equations determining y_1, y_2, y_3 are precisely the same as those for x_1, x_2, x_3; hence their primitive is

$$\left.\begin{aligned} y_1 &= \lambda' X_1 + \mu' Y_1 + \nu' Z_1 \\ y_2 &= \lambda' X_2 + \mu' Y_2 + \nu' Z_2 \\ y_3 &= \lambda' X_3 + \mu' Y_3 + \nu' Z_3 \end{aligned}\right\},$$

where λ', μ', ν' are arbitrary constants. Likewise as to the equations determining z_1, z_2, z_3; their primitive is

$$\left.\begin{aligned} z_1 &= \lambda'' X_1 + \mu'' Y_1 + \nu'' Z_1 \\ z_2 &= \lambda'' X_2 + \mu'' Y_2 + \nu'' Z_2 \\ z_3 &= \lambda'' X_3 + \mu'' Y_3 + \nu'' Z_3 \end{aligned}\right\},$$

where λ'', μ'', ν'' are arbitrary constants.

251. Thus the complete primitive of all the equations together appears to contain nine arbitrary constants. But these equations are not independent of one another; they are differential inferences from the earlier equations, viz. from

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= H_1^2, & x_2x_3 + y_2y_3 + z_2z_3 &= 0, \\ x_2^2 + y_2^2 + z_2^2 &= H_2^2, & x_3x_1 + y_3y_1 + z_3z_1 &= 0, \\ x_3^2 + y_3^2 + z_3^2 &= H_3^2, & x_1x_2 + y_1y_2 + z_1z_2 &= 0; \end{aligned}$$

consequently the complete primitive, which has been obtained, must satisfy these equations also. When substitution takes place in the first of them, we have

$$X_1^2\Sigma\lambda^2 + 2X_1Y_1\Sigma\lambda\mu + 2X_1Z_1\Sigma\lambda\nu + Y_1^2\Sigma\mu^2 + 2Y_1Z_1\Sigma\mu\nu + Z_1^2\Sigma\nu^2 = H_1^2.$$

But X_1, Y_1, $Z_1 = x_1$, y_1, z_1, constitute a special solution, so that

$$X_1^2 + Y_1^2 + Z_1^2 = H_1^2;$$

hence, writing

$$k_1 = \Sigma\lambda^2 - 1, \quad k_2 = \Sigma\lambda\mu, \quad k_3 = \Sigma\lambda\nu, \quad k_4 = \Sigma\mu^2 - 1, \quad k_5 = \Sigma\mu\nu, \quad k_6 = \Sigma\nu^2 - 1,$$

we have

$$X_1^2k_1 + 2X_1Y_1k_2 + 2X_1Z_1k_3 + Y_1^2k_4 + 2Y_1Z_1k_5 + Z_1^2k_6 = 0.$$

Treating the other equations in the same way, we find

$$X_1X_2k_1 + (X_1Y_2 + X_2Y_1)k_2 + (X_1Z_2 + X_2Z_1)k_3 + Y_1Y_2k_4 + (Y_1Z_2 + Y_2Z_1)k_5 + Z_1Z_2k_6 = 0,$$

$$X_1X_3k_1 + (X_1Y_3 + X_3Y_1)k_2 + (X_1Z_3 + X_3Z_1)k_3 + Y_1Y_3k_4 + (Y_1Z_3 + Y_3Z_1)k_5 + Z_1Z_3k_6 = 0,$$

$$X_2^2k_1 + 2X_2Y_2k_2 + 2X_2Z_2k_3 + Y_2^2k_4 + 2Y_2Z_2k_5 + Z_2^2k_6 = 0,$$

$$X_2X_3k_1 + (X_2Y_3 + X_3Y_2)k_2 + (X_2Z_3 + X_3Z_2)k_3 + Y_2Y_3k_4 + (Y_2Z_3 + Y_3Z_2)k_5 + Z_2Z_3k_6 = 0,$$

$$X_3^2k_1 + 2X_3Y_3k_2 + 2X_3Z_3k_3 + Y_3^2k_4 + 2Y_3Z_3k_5 + Z_3^2k_6 = 0.$$

Thus there are six equations, homogeneous and linear in the six quantities $k_1, k_2, k_3, k_4, k_5, k_6$. The determinant of the coefficients of the six quantities in these equations is equal to

$$\begin{vmatrix} X_1, & Y_1, & Z_1 \\ X_2, & Y_2, & Z_2 \\ X_3, & Y_3, & Z_3 \end{vmatrix}^4,$$

which does not vanish because the quantities X, Y, Z constitute three linearly independent solutions of our equations; hence we must have

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0, \quad k_4 = 0, \quad k_5 = 0, \quad k_6 = 0,$$

that is,

$$\lambda^2 + \lambda'^2 + \lambda''^2 = 1, \quad \mu\nu + \mu'\nu' + \mu''\nu'' = 0,$$

$$\mu^2 + \mu'^2 + \mu''^2 = 1, \quad \nu\lambda + \nu'\lambda' + \nu''\lambda'' = 0,$$

$$\nu^2 + \nu'^2 + \nu''^2 = 1, \quad \lambda\mu + \lambda'\mu' + \lambda''\mu'' = 0.$$

Thus the nine constants are limited by the six equations satisfied by the direction-cosines of any three directions in space that are perpendicular to one another. Now

$$dx = x_1du + x_2dv + x_3dw,$$

and similarly for dy and dz; hence

$$x - A = \int x_1du + x_2dv + x_3dw$$

$$\left.\begin{aligned} &= \lambda x' + \mu y' + \nu z' \\ y - B &= \lambda' x' + \mu' y' + \nu' z' \\ z - C &= \lambda'' x' + \mu'' y' + \nu'' z' \end{aligned}\right\},$$

where x', y', z' are definite functions of u, v, w, and A, B, C are arbitrary constants. The result can be enunciated in the form:—

Quantities H_1, H_2, H_3, satisfying the six characteristic equations, determine a triply orthogonal system of surfaces uniquely save as to position and orientation in space.

The theorem is the extension, to triply orthogonal surfaces in space, of Bonnet's theorem (§ 37) concerning the determination of a surface in general by its fundamental magnitudes. After the theorem, a knowledge of appropriate quantities H_1, H_2, H_3 is sufficient to ensure the existence of a triply orthogonal system; the difficulty is to obtain this knowledge.

Ex. As an illustration of the use of these equations, consider the conformal representation of space upon itself.

Let x, y, z be the coordinates of a point in space; and let u, v, w be the coordinates of the associated point in the conformal representation. The arc-elements are given by

$$ds^2 = dx^2 + dy^2 + dz^2, \quad ds'^2 = du^2 + dv^2 + dw^2.$$

As the representation is conformal, we must have

$$ds' = \lambda\, ds,$$

where λ is any variable function free from differential elements; hence

$$dx^2 + dy^2 + dz^2 = \frac{1}{\lambda^2}(du^2 + dv^2 + dw^2).$$

Consequently

$$H_1 = H_2 = H_3 = \frac{1}{\lambda}.$$

Let these values be substituted in the three relations of the type

$$\frac{\partial^2 H_1}{\partial v \partial w} - \frac{1}{H_2}\frac{\partial H_2}{\partial w}\frac{\partial H_1}{\partial v} - \frac{1}{H_3}\frac{\partial H_3}{\partial v}\frac{\partial H_1}{\partial w} = 0;$$

they give

$$\frac{\partial^2 \lambda}{\partial v \partial w} = 0, \qquad \frac{\partial^2 \lambda}{\partial w \partial u} = 0, \qquad \frac{\partial^2 \lambda}{\partial u \partial v} = 0,$$

so that

$$\lambda = U + V + W,$$

where (so far as these relations are concerned) U is any function of u alone, V of v alone, and W of w alone.

Let the values of H_1, H_2, H_3 be substituted in the three relations of the type

$$\frac{\partial}{\partial u}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial v}\right) + \frac{1}{H_3{}^2}\frac{\partial H_1}{\partial w}\frac{\partial H_2}{\partial w} = 0;$$

they give three equations of the form

$$\frac{\partial^2 \lambda}{\partial u^2} + \frac{\partial^2 \lambda}{\partial v^2} = \frac{1}{\lambda}\left\{\left(\frac{\partial \lambda}{\partial u}\right)^2 + \left(\frac{\partial \lambda}{\partial v}\right)^2 + \left(\frac{\partial \lambda}{\partial w}\right)^2\right\}.$$

Inserting the value of λ, we find

$$U'' + V'' = V'' + W'' = W'' + U''$$
$$= \frac{U'^2 + V'^2 + W'^2}{U + V + W}.$$

Hence

$$U'' = V'' = W'';$$

and therefore, as U, V, W are functions of u alone, v alone, and w alone, respectively, we must have

$$U'' = V'' = W'' = \frac{2}{a} \text{ or } 0,$$

where a is a finite constant.

Taking the common value of U'', V'', W'' to be $2/a$, we have

$$U=\frac{1}{a}(u-a_1)^2+a_2,$$

$$V=\frac{1}{a}(v-b_1)^2+b_2,$$

$$W=\frac{1}{a}(w-c_1)^2+c_2,$$

where the new quantities a, b, c are constants. But

$$U''+V''=\frac{U'^2+V'^2+W'^2}{U+V+W};$$

when the values are substituted, we must have

$$a_2+b_2+c_2=0.$$

Hence, changing the origin for u, v, w (which amounts only to a displacement in space), we have

$$\lambda=U+V+W=\frac{1}{a}(u^2+v^2+w^2),$$

while

$$H_1=H_2=H_3=\frac{1}{\lambda}.$$

Let this value, common for H_1, H_2, H_3, be substituted in the equations for the derivatives of a_1, a_2, a_3, obtained in § 244. They become

$$\frac{\partial a_1}{\partial u}=\frac{2}{a\lambda}(va_2+wa_3),\qquad \frac{\partial a_1}{\partial v}=-\frac{2}{a\lambda}ua_2,\qquad \frac{\partial a_1}{\partial w}=-\frac{2}{a\lambda}ua_3,$$

$$\frac{\partial a_2}{\partial u}=-\frac{2}{a\lambda}va_1,\qquad \frac{\partial a_2}{\partial v}=\frac{2}{a\lambda}(ua_1+wa_3),\qquad \frac{\partial a_2}{\partial w}=-\frac{2}{a\lambda}va_3,$$

$$\frac{\partial a_3}{\partial u}=-\frac{2}{a\lambda}wa_1,\qquad \frac{\partial a_3}{\partial v}=-\frac{2}{a\lambda}wa_2,\qquad \frac{\partial a_3}{\partial w}=\frac{2}{a\lambda}(ua_1+va_2);$$

together with similar equations for β_1, β_2, β_3 and γ_1, γ_2, γ_3.

Integrating, and maintaining the relations of the type

$$a_1^2+\beta_1^2+\gamma_1^2=1,\quad \beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3=0,$$

we have

$$a_1=1-\frac{2u^2}{a\lambda},\qquad a_2=-\frac{2uv}{a\lambda},\qquad a_3=-\frac{2uw}{a\lambda},$$

$$\beta_1=-\frac{2uv}{a\lambda},\qquad \beta_2=1-\frac{2v^2}{a\lambda},\qquad \beta_3=-\frac{2vw}{a\lambda},$$

$$\gamma_1=-\frac{2uw}{a\lambda},\qquad \gamma_2=-\frac{2vw}{a\lambda},\qquad \gamma_3=1-\frac{2w^2}{a\lambda}.$$

But

$$x_1=a_1H_1,\quad x_2=a_2H_2,\quad x_3=a_3H_3,$$

so that

$$dx=\frac{1}{\lambda}\left\{\left(1-\frac{2u^2}{a\lambda}\right)du-\frac{2uv}{\lambda}dv-\frac{2uw}{\lambda}dw\right\},$$

and therefore

$$x-A=\frac{au}{u^2+v^2+w^2}.$$

Similarly

$$y-B=\frac{av}{u^2+v^2+w^2},$$

$$z-C=\frac{aw}{u^2+v^2+w^2}.$$

These equations express an inversion with respect to the sphere

$$u^2+v^2+w^2=a.$$

Next, taking the common value of U'', V'', W'' to be zero, and noting the relation

$$U''+V''=\frac{U'^2+V'^2+W'^2}{U+V+W},$$

we have U', V', W' zero. Hence U, V, W, λ, H_1, H_2, H_3 are constant, and so the equations for x, y, z, when integrated, give

$$x-A,\ y-B,\ z-C=\left(\begin{array}{ccc} a, & a', & a'' \\ b, & b', & b'' \\ c, & c', & c'' \end{array}\right)(u,\ v,\ w),$$

where the constants a, b, c on the right-hand side are proportional to the direction-cosines of three perpendicular straight lines. These equations express displacement and rotation, with constant magnification.

Hence there are only two independent methods of representing space conformally upon itself, viz.

(i) by displacement and rotation, together with constant magnification,

(ii) by inversion.

The two methods can be repeated and combined in any manner and any number of times.

252. The difficulty of determining H_1, H_2, H_3 does not depend solely upon the fact that, by one method of procedure, we should be obliged to solve a number of simultaneous partial equations of the second order. An added complexity is caused by the fact that the number of independent equations in the system is greater than the number of dependent variables involved; and so even Cauchy's existence-theorem cannot be applied to the system.

A preliminary investigation reveals the degree of generality which is the utmost to be expected among such solutions as exist. In order that three surfaces

$$u(x,\ y,\ z)=u,\quad v(x,\ y,\ z)=v,\quad w(x,\ y,\ z)=w,$$

may be orthogonal to one another (the quantities u, v, w on the right-hand sides being parametric), the equations

$$\left.\begin{array}{r} v_1w_1+v_2w_2+v_3w_3=0 \\ w_1u_1+w_2u_2+w_3u_3=0 \\ u_1v_1+u_2v_2+u_3v_3=0 \end{array}\right\}$$

must be satisfied. Let

$$-S_1=v_1w_1+v_2w_2,\quad -S_2=w_1u_1+w_2u_2,\quad -S_3=u_1v_1+u_2v_2;$$

then resolving the equations for u_3, v_3, w_3, we find

$$u_3 = \left(\frac{S_2 S_3}{S_1}\right)^{\frac{1}{2}}, \quad v_3 = \left(\frac{S_3 S_1}{S_2}\right)^{\frac{1}{2}}, \quad w_3 = \left(\frac{S_1 S_2}{S_3}\right)^{\frac{1}{2}}.$$

This is a set of three partial equations of the first order in three dependent variables, and so we can apply Cauchy's theorem, as follows. Take any arbitrary (constant) value of z, say* $z = 0$; and let α, β, γ denote three arbitrary functions of x and y, such that no two of the curves

$$\alpha = \text{constant}, \quad \beta = \text{constant}, \quad \gamma = \text{constant},$$

in the plane of z cut orthogonally. Then the quantities S_1, S_2, S_3 do not vanish when $z = 0$; and so, within some range of values of z, the values of the branches of u_3, v_3, w_3 are uniform and continuous. Then Cauchy's theorem declares that unique uniform functions u, v, w exist, satisfying the partial equations, and acquiring the values α, β, γ when $z = 0$; in other words, *a triply orthogonal system exists, determined by the condition that three surfaces pass through any three assigned curves in the plane of z, provided no two of these three curves cut one another orthogonally.* (The same limitation must hold for any set of three curves in the plane through which any three of the surfaces would pass.)

We have taken any plane $z = 0$. A corresponding theorem, with corresponding limitations, would hold if we chose curves in any plane in space or curves on any surface in space.

The importance of the result is obvious. The utmost degree of generality that can be expected from solutions of the six characteristic equations corresponds to the generality represented by the assignment of three arbitrary functions of two variables.

The critical equation of the third order.

253. To make some nearer approach to the actual determination of triply orthogonal systems, we proceed once more from the conditions of orthogonality as follows. From two of them, we have

$$\frac{v_1}{\Psi_1} = \frac{v_2}{\Psi_2} = \frac{v_3}{\Psi_3},$$

where

$$\Psi_1 = w_2 u_3 - w_3 u_2, \quad \Psi_2 = w_3 u_1 - w_1 u_3, \quad \Psi_3 = w_1 u_2 - w_2 u_1.$$

In order that the function v may exist, it is necessary and sufficient that the equation

$$\Psi_1 dx + \Psi_2 dy + \Psi_3 dz = 0$$

* This particularisation involves no loss of generality; it only implies a change of origin of coordinates.

should have a single equation as its integral equivalent; hence the condition of integrability

$$\Psi_1\left(\frac{\partial\Psi_2}{\partial z}-\frac{\partial\Psi_3}{\partial y}\right)+\Psi_2\left(\frac{\partial\Psi_3}{\partial x}-\frac{\partial\Psi_1}{\partial z}\right)+\Psi_3\left(\frac{\partial\Psi_1}{\partial y}-\frac{\partial\Psi_2}{\partial x}\right)=0$$

must be satisfied. Now

$$\frac{\partial\Psi_2}{\partial z}-\frac{\partial\Psi_3}{\partial y}=u_1\nabla^2 w-w_1\nabla^2 u-(u_1w_{11}+u_2w_{12}+u_3w_{13})+(w_1u_{11}+w_2u_{12}+w_3u_{13}),$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

Moreover, the remaining equation of orthogonality is

$$u_1w_1+u_2w_2+u_3w_3=0,$$

so that

$$u_1w_{11}+u_2w_{12}+u_3w_{13}+w_1u_{11}+w_2u_{12}+w_3u_{13}=0;$$

hence

$$\frac{\partial\Psi_2}{\partial z}-\frac{\partial\Psi_3}{\partial y}=u_1\nabla^2 w-w_1\nabla^2 u+2(w_1u_{11}+w_2u_{12}+w_3u_{13}).$$

Similarly for the other quantities of the same kind. When the values are substituted, the condition of integrability becomes

$$\begin{vmatrix} w_1u_{11}+w_2u_{12}+w_3u_{13}, & w_1, & u_1 \\ w_1u_{12}+w_2u_{22}+w_3u_{23}, & w_2, & u_2 \\ w_1u_{13}+w_2u_{23}+w_3u_{33}, & w_3, & u_3 \end{vmatrix}=0,$$

which, on multiplication by 2, can be written

$$Aw_1^2+Bw_2^2+Cw_3^2+2Fw_2w_3+2Gw_3w_1+2Hw_1w_2=0,$$

where

$$\left.\begin{array}{ll} A=2(u_2u_{13}-u_3u_{12}), & F=u_1u_{22}-u_2u_{12}+u_3u_{13}-u_1u_{33} \\ B=2(u_3u_{21}-u_1u_{23}), & G=u_2u_{33}-u_3u_{23}+u_1u_{21}-u_2u_{11} \\ C=2(u_1u_{32}-u_2u_{31}), & H=u_3u_{11}-u_1u_{31}+u_2u_{32}-u_3u_{22} \end{array}\right\}.$$

Manifestly the same equation would be satisfied, if we write v_1, v_2, v_3 in place of w_1, w_2, w_3.

When we associate the equation

$$u_1w_1+u_2w_2+u_3w_3=0$$

with the preceding relation which is homogeneous and quadratic in w_1, w_2, w_3, and resolve the two equations for $w_1 : w_2 : w_3$, we find

$$w_1 : w_2 : w_3 = U : U' : U'',$$

where U, U', U'' are two-signed functions of A, B, C, F, G, H, u_1, u_2, u_3. (For one of the signs, we have $w_1 : w_2 : w_3$, while the other gives the values of $v_1 : v_2 : v_3$.) In order that the function w may exist, the equation

$$Udx+U'dy+U''dz=0$$

must satisfy the condition of integrability; hence

$$U\left(\frac{\partial U'}{\partial z}-\frac{\partial U''}{\partial y}\right)+U'\left(\frac{\partial U''}{\partial x}-\frac{\partial U}{\partial z}\right)+U''\left(\frac{\partial U}{\partial y}-\frac{\partial U'}{\partial x}\right)=0.$$

This relation is not evanescent. It remains as a partial differential equation of the third order satisfied by $u(x, y, z)$. Moreover, all the foregoing analysis is reversible; hence this condition is sufficient as well as necessary. So we have the theorem :—

In order that a family of surfaces, represented by

$$u(x, y, z) = \text{constant},$$

may form part of a triply orthogonal system, it is necessary and sufficient that u should satisfy a partial differential equation of the third order.

254. Should the equation be satisfied by $u(x, y, z)$, it still is necessary to determine $v(x, y, z)$ and $w(x, y, z)$, in order to have the full system. These two functions satisfy the same equation of the third order as $u(x, y, z)$; but it is unnecessary to take further solutions of that equation. We have seen that quantities U, U', U'' in the preceding analysis arise, as two-signed functions; let U, U', U'' denote one set, and Υ, Υ', Υ'' the other set, all of them involving derivatives of u alone. Then

$$w_1 : w_2 : w_3 = U : U' : U'',$$
$$v_1 : v_2 : v_3 = \Upsilon : \Upsilon' : \Upsilon''.$$

The condition of orthogonality ought to be satisfied, so that we ought to have

$$U\Upsilon + U'\Upsilon' + U''\Upsilon'' = 0.$$

Now the two sets of ratios are given by the equations

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta = 0,$$
$$\xi u_1 + \eta u_2 + \zeta u_3 = 0;$$

hence

$$\frac{U\Upsilon}{Bu_3^2 + Cu_2^2 - 2Fu_2u_3} = \frac{U'\Upsilon'}{Cu_1^2 + Au_3^2 - 2Gu_3u_1} = \frac{U''\Upsilon''}{Au_2^2 + Bu_1^2 - 2Hu_1u_2}.$$

It is easy to verify that

$$A + B + C = 0,$$

$$Au_1^2 + Bu_2^2 + Cu_3^2 + 2Fu_2u_3 + 2Gu_3u_1 + 2Hu_1u_2 = 0:$$

hence

$$U\Upsilon + U'\Upsilon' + U''\Upsilon'' = 0,$$

so that the condition of orthogonality is satisfied without any further conditions.

The surface, $w(x, y, z) = \text{constant}$, is obtained by the integration of the equation

$$Udx + U'dy + U''dz = 0;$$

and the surface, $v(x, y, z) =$ constant, is obtained by the integration of the equation

$$\Upsilon dx + \Upsilon' dy + \Upsilon'' dz = 0.$$

Hence, when one of the families of surfaces is known, the triply orthogonal system can be completed by the integration of two ordinary equations of the first order.

255. The equation of the third order, satisfied by any one of the families in the triple system, is

$$U\left(\frac{\partial U'}{\partial z} - \frac{\partial U''}{\partial y}\right) + U'\left(\frac{\partial U''}{\partial x} - \frac{\partial U}{\partial z}\right) + U''\left(\frac{\partial U}{\partial y} - \frac{\partial U'}{\partial x}\right) = 0.$$

When the values of U, U', U'' are substituted*, we have the equation required; but the analysis is long and laborious. In preference, we adopt the following method of constructing the partial equation of the third order to be satisfied by a family of surfaces forming part of a triply orthogonal system; it is due† to Darboux.

Let a denote any one of the three quantities u, v, w; the operator D_a is used, where

$$D_a = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}.$$

We have

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = 0.$$

Denoting by x_m any one of the variables x, y, z, we have

$$\frac{\partial}{\partial x_m}(u_1 v_1 + u_2 v_2 + u_3 v_3) = 0,$$

and therefore

$$D_u v_m + D_v u_m = 0;$$

and, similarly,

$$D_u w_m + D_w u_m = 0.$$

Again, we have

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0,$$

say,

$$\sum_m v_m w_m = 0;$$

hence

$$\sum_m w_m D_u v_m + \sum_m v_m D_u w_m = 0,$$

and therefore

$$\sum_m w_m D_v u_m + \sum_m v_m D_w u_m = 0.$$

When this is expanded and a superfluous factor 2 is removed, it becomes

$$\sum_m \sum_n v_m w_n u_{mn} = 0.$$

* This is the method adopted by Cayley; see *Coll. Math. Papers*, vol. viii, no. 518, where the equation is obtained with a superfluous factor; *ib.*, vol. viii, no. 519, where the superfluous factor has been removed from the equation.

† *Systèmes orthogonaux*, (1910), §§ 9—12.

To the last relation, we apply the operator D_u; and then, using the above relations, we find

$$\sum_m \sum_n v_m w_n D_u u_{mn} - \sum_m \sum_n v_m u_{mn} D_w u_n - \sum_m \sum_n w_n u_{mn} D_v u_m = 0.$$

Let

$$\begin{aligned} A_{mn} &= D_u u_{mn} - 2\,(u_{m1}u_{1n} + u_{m2}u_{2n} + u_{m3}u_{3n}) \\ &= \sum_l u_l u_{lmn} - 2\sum_l u_{ml} u_{ln}, \end{aligned}$$

so that

$$A_{mn} = A_{nm};$$

the new equation is

$$\sum_m \sum_n \beta_m \gamma_n A_{mn} = 0.$$

We thus have three equations, containing homogeneously and linearly the six quantities*

$$v_1 w_1, \quad v_1 w_2 + v_2 w_1, \quad v_2 w_2, \quad v_1 w_3 + v_3 w_1, \quad v_2 w_3 + v_3 w_2, \quad v_3 w_3;$$

they are

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0,$$

$$u_{11} v_1 w_1 + u_{12}(v_1 w_2 + v_2 w_1) + u_{22} v_2 w_2 + u_{13}(v_1 w_3 + v_3 w_1) + u_{23}(v_2 w_3 + v_3 w_2) + u_{33} v_3 w_3 = 0,$$

$$A_{11} v_1 w_1 + A_{12}(v_1 w_2 + v_2 w_1) + A_{22} v_2 w_2 + A_{13}(v_1 w_3 + v_3 w_1) + A_{23}(v_2 w_3 + v_3 w_2) + A_{33} v_3 w_3 = 0.$$

Further, we have the earlier equations

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = 0,$$

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0;$$

so, multiplying these by w_1 and v_1, w_2 and v_2, w_3 and v_3, and adding in each case, we have three further equations

$$2u_1 v_1 w_1 + u_2(v_1 w_2 + v_2 w_1) + u_3(v_1 w_3 + v_3 w_1) = 0,$$

$$u_1(v_1 w_2 + v_2 w_1) + 2u_2 v_2 w_2 + u_3(v_2 w_3 + v_3 w_2) = 0,$$

$$u_1(v_1 w_3 + v_3 w_1) + u_2(v_2 w_3 + v_3 w_2) + 2u_3 v_3 w_3 = 0,$$

homogeneous and linear in the same six quantities. Now these six quantities do not simultaneously vanish; hence the determinant of their coefficients in our six linear equations must vanish, so that we have

$$\begin{vmatrix} A_{11}, & A_{22}, & A_{33}, & A_{23}, & A_{31}, & A_{12} \\ u_{11}, & u_{22}, & u_{33}, & u_{23}, & u_{31}, & u_{12} \\ 1, & 1, & 1, & 0, & 0, & 0 \\ 2u_1, & 0, & 0, & 0, & u_3, & u_2 \\ 0, & 2u_2, & 0, & u_3, & 0, & u_1 \\ 0, & 0, & 2u_3, & u_2, & u_1, & 0 \end{vmatrix} = 0.$$

* It may be added that, save as to a common multiplier, these six quantities are equal to

$$U\Upsilon, \quad U\Upsilon' + U'\Upsilon, \quad U'\Upsilon', \quad U\Upsilon'' + U''\Upsilon, \quad U'\Upsilon'' + U''\Upsilon', \quad U''\Upsilon'',$$

in the notation of § 254; but these relations will not be used for our immediate purpose.

Expanding this determinant, and removing a factor 2, we find

$$\begin{aligned}\Sigma A_{11}\,&\{u_1u_2u_3(u_{22}-u_{33})-u_1u_{23}(u_2^2-u_3^2)+(u_2^2+u_3^2)(u_2u_{13}-u_3u_{12})\}\\&+\Sigma A_{12}\,[u_3\{u_1^2(u_{33}-u_{22})+u_2^2(u_{11}-u_{33})-u_3^2(u_{22}-u_{11})\}\\&\qquad+2\{(u_1^2+u_2^2)u_2u_{23}-(u_2^2+u_3^2)u_1u_{31}\}]=0,\end{aligned}$$

where the first summation is cyclical for 11, 22, 33, and the second summation is cyclical for 12, 23, 31.

As A_{11}, A_{22}, A_{33}, A_{23}, A_{31}, A_{12} contain the derivatives of the third order linearly, this equation has the form

$$\Theta+\Phi=0,$$

where Θ is linear in the derivatives of the third order,
linear .... second order,
quartic..... .... first order,

while Φ is cubic .... second order,
cubic .... first order.

So far, however, as concerns the simpler applications, it is easier to deal with the unexpanded form of equation.

256. Many forms can be given to the equation; among them is one which has a similar form, though with a different first row of constituents. Let

$$T=(u_1^2+u_2^2+u_3^2)^{-\frac{1}{2}},$$

so that $Th_1=1$; but we keep T as the variable* in preference to h_1. Then we have

$$T_{11}+\frac{A_{11}}{h_1^3}=\frac{3}{h_1^5}(u_1u_{11}+u_2u_{12}+u_3u_{13})^2-\frac{3}{h_1^3}(u_{11}^2+u_{12}^2+u_{13}^2),$$

$$\begin{aligned}T_{12}+\frac{A_{12}}{h_1^3}=\frac{3}{h_1^5}&(u_1u_{11}+u_2u_{12}+u_3u_{13})(u_1u_{12}+u_2u_{22}+u_3u_{23})\\&-\frac{3}{h_1^3}(u_{11}u_{12}+u_{12}u_{22}+u_{13}u_{23});\end{aligned}$$

the other second derivatives of T are given by cyclical interchange. Now

$$\begin{aligned}A_{11}v_1w_1+A_{12}(v_1w_2+v_2w_1)+A_{13}(v_1w_3+v_3w_1)+A_{22}v_2w_2&\\+A_{23}(v_2w_3+v_3w_2)+A_{33}v_3w_3&=0.\end{aligned}$$

Substitute from the above relations for the quantities A; there are three aggregates of terms.

* It is easy to prove that $T\dfrac{du}{dn}=1$, where dn is an element of arc along the normal. The quantity du/dn may be regarded as the dilatation of the surface u at the point.

In one aggregate, we have a set involving the second derivatives of T, the same in form as the left-hand side of our equation with a factor h_1^{-3}.

The aggregate of terms with the factor $3h_1^{-5}$ is

$$= \left(v_1 \frac{\partial h_1}{\partial x} + v_2 \frac{\partial h_1}{\partial y} + v_3 \frac{\partial h_1}{\partial z}\right)\left(w_1 \frac{\partial h_1}{\partial x} + w_2 \frac{\partial h_1}{\partial y} + w_3 \frac{\partial h_1}{\partial z}\right) h_1^2$$

$$= h_1^2 h_2^2 h_3^2 \left(x_2 \frac{\partial h_1}{\partial x} + y_2 \frac{\partial h_1}{\partial y} + z_2 \frac{\partial h_1}{\partial z}\right)\left(x_3 \frac{\partial h_1}{\partial x} + y_3 \frac{\partial h_1}{\partial y} + z_3 \frac{\partial h_1}{\partial z}\right)$$

$$= h_1^2 h_2^2 h_3^2 \frac{\partial h_1}{\partial v} \frac{\partial h_1}{\partial w}.$$

The aggregate of terms with the factor $-3h_1^{-3}$ is

$$\begin{aligned}
&(v_1 u_{11} + v_2 u_{21} + v_3 u_{31})(w_1 u_{11} + w_2 u_{21} + w_3 u_{31}) \\
&+ (v_1 u_{12} + v_2 u_{22} + v_3 u_{32})(w_1 u_{12} + w_2 u_{22} + w_3 u_{32}) \\
&+ (v_1 u_{13} + v_2 u_{23} + v_3 u_{33})(w_1 u_{13} + w_2 u_{23} + w_3 u_{33}).
\end{aligned}$$

Now

$$\begin{aligned}
v_1 u_{11} + v_2 u_{21} + v_3 u_{31} &= h_2^2 (x_2 u_{11} + y_2 u_{21} + z_2 u_{31}) \\
&= h_2^2 \frac{\partial u_1}{\partial v} \\
&= h_2^2 \frac{\partial}{\partial v}(x_1 h_1^2) \\
&= h_2^2 \left(x_{12} h_1^2 + 2x_1 h_1 \frac{\partial h_1}{\partial v}\right) \\
&= h_2^2 \left(x_1 h_1 \frac{\partial h_1}{\partial v} - x_2 \frac{h_1^2}{h_2} \frac{\partial h_2}{\partial u}\right);
\end{aligned}$$

also

$$w_1 u_{11} + w_2 u_{21} + w_3 u_{31} = h_3^2 \left(x_1 h_1 \frac{\partial h_1}{\partial w} - x_3 \frac{h_1^2}{h_3} \frac{\partial h_3}{\partial u}\right);$$

and similarly for the others. Hence the aggregate is

$$= h_2^2 h_3^2 \left\{h_1^2 \frac{\partial h_1}{\partial v} \frac{\partial h_1}{\partial w} \Sigma x_1^2 - \frac{h_1^3}{h_2} \frac{\partial h_2}{\partial u} \frac{\partial h_1}{\partial w} \Sigma x_1 x_2 - \frac{h_1^3}{h_3} \frac{\partial h_1}{\partial v} \frac{\partial h_3}{\partial u} \Sigma x_1 x_3 + \frac{h_1^4}{h_2 h_3} \frac{\partial h_2}{\partial v} \frac{\partial h_3}{\partial u} \Sigma x_2 x_3\right\}$$

$$= h_2^2 h_3^2 h_1^2 \frac{\partial h_1}{\partial v} \frac{\partial h_1}{\partial w} H_1^2$$

$$= h_2^2 h_3^2 \frac{\partial h_1}{\partial v} \frac{\partial h_1}{\partial w}.$$

Thus the second and third aggregates cancel; and so the equation becomes

$$T_{11} v_1 w_1 + T_{12}(v_1 w_2 + v_2 w_1) + T_{13}(v_1 w_3 + v_3 w_1) + T_{22} v_2 w_2 + T_{23}(v_2 w_3 + v_3 w_2) + T_{33} v_3 w_3 = 0.$$

Taking the other five equations which involve v_1w_1, $v_1w_2+v_2w_1$, ..., v_3w_3 linearly and homogeneously, and eliminating these six quantities, we have

$$\begin{vmatrix} T_{11}, & T_{22}, & T_{33}, & T_{23}, & T_{31}, & T_{12} \\ u_{11}, & u_{22}, & u_{33}, & u_{23}, & u_{31}, & u_{12} \\ 1\ , & 1\ , & 1\ , & 0\ , & 0\ , & 0 \\ 2u_1, & 0\ . & 0\ , & 0\ , & u_3\ , & u_2 \\ 0\ , & 2u_2, & 0\ , & u_3\ , & 0\ , & u_1 \\ 0\ , & 0\ , & 2u_3, & u_2\ , & u_1\ , & 0 \end{vmatrix} = 0,$$

the other form of the equation indicated.

If this form (which was obtained first by Cayley) be denoted by

$$\Omega' = 0,$$

and the earlier form by

$$\Omega = 0,$$

the foregoing analysis shews that

$$\Omega = -\Omega' h_1^3.$$

The earlier form is the more direct; the latter will be used to establish a theorem due to Darboux.

257. Still another form can be given to the general equation of the third order; and it is required when the family of surfaces, forming part of a triply orthogonal system, is given by an equation

$$\phi(x, y, z, u) = 0,$$

where ϕ is either not actually or not conveniently resoluble with regard to u. Such a case occurs with the triple families of confocal quadrics.

Instead of proceeding from the last form of the equation, where the value of T would now be

$$-\frac{\partial\phi}{\partial u}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}^{-\frac{1}{2}},$$

we proceed from the earlier form involving the quantities A. Let

$$\frac{\partial\phi}{\partial u} = \phi', \quad \frac{\partial\phi}{\partial x} = \phi_1, \quad \frac{\partial\phi}{\partial y} = \phi_2, \quad \frac{\partial\phi}{\partial z} = \phi_3,$$

as usual, and similarly for the derivatives of other quantities; then

$$\phi_r + u_r\phi' = 0,$$

for $r = 1, 2, 3$. Hence three of the equations in § 255 become

$$2\phi_1 v_1 w_1 + \phi_2(v_1w_2 + v_2w_1) + \phi_3(v_1w_3 + v_3w_1) = 0,$$

$$\phi_1(v_1w_2 + v_2w_1) + 2\phi_2 v_2 w_2 + \phi_3(v_2w_3 + v_3w_2) = 0,$$

$$\phi_1(v_1w_3 + v_3w_1) + \phi_2(v_2w_3 + v_3w_2) + 2\phi_3 v_3 w_3 = 0;$$

and the equation

$$v_1w_1 + v_2w_2 + v_3w_3 = 0$$

is unaltered.

Again, we have

$$-u_{rs}\phi' = \phi_{rs} + u_r\phi_s' + u_s\phi_r' + u_ru_s\phi'';$$

hence, substituting in the equation

$$u_{11}v_1w_1 + u_{12}(v_1w_2 + v_2w_1) + u_{13}(v_1w_3 + v_3w_1) + u_{22}v_2w_2 + u_{23}(v_2w_3 + v_3w_2) + u_{33}v_3w_3 = 0,$$

after multiplication by $-\phi'$, the coefficient of ϕ''

$$= \Sigma u_1v_1 \,.\, \Sigma u_1w_1,$$

and therefore is zero; the coefficient of ϕ_1'

$$= v_1\Sigma u_1w_1 + w_1\Sigma u_1v_1,$$

and therefore is zero. Likewise for the coefficients of ϕ_2' and ϕ_3'. Hence the equation becomes

$$\phi_{11}v_1w_1 + \phi_{12}(v_1w_2 + v_2w_1) + \phi_{13}(v_1w_3 + v_3w_1) + \phi_{22}v_2w_2 + \phi_{23}(v_2w_3 + v_3w_2) + \phi_{33}v_3w_3 = 0.$$

There remains the sixth equation

$$A_{11}v_1w_1 + A_{12}(v_1w_2 + v_2w_1) + A_{13}(v_1w_3 + v_3w_1) + A_{22}v_2w_2 + A_{23}(v_2w_3 + v_3w_2) + A_{33}v_3w_3 = 0.$$

Now

$$-u_{rst}\phi' = \phi_{rst} + u_{rs}\phi_t' + u_{st}\phi_r' + u_{tr}\phi_s' + (u_{rs}u_t + u_{st}u_r + u_{tr}u_s)\phi''$$
$$+ u_ru_su_t\phi''' + u_ru_s\phi_t'' + u_su_t\phi_r'' + u_tu_r\phi_s'' + u_r\phi_{st}' + u_s\phi_{rt}' + u_t\phi_{rs}';$$

as corresponding to the quantities A_{rs}, we introduce quantities Φ_{rs} under the definitions

$$\Phi_{rs} = \phi'\Sigma(\phi_t\phi_{rst} - 2\phi_{rt}\phi_{ts}) + \phi_r'\Sigma\phi_t\phi_{st} + \phi_s'\Sigma\phi_t\phi_{rt} - \phi_{rs}'\Sigma\phi_t^2,$$

where the summation in each case (as for the quantities A_{rs}) is for $t = 1, 2, 3$: and we substitute in the A-equation for the various quantities u_{rst}, u_{rs}, u_{rt}, u_{st}, after multiplying by ϕ'^3. In the resulting equation, the coefficient of ϕ''' is

$$\phi'\Sigma u_t^2\Sigma u_1v_1\Sigma u_1w_1,$$

which is zero; the coefficient of ϕ''^2 is

$$-2\phi'\Sigma u_t^2\Sigma u_1v_1\Sigma u_1w_1,$$

which also is zero; and similarly for all the aggregates of terms which contain ϕ'' as a factor. Gathering together the remainder, we have the resulting equation in the form

$$\Phi_{11}v_1w_1 + \Phi_{12}(v_1w_2 + v_2w_1) + \Phi_{13}(v_1w_3 + v_3w_1) + \Phi_{22}v_2w_2 + \Phi_{23}(v_2w_3 + v_3w_2) + \Phi_{33}v_3w_3 = 0.$$

Thus, as before, we have six equations, homogeneous and linear in the six

non-vanishing quantities $v_1w_1, \ldots, \ldots, v_3w_3$; hence the determinant of their coefficients vanishes, and so we have Darboux's equation* in the form

$$\begin{vmatrix} \Phi_{11}, & \Phi_{22}, & \Phi_{33}, & \Phi_{23}, & \Phi_{31}, & \Phi_{12} \\ \phi_{11}, & \phi_{22}, & \phi_{33}, & \phi_{23}, & \phi_{31}, & \phi_{12} \\ 1, & 1, & 1, & 0, & 0, & 0 \\ 2\phi_1, & 0, & 0, & 0, & \phi_3, & \phi_2 \\ 0, & 2\phi_2, & 0, & \phi_3, & 0, & \phi_1 \\ 0, & 0, & 2\phi_3, & \phi_2, & \phi_1, & 0 \end{vmatrix} = 0,$$

as the partial equation of the third order to be satisfied by a function $\phi(x, y, z, u)$, when

$$\phi(x, y, z, u) = 0$$

is a family of surfaces constituting part of a triply orthogonal system.

The invariance of form of the equation will be noticed, throughout its different shapes. It is through certain invariantive forms that Darboux constructs all the forms, after the first original equation of the third order has been obtained.

258. One modification of the fundamental equation is worthy of notice. It reduces the equation to an arithmetical test at a point in space on one of the surfaces, while the point is current upon the surface and the surface is any member of its family. The degenerate form of the equation is useful as a test for any particular family of surfaces, and some examples will be given; but it cannot be used for constructive purposes of integration.

Let the u-surface be referred to any point, that lies upon it within the region considered, as origin. Take the normal to the surface at the point as the axis of x; and take the directions of the lines of curvature at the point as the axes of y and z. Then we have (always at the point)

$$u_1 \gtrless 0, \quad u_2 = 0, \quad u_3 = 0, \quad u_{23} = 0;$$

and so the quantities, denoted (§ 254) by B, C, F, are such that

$$B = 0, \quad C = 0, \quad F = u_1(u_{22} - u_{33}).$$

Now the lines of curvature on the u-surface, being the intersections with the v-surface and the w-surface, are given by the equations

$$\xi u_1 + \eta u_2 + \zeta u_3 = 0,$$

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta = 0.$$

The former of these equations requires that $\xi = 0$, when $\xi = v_1$ or when $\xi = w_1$; the requirement is satisfied. The latter of the equations, now that $\xi = 0$ definitely, requires that the equation

$$u_1(u_{22} - u_{33})\eta\zeta = 0$$

* *l.c.*, p. 94.

shall be satisfied at the point; thus if $\xi = v_1$ so that $\eta = v_2$, or if $\xi = w_1$ so that $\eta = w_2$, the requirement is necessarily and sufficiently satisfied, unless

$$u_{22} - u_{33} = 0,$$

which also will satisfy the second equation, without limitations upon η and ζ.

Thus all the conditions required, in order to secure the lines selected as axes, are satisfied except only when

$$u_{22} - u_{33} = 0\,;$$

and the directions are definite and unique at the point, save for the possible existence of this relation (which will be found to be a current characteristic of a sphere).

In these circumstances, and with these values at the point, the terms involving A_{11}, A_{22}, A_{33}, A_{12}, A_{13} in the general expanded equation become evanescent as aggregate coefficients; and thus the general equation becomes

$$A_{23} u_1^3 (u_{22} - u_{33}) = 0.$$

Inserting the value of A_{23} at the point, and rejecting the non-vanishing power of u_1, the equation of the third order degenerates at the point into the critical test represented by the relation

$$(u_{22} - u_{33})(u_1 u_{123} - 2u_{12}u_{13}) = 0.$$

But, as already remarked, this test is arithmetical at a current point; it is not a differential equation.

Ex. 1. A family of parallel planes can belong to a triply orthogonal system. For the family can be taken in the form

$$x = u,$$

so that

$$u_{12} = 0, \quad u_{13} = 0, \quad u_{123} = 0\,;$$

the test is satisfied.

The determination of the other families in the system must be effected specially; for the general method of § 254 is ineffective, because all the quantities A, B, C, F, G, H vanish.

The u-surfaces are known; they are such that

$$u_1 = 1, \quad u_2 = 0, \quad u_3 = 0.$$

The v-surfaces must satisfy the equation

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = 0\,;$$

hence we must have

$$v_1 = 0,$$

and therefore

$$v = \phi(y, z),$$

where ϕ is any arbitrary function. Thus the v-surfaces are a family of cylinders having their generators perpendicular to the plane of x, that is, perpendicular to the u-surfaces.

The w-surfaces must satisfy the equations

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0, \quad v_1 w_1 + v_2 w_2 + v_3 w_3 = 0.$$

From the former, we have

$$w_1 = 0\,;$$

from the latter, we have

$$\frac{\partial \phi}{\partial y} w_2 + \frac{\partial \phi}{\partial z} w_3 = 0.$$

Let an integral of the equation

$$\frac{\partial \phi}{\partial y} dz - \frac{\partial \phi}{\partial z} dy = 0$$

be given by

$$\psi(y, z) = \text{constant};$$

then we can take the w-surfaces in the form

$$\psi(y, z) = w.$$

(We could take w equal to an arbitrary function of ψ; but no generality is gained thereby.) These w-surfaces also are cylinders having their generators perpendicular to the plane of x.

Ex. 2. A family of concentric spheres can belong to a triply orthogonal system.

The family can be taken in the form

$$x^2 + y^2 + z^2 = u,$$

so that

$$u_{12} = 0, \quad u_{13} = 0, \quad u_{123} = 0;$$

the test is satisfied.

As $u_{11} = u_{22} = u_{33} = 2$, $u_{12} = u_{23} = u_{31} = 0$, all the quantities A, B, C, F, G, H vanish; so again the determination of the other families in the system must be specially effected.

We have

$$u_1 = 2x, \quad u_2 = 2y, \quad u_3 = 2z;$$

hence the v-surfaces satisfy the equation

$$xv_1 + yv_2 + zv_3 = 0,$$

so that their equation is

$$v = \phi\left(\frac{y}{x}, \frac{z}{x}\right)$$
$$= \phi(\eta, \zeta),$$

say, where ϕ is an arbitrary function of its arguments.

The w-surfaces must satisfy the two equations

$$xw_1 + yw_2 + zw_3 = 0,$$
$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0.$$

From the former, it follows that w can be any function of η and ζ. Making η and ζ the independent variables, we have the second equation in the form

$$\frac{\partial w}{\partial \eta}\left\{(1+\eta^2)\frac{\partial \phi}{\partial \eta} + \eta\zeta\frac{\partial \phi}{\partial \zeta}\right\} + \frac{\partial w}{\partial \zeta}\left\{\eta\zeta\frac{\partial \phi}{\partial \eta} + (1+\zeta^2)\frac{\partial \phi}{\partial \zeta}\right\} = 0.$$

Let an integral of the equation

$$\left\{(1+\eta^2)\frac{\partial \phi}{\partial \eta} + \eta\zeta\frac{\partial \phi}{\partial \zeta}\right\} d\zeta - \left\{\eta\zeta\frac{\partial \phi}{\partial \eta} + (1+\zeta^2)\frac{\partial \phi}{\partial \zeta}\right\} d\eta = 0$$

be

$$\psi(\eta, \zeta) = \text{constant};$$

then we can take the w-surfaces in the form

$$\psi(\eta, \zeta) = w.$$

As an illustration, let $\phi(\eta, \zeta) = \eta$; then $\psi(\eta, \zeta) = (1+\eta^2)\zeta^{-2}$. The triple system then is

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= u \\ y &= vx \\ x^2 + y^2 &= wz^2 \end{aligned}\right\}.$$

Ex. 3. A family of spheres, touching one another at the same point, can belong to a triply orthogonal system.

The family can be taken in the form

$$\frac{x^2+y^2+z^2}{x}=u\,;$$

then

$$u_{22}=\frac{2}{x}=u_{33}\,;$$

the test is satisfied.

Once more, the method of § 254 cannot be applied*; for the equations are

$$-2\frac{y^2}{x^3}\eta^2+2\frac{y^2}{x^3}\zeta^2+2\frac{y^2-z^2}{x^3}\eta\zeta+\frac{1}{x^2}\left(1-\frac{y^2+z^2}{x^2}\right)(y\zeta-z\eta)\,\xi=0,$$

$$\left(1-\frac{y^2+z^2}{x^2}\right)\xi+\frac{2}{x}(y\eta+z\zeta)=0,$$

and the left-hand side of the latter is a factor of the left-hand side of the former, so that the equations do not determine two sets of values for $\xi:\eta:\zeta$.

The v-surfaces must satisfy the equation

$$u_1v_1+u_2v_2+u_3v_3=0,$$

that is,

$$\left(1-\frac{y^2+z^2}{x^2}\right)v_1+\frac{2y}{x}v_2+\frac{2z}{x}v_3=0.$$

Hence, writing

$$\frac{x^2+y^2+z^2}{y}=\eta,\qquad \frac{z}{y}=\zeta,$$

we have

$$\phi(\eta,\,\zeta)=v,$$

as the equation of the v-surfaces, ϕ being any arbitrary function of its arguments.

The w-surfaces must satisfy the two equations

$$u_1w_1+u_2w_2+u_3w_3=0,$$

$$v_1w_1+v_2w_2+v_3w_3=0.$$

The former equation is the same as the equation for the v-surfaces; hence w must be some function of η and ζ alone. When we take η and ζ as the independent variables, the second of these equations becomes

$$\left(\eta^2\frac{\partial\phi}{\partial\eta}+\eta\zeta\frac{\partial\phi}{\partial\zeta}\right)\frac{\partial w}{\partial\eta}+\left\{\eta\zeta\frac{\partial\phi}{\partial\eta}+(1+\zeta^2)\frac{\partial\phi}{\partial\zeta}\right\}\frac{\partial w}{\partial\zeta}=0.$$

Let an integral of the equation

$$\left\{\eta\zeta\frac{\partial\phi}{\partial\eta}+(1+\zeta^2)\frac{\partial\phi}{\partial\zeta}\right\}d\eta-\left(\eta^2\frac{\partial\phi}{\partial\eta}+\eta\zeta\frac{\partial\phi}{\partial\zeta}\right)d\zeta=0$$

be given by

$$\psi(\eta,\,\zeta)=\text{constant}\,;$$

then the w-surfaces are given by

$$\psi(\eta,\,\zeta)=w.$$

* The explanation of the failure in this example and in the preceding examples is simple. The two equations determine the directions of the lines of curvature on the u-surface; these are not definite when the surface is a plane or a sphere, and so the two equations must cease to be effective.

As a special illustration, let

$$\phi(\eta, \zeta) = \eta = \frac{x^2+y^2+z^2}{y};$$

then we find

$$\psi = \frac{\eta}{\zeta} = \frac{x^2+y^2+z^2}{z},$$

so that we have a triply orthogonal system given by

$$\left.\begin{aligned} \frac{x^2+y^2+z^2}{x} &= u \\ \frac{x^2+y^2+z^2}{y} &= v \\ \frac{x^2+y^2+z^2}{z} &= w \end{aligned}\right\}.$$

Another triply orthogonal system is given by

$$\left.\begin{aligned} \frac{x^2+y^2+z^2}{x} &= u \\ \frac{z}{y} &= v \\ \frac{(x^2+y^2+z^2)^2}{y^2+z^2} &= w \end{aligned}\right\}.$$

Ex. 4. As a last example for the present, consider a family of parallel surfaces.

The quantity u, in the equation of a family of parallel surfaces given by

$$u(x, y, z) = u,$$

satisfies a simple partial differential equation of the first order. To find it, measure a small constant distance ρ along the normal; the consecutive surface is given by the equation

$$\begin{aligned} u + du &= u\left(x + \frac{u_1}{h_1}\rho, \quad y + \frac{u_2}{h_1}\rho, \quad z + \frac{u_3}{h_1}\rho\right) \\ &= u + h_1\rho, \end{aligned}$$

so that

$$h_1 = \frac{du}{\rho}.$$

As ρ is constant, h_1 depends upon u alone; and so we may take

$$u_1^2 + u_2^2 + u_3^2 = f(u),$$

as an equation characteristic of parallel surfaces.

Hence

$$u_1 u_{12} + u_2 u_{22} + u_3 u_{32} = \tfrac{1}{2} f'(u)\, u_2,$$

$$u_1 u_{13} + u_2 u_{23} + u_3 u_{33} = \tfrac{1}{2} f'(u)\, u_3;$$

and therefore, for our arithmetical test at the point, we have

$$u_{12} = 0, \quad u_{13} = 0.$$

Again,

$$u_1 u_{123} + u_{13} u_{12} + u_{23} u_{22} + u_2 u_{223} + u_{33} u_{23} + u_3 u_{233} = \tfrac{1}{2} f''(u)\, u_2 u_3 + \tfrac{1}{2} f'(u)\, u_{23},$$

so that, at the point, we have

$$u_{123} = 0.$$

The arithmetical test is satisfied; and so any family* of parallel surfaces can belong to a triply orthogonal system.

It is easy to infer geometrically that the other members of the triple system are the two sets of developables generated by the normals along the lines of curvature.

259. Proceeding by another method, Puiseux obtains† a number of arithmetical results applicable at a current point—among them, the more important factor of the arithmetical test which has just been considered. He refers a triply orthogonal system to any point as origin, taking the normals to the three surfaces as the axes of reference; then, by adjusting the values of the parameters, he takes the surfaces near the origin in the form

$$\left.\begin{aligned} u &= x + ax^2 + dy^2 + gz^2 + 2Ayz + 2Fzx + 2Hxy + \text{terms of higher orders} \\ v &= y + hx^2 + by^2 + ez^2 + 2Iyz + 2Bzx + 2Dxy + \dots\dots\dots \\ w &= z + fx^2 + iy^2 + cz^2 + 2Eyz + 2Gzx + 2Cxy + \dots\dots\dots \end{aligned}\right\}.$$

Now the equations

$$u_1v_1 + u_2v_2 + u_3v_3 = 0,$$
$$v_1w_1 + v_2w_2 + v_3w_3 = 0,$$
$$w_1u_1 + w_2u_2 + w_3u_3 = 0,$$

have to be satisfied along the lines of intersection; hence

$$0 = (H+h)\,x + (D+d)\,y + (A+B)\,z + \text{terms of order higher than the first},$$
$$0 = (B+C)\,x + (I+i)\,y + (E+e)\,z + \dots\dots\dots,$$
$$0 = (F+f)\,x + (C+A)\,y + (G+g)\,z + \dots\dots\dots.$$

The terms of the various orders must vanish separately; in order that the terms of the first order may vanish, we must have

$$\begin{aligned} H+h&=0, & D+d&=0, & A+B&=0, \\ I+i&=0, & E+e&=0, & B+C&=0, \\ G+g&=0, & F+f&=0, & C+A&=0. \end{aligned}$$

The last column of three relations gives

$$A=0,\quad B=0,\quad C=0,$$

which effectively is Dupin's theorem. Using all the relations, we can take the surfaces in the form

$$\begin{aligned} u &= x + ax^2 + dy^2 + gz^2 - 2fzx - 2hxy \\ &\qquad + jx^3 + my^3 + pz^3 + \alpha y^2z + \delta yz^2 + \eta z^2x + \kappa zx^2 + \nu x^2y + \rho xy^2 + \upsilon xyz + \dots, \\ v &= y + hx^2 + by^2 + ez^2 - 2iyz - 2dxy \\ &\qquad + qx^3 + ky^3 + nz^3 + \xi y^2z + \sigma yz^2 + \beta z^2x + \epsilon zx^2 + \theta x^2y + \lambda xy^2 + \phi xyz + \dots, \\ w &= z + fx^2 + iy^2 + cz^2 - 2eyz - 2gzx \\ &\qquad + sx^3 + ry^3 + lz^3 + \omega y^2z + \mu yz^2 + \varpi z^2x + \tau zx^2 + \gamma x^2y + \zeta xy^2 + \chi xyz + \dots, \end{aligned}$$

* We already have had examples, in a family of parallel planes, a family of concentric spheres, and a family of Dupin cyclides.

† *Liouville*, 2me Sér., t. viii (1863), p. 336.

the unexpressed terms being of the fourth and higher orders. When we substitute in the equation

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0,$$

the terms of the first order disappear; the terms of the second order, which are unaffected by the unexpressed terms, are to vanish by themselves, and so we have

$$\begin{aligned}
0 &= 4fh + \gamma + \epsilon,\\
0 &= 4bi + 4ei + 3r + \xi,\\
0 &= 4ce + 4ei + 3n + \mu,\\
0 &= 2dg - 2be - 2ci - 2e^2 - 2i^2 + \omega + \sigma,\\
0 &= 4de - 4eg - 4gh + 2\beta + \chi,\\
0 &= 4gi - 4df - 4di + 2\zeta + \phi.
\end{aligned}$$

Similarly from $w_1 u_1 + w_2 u_2 + w_3 u_3 = 0$, we have

$$\begin{aligned}
0 &= 4di + \alpha + \zeta,\\
0 &= 4cg + 4fg + 3p + \varpi,\\
0 &= 4af + 4fg + 3s + \kappa,\\
0 &= 2eh - 2cf - 2ag - 2f^2 - 2g^2 + \eta + \tau,\\
0 &= 4ef - 4fh - 4hi + 2\gamma + \upsilon,\\
0 &= 4gh - 4de - 4eg + 2\delta + \chi;
\end{aligned}$$

and from $u_1 v_1 + u_2 v_2 + u_3 v_3 = 0$, we have

$$\begin{aligned}
0 &= 4eg + \beta + \delta,\\
0 &= 4ah + 4dh + 3q + \upsilon,\\
0 &= 4bd + 4dh + 3m + \lambda,\\
0 &= 2fi - 2ad - 2bh - 2d^2 - 2h^2 + \theta + \rho,\\
0 &= 4df - 4di - 4gi + 2\alpha + \phi,\\
0 &= 4hi - 4ef - 4fh + 2\epsilon + \upsilon.
\end{aligned}$$

From the fifth of the second set, we have

$$\gamma = -2ef + 2fh + 2hi - \tfrac{1}{2}\upsilon;$$

and from the sixth of the third set, we have

$$\epsilon = 2ef + 2fh - 2hi - \tfrac{1}{2}\upsilon.$$

When these are substituted in the first of the first set, it becomes

$$\upsilon = 8fh.$$

But in the present case, taking the values at the origin, we have

$$u_1 = 1, \quad f = -\tfrac{1}{2}u_{13}, \quad h = -\tfrac{1}{2}u_{12}, \quad \upsilon = u_{123};$$

so that the relation is

$$u_1 u_{123} - 2u_{12}u_{13} = 0,$$

being one of the factors of the arithmetical test in § 258.

Similarly

$$\phi = 8di, \quad \chi = 8eg,$$

which are the corresponding tests for the v-surface and the w-surface.

The equations are turned to other uses by Puiseux; for these uses, reference should be made to his memoir. In particular, he shews that they contain the Gauss and the Mainardi-Codazzi relations (§§ 34, 35).

Lamé families of surfaces.

260. Whichever form of the equation be adopted, we now have explicitly the partial equation of the third order which must be satisfied by the parameter of a family of surfaces belonging to a triply orthogonal system. Such a family is called* a *Lamé family.*

As the equation is of the third order, it is to be expected (from the general theory of partial differential equations) that its primitive will contain three arbitrary functions† which (after Cauchy's existence-theorem) may be taken as (say) the values of u, u_3, u_{33} when $z = 0$, so that they are then arbitrary functions of x and y. But the general equation appears too complicated to admit of explicit integration in finite terms; so we have to deal with specialised cases. Nevertheless, these cases have some real degree of generality. Among them, one of the most important is contained in a theorem‡ by Darboux, dealing with a large family of Lamé surfaces.

Having regard to the form of the partial equation of the third order satisfied by such a family $u(x, y, z) = u$, we consider the equation

$$\begin{vmatrix} \theta_{11}, & \theta_{22}, & \theta_{33}, & \theta_{23}, & \theta_{31}, & \theta_{12} \\ u_{11}, & u_{22}, & u_{33}, & u_{23}, & u_{31}, & u_{12} \\ 1, & 1, & 1, & 0, & 0, & 0 \\ 2u_1, & 0, & 0, & 0, & u_3, & u_2 \\ 0, & 2u_2, & 0, & u_3, & 0, & u_1 \\ 0, & 0, & 2u_3, & u_2, & u_1, & 0 \end{vmatrix} = 0.$$

By § 256, we know that it is satisfied by

$$\theta = T = (u_1^2 + u_2^2 + u_3^2)^{-\frac{1}{2}}.$$

We verify at once that it is satisfied by

$$\theta = f(u);$$

* By Darboux, on account of the importance of Lamé's work on curvilinear coordinates.

† This is in accord with the alternative form of statement in § 252.

‡ *l.c.*, Book i, ch. iii, where the integral is discussed in some detail.

for substituting, multiplying the second row by f'', the fourth by $\frac{1}{2}u_1 f'$, the fifth by $\frac{1}{2}u_2 f'$, the sixth by $\frac{1}{2}u_3 f'$, and subtracting the sum of these multiplied rows from the first, we have a row of zeros.

Similarly, it is satisfied by

$$\theta = xg(u), \quad \theta = yh(u), \quad \theta = zk(u);$$

the process of verification is the same, save that the respective factors are

xg', yh', zk', for the second row;

$\frac{1}{2}xu_1 g'' + g'$, $\frac{1}{2}yu_2 h'' + h'$, $\frac{1}{2}zu_3 k'' + k'$, for the fourth row;

$\frac{1}{2}xu_2 g''$, $\frac{1}{2}yu_2 h''$, $\frac{1}{2}zu_2 k''$, for the fifth row;

$\frac{1}{2}xu_3 g''$, $\frac{1}{2}yu_3 h''$, $\frac{1}{2}zu_3 k''$, for the sixth row.

Similarly, it is satisfied by

$$\theta = (x^2 + y^2 + z^2)\, l(u),$$

the process of verification being the same, but with the multipliers

$(x^2 + y^2 + z^2)\, l'$, for the second row,

$2l$, for the third row,

$\frac{1}{2}(x^2 + y^2 + z^2)\, u_1 l'' + 2xl'$, for the fourth row,

$\frac{1}{2}(x^2 + y^2 + z^2)\, u_2 l'' + 2yl'$, for the fifth row,

$\frac{1}{2}(x^2 + y^2 + z^2)\, u_3 l'' + 2zl'$, for the sixth row.

Now the equation quoted is linear in the derivatives of θ, and therefore the sum of any number of integrals with constant coefficients is an integral; thus

$$\Theta = (u_1^2 + u_2^2 + u_3^2)^{-\frac{1}{2}} - (x^2 + y^2 + z^2)\, l - xg - yh - zk - f$$

is an integral. But

$$\Theta = 0$$

satisfies the equation; and so we have Darboux's theorem*:—

Any family of surfaces $u(x, y, z) = u$, satisfying the equation

$$\frac{1}{(u_1^2 + u_2^2 + u_3^2)^{\frac{1}{2}}} = (x^2 + y^2 + z^2)\, l + xg + yh + zk + f,$$

where f, g, h, k, l are any arbitrary functions of u, is a Lamé family belonging to a triply orthogonal system.

For the development, and for some applications, of the theorem, reference should be made to Darboux's treatise†.

* Apparently, there are five arbitrary functions in the integral, instead of three; but the five can be reduced to four, by taking a new variable u', such that $ldu = du'$. The four functions involve only one parameter u; each of the three functions in the primitive (after the statement of Cauchy's existence-theorem in § 252) contains two independent parametric variables.

† *Systèmes orthogonaux*, Book i, chap. iii.

Bouquet surfaces.

261. As the general primitive of the equation of the third order has not been obtained, it is worth while considering some special classes of surfaces that belong to triply orthogonal systems. Among these are the u-surfaces, whose equation* has, or can be made to have, the form

$$u = X + Y + Z,$$

where X is a function of x only, Y of y only, Z of z only. Then

$$u_{11} = X'', \quad u_{23} = 0,$$
$$A_{11} = X'X''' - 2X''^2, \quad A_{23} = 0,$$

and similarly for the others; hence the equation of the third order, when expanded, is

$$X'Y'Z'\Sigma(X'X''' - 2X''^2)(Y'' - Z'') = 0.$$

Manifestly the factor $X'Y'Z'$ can be dropped; and so the equation becomes

$$(X'X''' - 2X''^2)(Y'' - Z'') + (Y'Y''' - 2Y''^2)(Z'' - X'') \\ + (Z'Z''' - 2Z''^2)(X'' - Y'') = 0.$$

The equation can be established *ab initio* by the following method which also contributes some knowledge of the other families in the triple system. Assuming that the u-surface does belong to a triply orthogonal system, the other families are given by

$$X'v_1 + Y'v_2 + Z'v_3 = 0,$$
$$X'w_1 + Y'w_2 + Z'w_3 = 0,$$
$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0.$$

As regards the first two of these equations, we take two independent integrals of the subsidiary set

$$\frac{dx}{X'} = \frac{dy}{Y'} = \frac{dz}{Z'},$$

say

$$\alpha = \int \frac{dx}{X'} - \int \frac{dy}{Y'}, \qquad \beta = \int \frac{dx}{X'} - \int \frac{dz}{Z'};$$

and then the first two equations are satisfied in complete generality, by taking v and w as any two functions of α and β only. Let these functions be substituted in the third equation; it becomes

$$\frac{1}{X'^2}\left(\frac{\partial v}{\partial \alpha} + \frac{\partial v}{\partial \beta}\right)\left(\frac{\partial w}{\partial \alpha} + \frac{\partial w}{\partial \beta}\right) + \frac{1}{Y'^2}\frac{\partial v}{\partial \alpha}\frac{\partial w}{\partial \alpha} + \frac{1}{Z'^2}\frac{\partial v}{\partial \beta}\frac{\partial w}{\partial \beta} = 0,$$

that is,

$$\left(\frac{\partial v}{\partial \alpha} + \frac{\partial v}{\partial \beta}\right)\left(\frac{\partial w}{\partial \alpha} + \frac{\partial w}{\partial \beta}\right) + \frac{X'^2}{Y'^2}\frac{\partial v}{\partial \alpha}\frac{\partial w}{\partial \alpha} + \frac{X'^2}{Z'^2}\frac{\partial v}{\partial \beta}\frac{\partial w}{\partial \beta} = 0.$$

* They were first considered by Bouquet, *Liouville*, t. xi (1846), pp. 446—450.

There are three independent variables, though v and w are functions of only two independent combinations of them. Change the independent variables, choosing them to be α, β, x; then y is a function of α and x, while z is a function of β and x, such that

$$\frac{\partial y}{\partial x}=\frac{Y'}{X'},\qquad \frac{\partial z}{\partial x}=\frac{Z'}{X'}.$$

Differentiating the modified equation with respect to x, and noting that v and w now do not involve x but only α and β, we have

$$\frac{X''-Y''}{Y'^2}\frac{\partial v}{\partial\alpha}\frac{\partial w}{\partial\alpha}+\frac{X''-Z''}{Z'^2}\frac{\partial v}{\partial\beta}\frac{\partial w}{\partial\beta}=0.$$

Differentiating again, we have

$$\frac{X'X'''-Y'Y'''-2Y''(X''-Y'')}{Y'^2}\frac{\partial v}{\partial\alpha}\frac{\partial w}{\partial\alpha}\\+\frac{X'X'''-Z'Z'''-2Z''(X''-Z'')}{Z'^2}\frac{\partial v}{\partial\beta}\frac{\partial w}{\partial\beta}=0.$$

Eliminating the derivatives of v and w between the last two relations, we find

$$(X'X'''-2X''^2)(Y''-Z'')+(Y'Y'''-2Y''^2)(Z''-X'')\\+(Z'Z'''-2Z''^2)(X''-Y'')=0,$$

which is the equation in question.

When we use the earlier of the two derived relations to eliminate Y'^2 from the α-β equation, we find

$$\left(\frac{\partial v}{\partial\alpha}+\frac{\partial v}{\partial\beta}\right)\left(\frac{\partial w}{\partial\alpha}+\frac{\partial w}{\partial\beta}\right)+P\frac{\partial v}{\partial\beta}\frac{\partial w}{\partial\beta}=0,$$

where

$$P=\frac{X'^2}{Z'^2}\frac{Z''-Y''}{X''-Y''}.$$

It is not difficult to verify, through the critical equation of the third order, that P is a function of α and β only.

Suppose that the u-surface does satisfy the Bouquet form of the critical equation. Then, for the other two families, we have

$$\frac{\partial v}{\partial\alpha}\frac{\partial w}{\partial\alpha}=MY'^2(Z''-X''),$$

$$\frac{\partial v}{\partial\beta}\frac{\partial w}{\partial\beta}=MZ'^2(X''-Y''),$$

$$\left(\frac{\partial v}{\partial\alpha}+\frac{\partial v}{\partial\beta}\right)\left(\frac{\partial w}{\partial\alpha}+\frac{\partial w}{\partial\beta}\right)=MX'^2(Y''-Z''),$$

where M is a quantity whose exact value is not required. Thus $\dfrac{\partial v}{\partial\alpha}\Big/\dfrac{\partial w}{\partial\alpha}$ and $\dfrac{\partial v}{\partial\beta}\Big/\dfrac{\partial w}{\partial\beta}$ are the roots of the quadratic equation

$$\theta^2Y'^2(Z''-X'')+\theta\{Y'^2(Z''-X'')+Z'^2(X''-Y'')-X'^2(Y''-Z'')\}\\+Z'^2(X''-Y'')=0.$$

One of the linear factors gives a homogeneous linear equation of the first order for v; the other of the linear factors similarly supplies w. When these equations are integrated, the values of v and of w are known; and so we have the triple system.

Accordingly, the first step is the determination of values of X, Y, Z which shall satisfy the critical equation of the third order. It may be written

$$\xi + AX'' + B = 0,$$

where A and B are independent of x, while

$$\xi = X'X''' - 2X''^2.$$

Differentiating with respect to x, we have

$$\xi' + AX''' = 0,$$

so that

$$\frac{\xi'}{X'''} = -A;$$

consequently both sides of this equation must be constant, so that

$$\xi' = aX''', \quad A = -a.$$

The former gives

$$X'X''' - 2X''^2 = \xi = aX'' + b,$$

where b is an arbitrary constant; and then

$$B = b.$$

The relations $A + a = 0$, $B - b = 0$, give

$$Y'Y''' - 2Y''^2 = aY'' + b, \quad Z'Z''' - 2Z''^2 = aZ'' + b.$$

Hence the most general resolution of the critical equation is constituted by the set of equations

$$\left.\begin{aligned} X'X''' - 2X''^2 &= aX'' + b \\ Y'Y''' - 2Y''^2 &= aY'' + b \\ Z'Z''' - 2Z''^2 &= aZ'' + b \end{aligned}\right\}.$$

Ex. 1. Consider the case when $a=0$, $b=0$. Then

$$X'X''' - 2X''^2 = 0, \quad Y'Y''' - 2Y''^2 = 0, \quad Z'Z''' - Z''^2 = 0.$$

The primitives of these equations are

$$X = m\log(x-m') + m'',$$

$$Y = n\log(y-n') + n'',$$

$$Z = p\log(z-p') + p'',$$

where all the quantities $m, \ldots, p''$ are arbitrary constants. No generality is lost by annihilating m', n', p', m'', n'', p''; so we have

$$X = m\log x, \quad Y = n\log y, \quad Z = p\log z.$$

The u-surface has therefore the simple form

$$U = e^u = e^{X+Y+Z} = x^m y^n z^p.$$

For the other two families in the triple system, we first deal with the equations

$$\frac{m}{x}v_1+\frac{n}{y}v_2+\frac{p}{z}v_3=0,$$

$$\frac{m}{x}w_1+\frac{n}{y}w_2+\frac{p}{z}w_3=0.$$

Hence, if

$$\alpha=nx^2-my^2, \quad \beta=px^2-mz^2,$$

both v and w are functions of α and β alone. The equation

$$v_1w_1+v_2w_2+v_3w_3=0$$

then becomes

$$\left\{\left(n\frac{\partial v}{\partial \alpha}+p\frac{\partial v}{\partial \beta}\right)\left(n\frac{\partial w}{\partial \alpha}+p\frac{\partial w}{\partial \beta}\right)+mn\frac{\partial v}{\partial \alpha}\frac{\partial w}{\partial \alpha}+mp\frac{\partial v}{\partial \beta}\frac{\partial w}{\partial \beta}\right\}x^2$$

$$=m\left(\alpha\frac{\partial v}{\partial \alpha}\frac{\partial w}{\partial \alpha}+\beta\frac{\partial v}{\partial \beta}\frac{\partial w}{\partial \beta}\right).$$

Hence

$$(mn+n^2)\frac{\partial v}{\partial \alpha}\frac{\partial w}{\partial \alpha}+np\left(\frac{\partial v}{\partial \alpha}\frac{\partial w}{\partial \beta}+\frac{\partial v}{\partial \beta}\frac{\partial w}{\partial \alpha}\right)+(mp+p^2)\frac{\partial v}{\partial \beta}\frac{\partial w}{\partial \beta}=0,$$

$$\alpha\frac{\partial v}{\partial \alpha}\frac{\partial w}{\partial \alpha}+\beta\frac{\partial v}{\partial \beta}\frac{\partial w}{\partial \beta}=0.$$

The equation satisfied by v is

$$\alpha\left(\frac{\partial v}{\partial \alpha}\right)^2+\left(\frac{p+m}{n}\alpha-\frac{n+m}{p}\beta\right)\frac{\partial v}{\partial \alpha}\frac{\partial v}{\partial \beta}-\beta\left(\frac{\partial v}{\partial \beta}\right)^2=0;$$

and the same equation is satisfied by w. Let

$$\Delta=\left(\frac{p+m}{n}\alpha-\frac{n+m}{p}\beta\right)^2+4\alpha\beta;$$

then we can take

$$2\alpha\frac{\partial v}{\partial \alpha}+\left(\frac{p+m}{n}\alpha-\frac{n+m}{p}\beta+\Delta^{\frac{1}{2}}\right)\frac{\partial v}{\partial \beta}=0,$$

$$2\alpha\frac{\partial w}{\partial \alpha}+\left(\frac{p+m}{n}\alpha-\frac{n+m}{p}\beta-\Delta^{\frac{1}{2}}\right)\frac{\partial w}{\partial \beta}=0.$$

Let integrals of

$$\left(\frac{p+m}{n}\alpha-\frac{n+m}{p}\beta+\Delta^{\frac{1}{2}}\right)d\alpha-2\alpha d\beta=0,$$

and

$$\left(\frac{p+m}{n}\alpha-\frac{n+m}{p}\beta-\Delta^{\frac{1}{2}}\right)d\alpha-2\alpha d\beta=0,$$

be

$$g(\alpha, \beta)=\text{constant}, \quad h(\alpha, \beta)=\text{constant},$$

respectively; the two other families of the system are

$$g(\alpha, \beta)=v,$$

$$h(\alpha, \beta)=w.$$

Ex. 2. Shew that a triply orthogonal system is given by the equations:—

(i) the hyperbolic paraboloids $\dfrac{yz}{x}=u$;

(ii) the closed sheets of the surface

$$(y^2-z^2)^2-2a(y^2+z^2+2x^2)+a^2=0;$$

(iii) the open sheets of the same surface.

Ex. 3. Obtain the families, to be associated with the family

$$xyz=u$$

so as to give a triply orthogonal system, in the form

$$v=(x^2+ay^2+a^2z^2)^{\frac{3}{2}}+(x^2+a^2y^2+az^2)^{\frac{3}{2}},$$

$$w=(x^2+ay^2+a^2z^2)^{\frac{3}{2}}-(x^2+a^2y^2+az^2)^{\frac{3}{2}},$$

where a is an imaginary cube root of unity.

Ex. 4. Shew that the critical equation of the third order is satisfied for the surface

$$u=X+c(y^2+z^2),$$

where X is any function of x. Shew also that the families of surfaces to be associated with it in a triply orthogonal system are

$$v=\frac{y}{z},$$

$$w=(y^2+z^2)\,e^{-4c\int\frac{dx}{X'}}.$$

Ex. 5. It was shewn that a general resolution of the critical equation leads to three equations of the form

$$X'X'''-2X''^2=aX''+b.$$

Thus we have

$$X'X'''=2X''^2+aX''+b$$

$$=2(X''-\rho)(X''-\sigma),$$

where ρ and σ are constants. Assuming them unequal, let

$$\frac{\lambda}{\rho}=\frac{\mu}{-\sigma}=\frac{1}{\rho-\sigma};$$

then

$$\left(\frac{\lambda}{X''-\rho}+\frac{\mu}{X''-\sigma}\right)X'''=2\frac{X''}{X'},$$

so that

$$(X''-\rho)^{\lambda}(X''-\sigma)^{\mu}=c^2X'^2,$$

an equation that usually is transcendental. Now

$$dx=\frac{dX''}{X'''}$$

$$=\frac{1}{2}\frac{X'}{(X''-\rho)(X''-\sigma)}dX''$$

$$=\frac{1}{2c}(X''-\rho)^{\frac{1}{2}\lambda-1}(X''-\sigma)^{\frac{1}{2}\mu-1}dX'',$$

so that

$$x-x_0=\frac{1}{2c}\int(X''-\rho)^{\frac{1}{2}\lambda-1}(X''-\sigma)^{\frac{1}{2}\mu-1}dX''.$$

If it were possible to invert this relation, so that

$$X''=\frac{d^2f(x-x_0,\,c)}{dx^2},$$

then

$$X=f(x-x_0,\,c)+Ax.$$

Similarly

$$Y=f(y-y_0,\,c')+A'y,$$

$$Z=f(z-z_0,\,c'')+A''z.$$

But, in general, the relation cannot be inverted so as to give an explicit form for f.

Ex. 6. Integrate the equation, so as to obtain the u-surface in the following cases:—

(i) when $b=0$,

(ii) when $a=0$,

(iii) when $a^2=8b$.

Quadrics.

262. Now consider the possibility of having a family of central coaxial (but not necessarily confocal) quadrics as a family belonging to a triply orthogonal system. Their equation will have the form

$$\phi(x, y, z, u) = \frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} - 1 = 0,$$

where A, B, C are functions of the family parameter u. We have

$$\phi_{11} = \frac{2}{A}, \quad \phi_{22} = \frac{2}{B}, \quad \phi_{33} = \frac{2}{C},$$

$$\phi_{23} = 0, \quad \phi_{31} = 0, \quad \phi_{12} = 0;$$

$$\phi_1 = 2\frac{x}{A}, \quad \phi_2 = 2\frac{y}{B}, \quad \phi_3 = 2\frac{z}{C};$$

while

$$\Phi_{11} = -8\frac{\phi'}{A^2} + 8\phi_1'\frac{x}{A^2} - 4\phi_{11}'\left(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\right),$$

$$\Phi_{23} = 4\left(\phi_2'\frac{z}{C^2} + \phi_3'\frac{y}{B^2}\right),$$

and so for the values of the other quantities Φ_{rs} by cyclical rotation of the indices. Substituting in the Darboux equation of § 257, and evaluating, we find (on the rejection of a merely numerical factor)

$$\frac{xyz}{A^2B^2C^2}\left(\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C}\right)\left\{\left(\frac{1}{B} - \frac{1}{C}\right)A' + \left(\frac{1}{C} - \frac{1}{A}\right)B' + \left(\frac{1}{A} - \frac{1}{B}\right)C'\right\} = 0.$$

Hence, rejecting irrelevant factors and non-vanishing factors, we have

$$A(B-C)A' + B(C-A)B' + C(A-B)C' = 0,$$

as a differential equation to be satisfied by the quantities A, B, C, which are functions of the family parameter u. And then, for any values of A, B, C as functions of the parameter u which satisfy the equation, the family of quadrics

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} - 1 = 0$$

belongs to a triply orthogonal system.

Ex. 1. The simplest solution is given by

$$A'=1, \quad B'=1, \quad C'=1,$$

so that

$$A=a+u, \quad B=b+u, \quad C=c+u,$$

where a, b, c are any constants. The equation of the family of surfaces is

$$\frac{x^2}{a+u}+\frac{y^2}{b+u}+\frac{z^2}{c+u}=1;$$

the family is composed of confocal quadrics of the same kind. The other two families in the system are the two sets of confocal quadrics of the other two kinds.

Ex. 2. Manifestly, a somewhat general solution is given by

$$\left.\begin{aligned} AA'&=Ah+g\\ BB'&=Bh+g\\ CC'&=Ch+g\end{aligned}\right\},$$

where h and g are any disposable functions of u. We have one set of solutions of this character by taking

$$A=(u+b)(u+c), \quad B=(u+c)(u+a), \quad C=(u+a)(u+b),$$

if

$$h=3u+a+b+c, \quad g=(u+a)(u+b)(u+c);$$

and the family of quadrics, forming part of the triply orthogonal system, has the equation

$$(u+a)x^2+(u+b)y^2+(u+c)z^2=(u+a)(u+b)(u+c).$$

The other families of the triple system satisfy the equations

$$\left.\begin{aligned} x(u+a)\,v_1+y(u+b)\,v_2+z(u+c)\,v_3&=0\\ x(u+a)\,w_1+y(u+b)\,w_2+z(u+c)\,w_3&=0\\ v_1w_1+v_2w_2+v_3w_3&=0\end{aligned}\right\}.$$

The first step in the construction of these families is made by integrating the equations

$$\frac{dx}{x(u+a)}=\frac{dy}{y(u+b)}=\frac{dz}{z(u+c)},$$

being the characteristics of the first two partial equations of the first order. When we equate each of these functions to $f''(u)\,du$, where f is another unknown function, the integral of these characteristics can be taken in the form

$$\left.\begin{aligned} x&=Ae^{(u+a)f'(u)-f(u)}\\ y&=Be^{(u+b)f'(u)-f(u)}\\ z&=Ce^{(u+c)f'(u)-f(u)}\end{aligned}\right\},$$

where A, B, C are arbitrary constants; and then the value of $f(u)$ satisfies the equation

$$A^2(u+a)e^{2(u+a)f'(u)}+B^2(u+b)e^{2(u+b)f'(u)}+C^2(u+c)e^{2(u+c)f'(u)}$$
$$=(u+a)(u+b)(u+c)e^{2f(u)}.$$

When this value of $f(u)$ is regarded as known, we can take the two integrals α and β of the subsidiary characteristic equations in the form

$$\alpha=x^{b-c}y^{c-a}z^{a-b},$$
$$\beta=x^{a(b-c)}y^{b(c-a)}z^{c(a-b)}e^{(a-b)(b-c)(c-a)f'(u)};$$

and then v and w are appropriate functions of α and β, satisfying the equation

$$\Sigma\frac{(b-c)^2}{x^2}\left\{\alpha\frac{\partial v}{\partial\alpha}+\beta\frac{\partial v}{\partial\beta}\,\frac{a(u+b+c)-bc}{u+a}\right\}\left\{\alpha\frac{\partial w}{\partial\alpha}+\beta\frac{\partial w}{\partial\beta}\,\frac{a(u+b+c)-bc}{u+a}\right\}=0.$$

263. Next, consider the paraboloids represented by

$$2x + \frac{y^2}{M} + \frac{z^2}{N} + L = 0,$$

where L, M, N are functions of a parameter u; can the paraboloids, for some appropriate functions as values of L, M, N, be a family of surfaces belonging to a triply orthogonal system? Writing

$$\phi(x, y, z, u) = 2x + \frac{y^2}{M} + \frac{z^2}{N} + L,$$

we have

$$\phi_1 = 2, \quad \phi_2 = 2\frac{y}{M}, \quad \phi_3 = 2\frac{z}{N},$$

$$\phi_{11} = 0, \quad \phi_{22} = \frac{2}{M}, \quad \phi_{33} = \frac{2}{N},$$

$$\phi_{23} = 0, \quad \phi_{31} = 0, \quad \phi_{12} = 0;$$

and then

$$\Phi_{22} = \phi'\left(-\frac{8}{M^2}\right) + \phi_2'\left(8\frac{y}{M^2}\right) - 4\left(\frac{y^2}{M^2} + \frac{z^2}{N^2}\right)\phi_{22}',$$

$$\Phi_{33} = \phi'\left(-\frac{8}{N^2}\right) + \phi_3'\left(8\frac{z}{N^2}\right) - 4\left(\frac{y^2}{M^2} + \frac{z^2}{N^2}\right)\phi_{33}',$$

$$\Phi_{23} = 4\left(\frac{z}{N^2}\phi_2' + \frac{y}{M^2}\phi_3'\right),$$

$$\Phi_{11} = 0, \quad \Phi_{12} = 0, \quad \Phi_{13} = 0.$$

Thus (removing the merely numerical factor 16) the critical equation is

$$\begin{vmatrix} 0, & \Phi_{22}, & \Phi_{33}, & \Phi_{23}, & 0, & 0 \\ 0, & \dfrac{1}{M}, & \dfrac{1}{N}, & 0, & 0, & 0 \\ 1, & 1, & 1, & 0, & 0, & 0 \\ 2, & 0, & 0, & 0, & \dfrac{z}{N}, & \dfrac{y}{M} \\ 0, & 2\dfrac{y}{M}, & 0, & \dfrac{z}{N}, & 0, & 1 \\ 0, & 0, & 2\dfrac{z}{N}, & \dfrac{y}{M}, & 1, & 0 \end{vmatrix} = 0,$$

which, when expanded, becomes

$$2\frac{yz}{MN}\left(\Phi_{22}\frac{1}{N} - \Phi_{33}\frac{1}{M}\right) + 2\Phi_{23}\left\{\frac{1}{M}\left(\frac{z^2}{N^2} + 1\right) - \frac{1}{N}\left(\frac{y^2}{M^2} + 1\right)\right\} = 0.$$

Substituting and collecting terms, we find that this equation takes the form

$$16\frac{yz}{M^2N^2}\left(\frac{1}{N} - \frac{1}{M}\right)(L' + M' + N') = 0.$$

We may set aside, as a particular family, the surfaces for which $M = N$; they are paraboloids of revolution. Thus the critical equation becomes

$$L' + M' + N' = 0,$$

so that

$$L + M + N = \text{constant}.$$

Ex. A particular family is given by

$$M = u, \quad N = -u, \quad L = \text{constant}.$$

The surfaces are

$$\frac{y^2 - z^2}{u} + 2x + c = 0$$

which (by change of origin and axes) become the surfaces

$$YZ = uX.$$

We have already (§ 261, Ex. 2) dealt with the triple system to which this family of paraboloids belongs.

Isometric systems.

264. Among triply orthogonal systems, there is one special class of surfaces of particular importance. They arise in two ways.

In the first way, they were connected (by Lamé, to whom their earliest consideration is due) with the equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = 0,$$

which has many physical interpretations—among them, that of representing the temperature of space in a state of heat equilibrium. If a family of surfaces

$$\phi(x, y, z, u) = 0$$

is isothermic, we must have

$$\theta = f(u),$$

where θ satisfies the foregoing equation. Then

$$\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial x} f'(u), \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} f'(u) + \left(\frac{\partial u}{\partial x}\right)^2 f''(u);$$

and so the equation becomes

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) f'(u) + \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\} f''(u) = 0.$$

It follows that, if the family of surfaces is isothermic, the parameter u of the surfaces (when regarded as a function of the variables) must satisfy an equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\} g(u),$$

where $g(u)$ is a function of u alone.

Should the condition be satisfied, the temperature of the surface is given by the equation

$$f''(u) + f'(u)\,g(u) = 0,$$

so that

$$f(u) = A + B\int e^{-\int g(u)\,du}\,du.$$

That the necessary condition is satisfied for a family of confocal quadrics

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$$

can easily be verified. We have

$$\frac{\partial u}{\partial x}\Sigma\frac{x^2}{(a^2+u)^2} = \frac{2x}{a^2+u},$$

and therefore

$$\Sigma\left(\frac{\partial u}{\partial x}\right)^2 \Sigma\frac{x^2}{(a^2+u)^2} = 4.$$

Again,

$$\frac{\partial^2 u}{\partial x^2}\Sigma\frac{x^2}{(a^2+u)^2} + \frac{\partial u}{\partial x}\frac{2x}{(a^2+u)^2} - 2\left(\frac{\partial u}{\partial x}\right)^2\Sigma\frac{x^2}{(a^2+u)^3} = \frac{2}{a^2+u} - \frac{2x}{(a^2+u)^2}\frac{\partial u}{\partial x},$$

so that

$$\frac{\partial^2 u}{\partial x^2}\Sigma\frac{x^2}{(a^2+u)^2} + \frac{4x}{(a^2+u)^2}\frac{\partial u}{\partial x} - 2\left(\frac{\partial u}{\partial x}\right)^2\Sigma\frac{x^2}{(a^2+u)^3} = \frac{2}{a^2+u};$$

and similarly for $\dfrac{\partial^2 u}{\partial y^2}$, $\dfrac{\partial^2 u}{\partial z^2}$. Now

$$\left\{\Sigma\frac{x^2}{(a^2+u)^2}\right\}\left\{\Sigma\frac{4x}{(a^2+u)^2}\frac{\partial u}{\partial x}\right\} = \Sigma\frac{8x^2}{(a^2+u)^3},$$

$$\left\{\Sigma\frac{x^2}{(a^2+u)^2}\right\}\left\{\Sigma\left(\frac{\partial u}{\partial x}\right)^2\right\}\Sigma\frac{x^2}{(a^2+u)^3} = 4\Sigma\frac{x^2}{(a^2+u)^3};$$

hence

$$\left(\Sigma\frac{\partial^2 u}{\partial x^2}\right)\Sigma\frac{x^2}{(a^2+u)^2} = 2\left(\frac{1}{a^2+u} + \frac{1}{b^2+u} + \frac{1}{c^2+u}\right),$$

and therefore

$$\frac{\Sigma\dfrac{\partial^2 u}{\partial x^2}}{\Sigma\left(\dfrac{\partial u}{\partial x}\right)^2} = \frac{1}{2}\left(\frac{1}{a^2+u} + \frac{1}{b^2+u} + \frac{1}{c^2+u}\right),$$

so that the condition is satisfied. Also

$$f(u) = A\int\{(a^2+u)(b^2+u)(c^2+u)\}^{-\frac{1}{2}}\,du + B.$$

265. The other method of proceeding deals simultaneously with the three families in the triple system, as follows.

It was seen (in Chap. III) that some surfaces have their lines of curvature of the isometric orthogonal type; so it is natural to enquire whether a triply orthogonal system of surfaces exists, in which each family is of that type. Naturally, the parametric curves of isometric division will be the lines of curvature on every member of each of the families. Hence, on the u-surface, we must have

$$H_2^2 : H_3^2 = \text{function of } v \text{ only} : \text{function of } w \text{ only};$$

on the v-surface,

$$H_3^2 : H_1^2 = \text{function of } w \text{ only} : \text{function of } u \text{ only};$$

and, on the w-surface,

$$H_1^2 : H_2^2 = \text{function of } u \text{ only} : \text{function of } v \text{ only};$$

the three ratios being subject to the conditions

$$u = \text{constant}, \quad v = \text{constant}, \quad w = \text{constant},$$

respectively. Now let

$$A' = \text{any function of } v \text{ and } w, \text{ independent of } u;$$
$$B' = \ldots\ldots\ldots\ldots\ldots\ldots \; w \;\ldots\; u, \;\ldots\ldots\ldots\ldots\ldots\ldots \; v;$$
$$C' = \ldots\ldots\ldots\ldots\ldots\ldots \; u \;\ldots\; v, \;\ldots\ldots\ldots\ldots\ldots\ldots \; w;$$

then we can take

$$H_1 = B'C'\Omega^{-1}, \quad H_2 = C'A'\Omega^{-1}, \quad H_3 = A'B'\Omega^{-1},$$

where Ω is any function of u, v, w, so far as concerns the foregoing ratio-conditions.

Ex. It was stated (in the example in § 245) that, for a triply orthogonal system constituted by three families of confocal quadrics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$

for $\lambda = u, v, w$, the values of H_1, H_2, H_3 are given by

$$4H_1^2 = \frac{(u-v)(u-w)}{(a+u)(b+u)(c+u)} = \frac{(u-v)(u-w)}{U},$$

$$4H_2^2 = \frac{(v-w)(v-u)}{(a+v)(b+v)(c+v)} = \frac{(v-w)(v-u)}{V},$$

$$4H_3^2 = \frac{(w-u)(w-v)}{(a+w)(b+w)(c+w)} = \frac{(w-u)(w-v)}{W}.$$

When we take

$$A'^2 = \frac{v-w}{VW}, \quad B'^2 = \frac{w-u}{WU}, \quad C'^2 = \frac{u-v}{UV},$$

which satisfies the restrictions upon A', B', C', and when we also take

$$4\Omega^{-2} = UVW,$$

all the conditions are satisfied. Hence the triple system is of the isometric type.

The quantities H_1, H_2, H_3 have to satisfy the Lamé relations. Following* Darboux, we write

$$H_1 = e^{B+C-\log \Omega}, \quad H_2 = e^{C+A-\log \Omega}, \quad H_3 = e^{A+B-\log \Omega},$$

where

$$\log A' = A, \quad \log B' = B, \quad \log C' = C.$$

When we substitute in the second set of these relations, we have

$$\left.\begin{aligned} \Omega_{23} &= \Omega_2 A_3 + \Omega_3 A_2 + \Omega S_1 \\ \Omega_{31} &= \Omega_3 B_1 + \Omega_1 B_3 + \Omega S_2 \\ \Omega_{12} &= \Omega_1 C_2 + \Omega_2 C_1 + \Omega S_3 \end{aligned}\right\},$$

where

$$\left.\begin{aligned} S_1 &= (B_3 - A_3)(C_2 - A_2) - A_3 A_2 \\ S_2 &= (C_1 - B_1)(A_3 - B_3) - B_1 B_3 \\ S_3 &= (A_2 - C_2)(B_1 - C_1) - C_2 C_1 \end{aligned}\right\}.$$

The equations for $\Omega_{23}, \Omega_{31}, \Omega_{12}$ should lead to one and the same value for Ω_{123}. Differentiating the first with respect to u, and substituting for Ω_{12} and Ω_{13} from the second and third, we have

$$\Omega_{123} = \Omega_1 B_3 C_2 + \Omega_2 C_1 A_3 + \Omega_3 A_2 B_1 + \Omega \left(\frac{\partial S_1}{\partial u} + S_2 A_2 + S_3 A_3\right);$$

and similarly from the others. Hence

$$\frac{\partial S_1}{\partial u} + S_2 A_2 + S_3 A_3 = \frac{\partial S_2}{\partial v} + S_3 B_3 + S_1 B_1 = \frac{\partial S_3}{\partial w} + S_1 C_1 + S_2 C_2.$$

Now

$$\begin{aligned} S_2 A_2 + S_3 A_3 &= -A_2 B_3 C_1 - A_3 B_1 C_2 \\ &= S_3 B_3 + S_1 B_1 \\ &= S_1 C_1 + S_2 C_2; \end{aligned}$$

thus the foregoing conditions become

$$\frac{\partial S_1}{\partial u} = \frac{\partial S_2}{\partial v} = \frac{\partial S_3}{\partial w}.$$

Let the values of S_1, S_2, S_3 be substituted in these conditions; then

$$\begin{aligned} & C_{12}(B_3 - A_3) + B_{31}(C_2 - A_2) \\ &= C_{12}(A_3 - B_3) + A_{23}(C_3 - B_3) \\ &= A_{23}(B_3 - C_3) + B_{13}(A_2 - C_2) \\ &= 0, \end{aligned}$$

by taking each of the three equal expressions as equal to one-third of their sum. Thus

$$\frac{\partial S_1}{\partial u} = 0, \quad \frac{\partial S_2}{\partial v} = 0, \quad \frac{\partial S_3}{\partial w} = 0,$$

* *Ann. Éc. Norm. Sup.*, 1re Sér., t. iii (1866), p. 131.

so that S_1 is a function of v and w only, S_2 of w and u only, and S_3 of u and v only. Thus

$$(B_3 - A_3)(C_2 - A_2) = K_1, \quad (C_1 - B_1)(A_3 - B_3) = K_2, \quad (A_2 - C_2)(B_1 - C_1) = K_3,$$

where K_1 is a function of v and w only, K_2 of w and u only, and K_3 of u and v only.

266. To find K_1, let u have any constant value, say zero, an assignment which does not affect K_1. Then

B_3 becomes a function of w only, $= w_3$ say,

C_2 .... v only, $= v_2$ say;

and let

$$w_3 - A_3 = J, \quad v_2 - A_2 = I,$$

$$B_3 - w_3 = T_2, \quad C_2 - v_2 = T_3.$$

Then

$$JI = K_1, \quad (J + T_2)(I + T_3) = K_1,$$

so that

$$\frac{I}{T_3} + \frac{J}{T_2} + 1 = 0.$$

Now I and J do not involve u; hence

$$\frac{\dfrac{\partial^2}{\partial u^2}\left(\dfrac{1}{T_3}\right)}{\dfrac{\partial}{\partial u}\left(\dfrac{1}{T_3}\right)} = \frac{\dfrac{\partial^2}{\partial u^2}\left(\dfrac{1}{T_2}\right)}{\dfrac{\partial}{\partial u}\left(\dfrac{1}{T_2}\right)}.$$

But T_2 does not contain v, and T_3 does not contain w; hence both fractions are functions of u only, say

$$\frac{b''(u)}{b'(u)};$$

then

$$\frac{1}{T_3} = b(u)\,c(v) + a(v) = b_1c_2 + a_2,$$

$$\frac{1}{T_2} = b(u)\,c'(w) + a'(w) = b_1c_3 + a_3.$$

Substituting in

$$\frac{I}{T_3} + \frac{J}{T_2} + 1 = 0,$$

we have

$$b_1(c_2I + c_3J) + a_2I + a_3J + 1 = 0.$$

Now I and J are functions of v and w only; hence

$$a_2I + a_3J + 1 = 0, \quad c_2I + c_3J = 0,$$

so that

$$\frac{I}{-c_3} = \frac{J}{c_2} = \frac{1}{a_2c_3 - a_3c_2}.$$

Consequently

$$A_2 = v_2 - I \ = v_2 + \frac{c_3}{a_2c_3 - a_3c_2},$$

$$A_3 = w_3 - J \ = w_3 - \frac{c_2}{a_2c_3 - a_3c_2},$$

$$B_3 = w_3 + T_2 = w_3 + \frac{1}{b_1c_3 + a_3},$$

$$C_2 = v_2 + T_3 = v_2 + \frac{1}{b_1c_2 + a_2}.$$

Change the variables from u, v, w to U, V, W, where

$$b_1 = U, \quad \frac{a_2}{c_2} = -V, \quad \frac{a_3}{c_3} = -W;$$

then

$$\frac{\partial A}{\partial V} = V_2' + \frac{\theta}{W - V}, \qquad \frac{\partial A}{\partial W} = W_3' - \frac{\phi}{W - V},$$

$$\frac{\partial C}{\partial V} = V_2' + \frac{\theta}{U - V}, \qquad \frac{\partial B}{\partial W} = W_3' + \frac{\phi}{W - V},$$

where

$$V_2' = v_2 \frac{\partial v}{\partial V}, \quad \text{a function of } V \text{ only,}$$

$$W_3' = w_3 \frac{\partial w}{\partial W}, \quad \ldots\ldots\ldots\ldots\ldots \quad W \text{ only,}$$

$$\theta = \frac{1}{c_2}\frac{\partial v}{\partial V}, \quad \ldots\ldots\ldots\ldots\ldots \quad V \text{ only at the utmost,}$$

$$\phi = \frac{1}{c_3}\frac{\partial w}{\partial W}, \quad \ldots\ldots\ldots\ldots\ldots \quad W \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

But

$$\frac{\partial}{\partial W}\left(\frac{\partial A}{\partial V}\right) = \frac{\partial}{\partial V}\left(\frac{\partial A}{\partial W}\right);$$

therefore

$$\theta = \phi = h,$$

where h is a pure constant. Thus

$$A = V_2 + W_3 - h \log (W - V).$$

Similarly

$$B = W_3 + U_1 - h \log (U - W),$$

$$C = U_1 + V_2 - h \log (V - U);$$

and so

$$H_1 = e^{B + C - \log \Omega}$$

$$= \frac{1}{\Omega} e^{U_1 + V_2 + W_3 - h\pi i} (V - U)^{-h} (W - U)^{-h} e^{U_1},$$

$$H_2 = \frac{1}{\Omega} e^{U_1 + V_2 + W_3 - h\pi i} (W - V)^{-h} (U - V)^{-h} e^{V_2},$$

$$H_3 = \frac{1}{\Omega} e^{U_1 + V_2 + W_3 - h\pi i} (U - W)^{-h} (V - W)^{-h} e^{W_3}.$$

Returning now (as is permissible) to our old variables, we have

$$H_1 = \frac{(v-u)^{-h}(w-u)^{-h}}{QU},$$

$$H_2 = \frac{(u-v)^{-h}(w-v)^{-h}}{QV},$$

$$H_3 = \frac{(u-w)^{-h}(v-w)^{-h}}{QW}.$$

Instead of modifying the equations, we substitute once more in the second set of Lamé relations; then

$$\left.\begin{aligned}(v-w)\,Q_{23} &= h\,(Q_2 - Q_3)\\ (w-u)\,Q_{31} &= h\,(Q_3 - Q_1)\\ (u-v)\,Q_{12} &= h\,(Q_1 - Q_2)\end{aligned}\right\},$$

the conditions of compatibility being satisfied.

Thus, for confocal quadrics

$$Q = 2, \quad h = -\tfrac{1}{2},$$

where

$$U^2 = (a+u)(b+u)(c+u);$$

and V and W are the same functions of v and of w as U is of u.

267. We now must have regard to the first set of three Lamé relations of § 246 which have to be satisfied by H_1, H_2, H_3. For simplicity, we shall take

$$Q = \text{constant} = 1,$$

so that the equations for Q are satisfied. One of the relations is

$$\frac{\partial}{\partial u}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial v}\right) + \frac{1}{H_3^2}\frac{\partial H_1}{\partial w}\frac{\partial H_2}{\partial w} = 0,$$

that is,

$$\frac{1}{H_1}\frac{\partial^2 H_2}{\partial u^2} + \frac{1}{H_2}\frac{\partial^2 H_1}{\partial v^2} - \frac{1}{H_1^2}\frac{\partial H_1}{\partial u}\frac{\partial H_2}{\partial u} - \frac{1}{H_2^2}\frac{\partial H_1}{\partial v}\frac{\partial H_2}{\partial v} + \frac{1}{H_3^2}\frac{\partial H_1}{\partial w}\frac{\partial H_2}{\partial w} = 0.$$

Now

$$\frac{1}{H_1}\frac{\partial H_1}{\partial u} = \frac{h}{v-u} + \frac{h}{w-u} - \frac{U'}{U}, \quad \frac{\partial H_1}{\partial w} = -\frac{h}{w-u}H_1,$$

$$\frac{\partial H_1}{\partial v} = -\frac{h}{v-u}H_1, \quad \frac{\partial^2 H_1}{\partial v^2} = \frac{h^2+h}{(v-u)^2}H_1;$$

and similarly for the others. Inserting these values in the foregoing relation, and reducing, we have

$$\frac{1}{H_1^2}\frac{U'}{U} - \frac{1}{H_2^2}\frac{V'}{V}$$

$$= \left(\frac{h}{w-u} - \frac{h}{w-v}\right)\frac{1}{H_3^2} + \left(\frac{1}{u-v} - \frac{h}{u-w}\right)\frac{1}{H_1^2} + \left(\frac{h}{v-w} - \frac{1}{v-u}\right)\frac{1}{H_2^2}.$$

The other two relations similarly give

$$\frac{1}{H_2^2}\frac{V'}{V}-\frac{1}{H_3^2}\frac{W'}{W}$$
$$=\left(\frac{h}{u-v}-\frac{h}{u-w}\right)\frac{1}{H_1^2}+\left(\frac{1}{v-w}-\frac{h}{v-u}\right)\frac{1}{H_2^2}+\left(\frac{h}{w-u}-\frac{1}{w-v}\right)\frac{1}{H_3^2},$$

$$\frac{1}{H_3^2}\frac{W'}{W}-\frac{1}{H_1^2}\frac{U'}{U}$$
$$=\left(\frac{h}{v-w}-\frac{h}{v-u}\right)\frac{1}{H_2^2}+\left(\frac{1}{w-u}-\frac{h}{w-v}\right)\frac{1}{H_3^2}+\left(\frac{h}{u-v}-\frac{1}{u-w}\right)\frac{1}{H_1^2}.$$

The sum of the left-hand sides of these three equations is zero; hence the sum of their right-hand sides must be zero, that is,

$$(1+2h)\left\{\left(\frac{1}{w-u}-\frac{1}{w-v}\right)\frac{1}{H_3^2}+\left(\frac{1}{u-v}-\frac{1}{u-w}\right)\frac{1}{H_1^2}+\left(\frac{1}{v-w}-\frac{1}{v-u}\right)\frac{1}{H_2^2}\right\}=0.$$

Now the variable factor, being

$$\frac{-1}{(u-v)(v-w)(w-u)}\left\{\left(\frac{u-v}{H_3}\right)^2+\left(\frac{v-w}{H_1}\right)^2+\left(\frac{w-u}{H_2}\right)^2\right\},$$

is not zero. Hence

$$1+2h=0,$$

so that $h=-\frac{1}{2}$; and therefore

$$H_1^2=\frac{(v-u)(w-u)}{U^2},\quad H_2^2=\frac{(u-v)(w-v)}{V^2},\quad H_3^2=\frac{(u-w)(v-w)}{W^2}.$$

The first of the three equations now is

$$(w-v)\,UU'+(u-w)\,VV'$$
$$=-\frac{1}{2}\frac{(u-v)^2}{(v-w)(w-u)}W^2$$
$$+\frac{1}{2}\frac{(3u-v-2w)(v-w)}{(u-v)(w-u)}U^2-\frac{1}{2}\frac{(3v-2w-u)(w-u)}{(v-w)(u-v)}V^2;$$

and the other two are obtainable by cyclical interchange of the three variables.

The equations, in this form, are linear and homogeneous in U^2, V^2, W^2 and their derivatives of the first order. The symmetry suggests an expectation that U^2, V^2, W^2 are similar polynomials in u, v, w or, at least, that solutions of this type certainly exist. If n be the common degree of these polynomials, it is easy to see that

$$n=3;$$

and it is easy to verify that

$$\left.\begin{aligned}U^2&=\kappa u^3+\alpha u^2+\beta u+\gamma\\ V^2&=\kappa v^3+\alpha v^2+\beta v+\gamma\\ W^2&=\kappa w^3+\alpha w^2+\beta w+\gamma\end{aligned}\right\},$$

where κ, α, β, γ are arbitrary constants, of which only the ratios are essential. As a matter of fact, these relations constitute the primitive of the three equations.

The values of H_1, H_2, H_3 thus obtained belong to the triply orthogonal system of confocal quadrics.

Darboux shews that a general triply orthogonal isometric system is given by the triple set of families of surfaces having the equation

$$(x^2+y^2+z^2)^2+ax^2+by^2+cz^2+d^2+\frac{4d^2-a^2}{a+\lambda}x^2+\frac{4d^2-b^2}{b+\lambda}y^2+\frac{4d^2-c^2}{c+\lambda}z^2=0,$$

for $\lambda = u, v, w$. The equation for the arc-element in space is

$$ds^2 = M\left\{\frac{(u-v)(u-w)}{U}du^2+\frac{(v-w)(v-u)}{V}dv^2+\frac{(w-u)(w-v)}{W}dw^2\right\},$$

where

$$\left.\begin{aligned} U &= (u+a)\,(u+b)(u+c)\,(u^2-4d^2)\\ V &= (v+a)\,(v+b)\,(v+c)\,(v^2-4d^2)\\ W &= (w+a)(w+b)(w+c)(w^2-4d^2)\end{aligned}\right\}.$$

For this result, and for further developments, reference should be made to his memoir already cited and, above all, to his treatise on orthogonal systems.

EXAMPLES.

1. Let ρ and σ be the radii of curvature and torsion of the intersection of the surfaces

$$v(x, y, z)=v, \quad w(x, y, z)=w,$$

in a triply orthogonal system; prove that

$$\frac{H_1^2}{\rho^2}=\frac{1}{H_2^2}\left(\frac{\partial H_1}{\partial v}\right)^2+\frac{1}{H_3^2}\left(\frac{\partial H_1}{\partial w}\right)^2,$$

$$\frac{H_1}{\sigma}=\frac{\partial}{\partial u}\tan^{-1}\left(\frac{H_3}{H_2}\frac{\partial H_1}{\partial v}\div\frac{\partial H_1}{\partial w}\right).$$

2. A family of surfaces is given by the equation

$$\phi(x, y, z, u)=0,$$

and x and y are regarded as the independent variables. Shew that the critical equation of the third order, which must be satisfied if the family belongs to a triply orthogonal system, is

$$\{(1+q^2)s-pqt\}\frac{\partial^2 T}{\partial x^2}+\{(1+p^2)t-(1+q^2)r\}\frac{\partial^2 T}{\partial x\partial y}+\{pqr-(1+p^2)s\}\frac{\partial^2 T}{\partial y^2}=0,$$

where

$$T=(1+p^2+q^2)^{-\frac{1}{2}}\frac{\partial z}{\partial u}.$$

3. Shew that any family of planes, the equation of which contains one parameter, can form part of a triply orthogonal system.

In any one of the planes, let two sets of curves be drawn cutting one another orthogonally. Let the plane be moved so as always to touch the developable surface which is the envelope of the family of planes; prove that the two families of surfaces generated by the two sets of curves complete a triply orthogonal system.

4. Circles are drawn, cutting a given surface orthogonally and also a given plane orthogonally. Shew that there is a Lamé family of surfaces cutting the circles orthogonally.

5. In the Bouquet surface $u=X+Y+Z$, denote the equation which X must satisfy by

$$X'X'''=2(X''-a)(X''-b);$$

and let λ and μ be the roots of the equation

$$\frac{X'^2}{\theta-X}+\frac{Y'^2}{\theta-Y}+\frac{Z'^2}{\theta-Z}=0.$$

Also let

$$v=\int^{\lambda}(\theta-a)^{\frac{a}{b-a}}(\theta-b)^{\frac{b}{a-b}}\left(\frac{X'^2}{\theta-X}+\frac{Y'^2}{\theta-Y}+\frac{Z'^2}{\theta-Z}\right)d\theta,$$

$$w=\int^{\mu}(\theta-a)^{\frac{a}{b-a}}(\theta-b)^{\frac{b}{a-b}}\left(\frac{X'^2}{\theta-X}+\frac{Y'^2}{\theta-Y}+\frac{Z'^2}{\theta-Z}\right)d\theta;$$

shew that the u-surface, the v-surface, and the w-surface thus obtained, are a triply orthogonal system.

6. Obtain equations for the orthogonal systems of which the surfaces

$$ax^3+by^3+cz^3+\lambda(x^2+y^2+z^2)=u,$$

$$ax^4+by^4+cz^4+\lambda(x^2+y^2+z^2)=u,$$

(where a, b, c, λ are constants) respectively form part.

7. The envelope of the family of surfaces

$$\left(\lambda+\frac{x^2}{m}\right)^m\left(\lambda+\frac{y^2}{n}\right)^n\left(\lambda+\frac{z^2}{p}\right)^p\lambda^q=\text{constant},$$

where λ is the parameter for the family, constitutes a triply orthogonal system, provided $m+n+p+q$ is not zero.

Discuss the case when $m+n+p+q=0$, obtaining a triply orthogonal system partly through the envelope.

8. A particular Lamé family consists of surfaces of constant negative curvature $-\frac{1}{U^2}$, where U may be a function of u. Shew that we may take

$$H_1=U\frac{\partial\Omega}{\partial u},\quad H_2=\cos\Omega,\quad H_3=\sin\Omega,$$

where Ω must satisfy the four equations (due to Bianchi)

$$\left.\begin{aligned}\frac{\partial^2\Omega}{\partial v^2}-\frac{\partial^2\Omega}{\partial w^2}-\frac{\sin\Omega\cos\Omega}{U^2}=0\\ \frac{\partial^3\Omega}{\partial u\partial v\partial w}-\frac{\partial\Omega}{\partial v}\frac{\partial^2\Omega}{\partial w\partial u}\cot\Omega+\frac{\partial\Omega}{\partial w}\frac{\partial^2\Omega}{\partial u\partial v}\tan\Omega=0\\ \frac{1}{U}\frac{\partial}{\partial u}\left(\frac{\sin\Omega}{U}\right)-\frac{\partial}{\partial v}\left(\frac{\partial^2\Omega}{\partial v\partial u}\sec\Omega\right)+\frac{\partial\Omega}{\partial w}\frac{\partial^2\Omega}{\partial w\partial u}\operatorname{cosec}\Omega=0\\ \frac{1}{U}\frac{\partial}{\partial u}\left(\frac{\cos\Omega}{U}\right)+\frac{\partial}{\partial w}\left(\frac{\partial^2\Omega}{\partial w\partial u}\operatorname{cosec}\Omega\right)+\frac{\partial\Omega}{\partial v}\frac{\partial^2\Omega}{\partial v\partial u}\sec\Omega=0\end{aligned}\right\}.$$

Prove that a solution of these equations is given by

$$\tfrac{1}{2}\pi-\Omega=\operatorname{am}\{cv+f(u)\},$$

where $f(u)$ is any arbitrary function of u, c is a pure constant, and the modulus of the elliptic functions is $1/cU$; and verify that these u-surfaces are surfaces of rotation.

CHAPTER XII.

Congruences of Curves.

The present chapter deals solely with the elements of the theory of congruences of curves; and within the range of that theory, attention is restricted to curves which are either straight lines or circles.

The simplest example of rectilinear congruences (in which, moreover, there is direct application to a physical subject) occurs when the straight lines are composed of a set of lines that can be cut orthogonally by a family of surfaces—such as the rays of light issuing from a centre. They were considered at an early stage by Malus, Dupin, and Hamilton.

The theory of congruences of plane curves, and particularly of circles, owes its early systematic development to Ribaucour.

Detailed references to many of the numerous writers on the subject will be found in the second volume of Darboux's *Théorie générale des surfaces*, book iv, chapters i, xii, xiii, xv, and in chapters x and xviii of Bianchi's *Geometria Differenziale.*

268. In almost all the preceding investigations, whether surfaces or space constituted the subject of investigation, the discussion has been based upon point-coordinates by taking a point as the initial element. Two exceptions arose; for each of them, the discussion was based upon plane-coordinates, by taking a plane as the initial element. In one of these exceptions, the equation of the osculating plane of a skew curve was taken as the analytical definition of the curve (§ 16); in the other of them, the coordinates of the tangent plane to a surface were used, to complete the spherical representation of the surface (§ 162).

Now, in algebraic geometry, it proves convenient to use line-coordinates by taking a straight line as the element of space, instead of a point or a plane; more generally, we could take a curve, plane or skew, as the element of space. For this purpose, we note that space may be regarded as containing ∞^3 points. For our purposes, a curve through a point will have a definite direction (or one of a limited number of definite directions); so that the curve will associate, with the point, ∞^1 other points or a finite multiple of ∞^1 other points; consequently, we should have ∞^2 curves for our investigation. As they are curves in space, they require two independent equations for their analytical expression; as they are ∞^2 in numerical range, these equations must involve two independent parameters. Such an aggregate of curves is

called a *congruence of curves*, sometimes a *congruence*, sometimes a congruence with a prefixed epithet (rectilinear, cyclical, or the like).

Examples of congruences of curves are frequent enough. Thus the aggregate of the normals to a surface is a rectilinear congruence, as is the aggregate of tangents from a twisted curve to a surface in space. Systems of rays in theoretical optics have been the subject of many investigations; and the importance of the congruence of characteristic curves in connection with the primitives of partial differential equations is well known.

To illustrate the origin of a congruence, consider two similar problems, one of which leads to a congruence, while the other does not. Take any two algebraical surfaces and, for greater definiteness, suppose that they are not parallel; let it be required to find the aggregate, (i) of straight lines which are orthogonal to both surfaces, (ii) of circles which are orthogonal to both surfaces. The equations of the surfaces are taken to be

$$f(x, y, z) = 0, \quad g(x, y, z) = 0.$$

(i) If x_0, y_0, z_0 be a point on the former surface; and if x_1, y_1, z_1 be a point on the latter surface; where the straight line

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$$

is normal to both surfaces, we have

$$f(x_0, y_0, z_0) = 0, \quad g(x_1, y_1, z_1) = 0,$$
$$\frac{1}{x_1-x_0}\frac{\partial f}{\partial x_0} = \frac{1}{y_1-y_0}\frac{\partial f}{\partial y_0} = \frac{1}{z_1-z_0}\frac{\partial f}{\partial z_0},$$
$$\frac{1}{x_1-x_0}\frac{\partial g}{\partial x_1} = \frac{1}{y_1-y_0}\frac{\partial g}{\partial y_1} = \frac{1}{z_1-z_0}\frac{\partial g}{\partial z_1}.$$

Thus there are six equations for the determination of the six quantities $x_0, y_0, z_0, x_1, y_1, z_1$. In the absence of special relations between the surfaces (such as parallelism, which would identify the last two pairs of equations), it can be inferred that they furnish a limited number of solutions, real or imaginary. Thus there is only a limited number of straight lines, normal to the two surfaces; they do not constitute a congruence.

(ii) Any circle in space can be represented by equations

$$\left.\begin{array}{r}(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \rho^2\\ l(x-\alpha) + m(y-\beta) + n(z-\gamma) = 0\end{array}\right\}.$$

Let this circle cut the surface $f(x, y, z) = 0$ orthogonally at x_0, y_0, z_0, and the surface $g(x, y, z) = 0$ orthogonally at x_1, y_1, z_1. Then the equations

$$f(x_0, y_0, z_0) = 0, \quad g(x_1, y_1, z_1) = 0,$$
$$(x_0-\alpha)^2 + (y_0-\beta)^2 + (z_0-\gamma)^2 = \rho^2, \quad l(x_0-\alpha) + m(y_0-\beta) + n(z_0-\gamma) = 0,$$
$$(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2 = \rho^2, \quad l(x_1-\alpha) + m(y_1-\beta) + n(z_1-\gamma) = 0,$$

$$\frac{\partial f}{\partial x_0} : \frac{\partial f}{\partial y_0} : \frac{\partial f}{\partial z_0} = \left\| \begin{array}{ccc} x_0 - \alpha, & y_0 - \beta, & z_0 - \gamma \\ l\ , & m\ , & n \end{array} \right\|,$$

$$\frac{\partial g}{\partial x_1} : \frac{\partial g}{\partial y_1} : \frac{\partial g}{\partial z_1} = \left\| \begin{array}{ccc} x_1 - \alpha, & y_1 - \beta, & z_1 - \gamma \\ l\ , & m\ , & n \end{array} \right\|,$$

must be satisfied. Thus, in general, there are ten equations (some of them not homogeneous) involving the twelve quantities x_0, y_0, z_0, x_1, y_1, z_1, $\alpha, \beta, \gamma, \rho$, $l : m$, $l : n$; hence two of these quantities may be regarded as ultimately independent parameters. They do constitute a congruence; such a double system of circles is often called a *cyclical congruence*.

But it does not, of course, follow that a congruence of circles is necessarily orthogonal to two independent surfaces.

269. Accordingly, we take a congruence of curves represented by two equations

$$f(x, y, z, p, q) = 0, \quad g(x, y, z, p, q) = 0,$$

where p and q are two parameters; and we shall assume that the equations are algebraical. When full variation is allowed to p and q independently of one another, we have a double infinitude of curves in the congruence. The curves, passing through a given point x_0, y_0, z_0 in space, are determined by values of p and q which satisfy the equations

$$f(x_0, y_0, z_0, p, q) = 0, \quad g(x_0, y_0, z_0, p, q) = 0.$$

Usually, these provide only a limited number of values of p and q, so that then the number of curves passing through the assigned point in space is limited. But it may happen that the two equations only determine a relation between p and q, so that p is not restricted to any definite value or values; in that case, a simple infinitude of curves pass through the point.

The double infinitude of curves can be grouped so as to constitute surfaces. Taking any relation

$$q = \phi(p),$$

and eliminating p and q between this equation and the equations of the curves, we have a surface; and by taking an infinitude of forms for ϕ, so as to exhaust the congruence, we obtain a simple infinitude of surfaces. These are called the *surfaces* of the congruence.

When any direction dx, dy, dz is taken at a point on the surface in the tangent plane, we have

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \left\{\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q} \phi'(p)\right\} dp = 0,$$

$$\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz + \left\{\frac{\partial g}{\partial p} + \frac{\partial g}{\partial q} \phi'(p)\right\} dp = 0;$$

and therefore

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=\mu\left(\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy+\frac{\partial g}{\partial z}dz\right),$$

where

$$\mu=\frac{\dfrac{\partial f}{\partial p}+\dfrac{\partial f}{\partial q}\phi'(p)}{\dfrac{\partial g}{\partial p}+\dfrac{\partial g}{\partial q}\phi'(p)}.$$

Thus the direction-cosines of the tangent plane are proportional to

$$\mu\frac{\partial g}{\partial x}-\frac{\partial f}{\partial x},\quad \mu\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y},\quad \mu\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}.$$

The curve of the congruence, passing through the point of contact of this tangent plane, lies on the surface; and so its tangent lies in the tangent plane to the surface.

270. The equations $f=0$ and $g=0$ are independent of one another, so far as concerns variables and parameters; hence their Jacobian with regard to any two of the arguments involved (*e.g.* with regard to p and q) does not vanish identically. Thus the equation

$$\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}=0$$

is usually a new equation, satisfied independently of the equations of the curve $f=0$ and $g=0$; taken simultaneously with them, it determines a finite number of sets of values of x, y, z, that is, it determines a finite number of points on the particular curve, which are independent of the existence of any assigned relation between p and q. For all such points, the value of μ is independent of the form of $\phi'(p)$; and so all the surfaces of the congruence, which pass through the particular curve, have the same tangent plane at each of the points in question. These points upon the curve are called its *focal points*.

It has been remarked that the number of focal points is limited, being the points given by the (usually) limited number of sets of simultaneous solutions of

$$f=0,\quad g=0,\quad \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}=0.$$

For a rectilinear congruence, $f=0$ and $g=0$ are planes; so the equations are of order 1, 1, 2 respectively. The number of focal points upon any line of the congruence usually is two; on particular lines, the two focal points may coincide.

For a congruence of circles, $f=0$ is a sphere and $g=0$ is a plane; so the equations are of order 2, 1, 2 respectively. The number of focal points upon any circle of the congruence usually is four; for particular circles, coincidences among the focal points may occur.

For a congruence of conics, the number of focal points upon each conic effectively is six. For sphero-conics, the number is twelve. For quadri-quadric curves, the number is sixteen. For plane curves of order n, the number is $n(n+1)$. In the last case, there may be decrease of the number of focal points in the finite part of space, owing to some speciality in the form of the curve. In all the cases, there may be apparent decrease in the number owing to coincidences among the focal points in the general aggregate.

The focal points of any curve are given by the equations

$$f=0, \quad g=0, \quad J=\frac{\partial(f, g)}{\partial(p, q)}=0,$$

whatever law between p and q is postulated. When p and q are eliminated between the three equations, usually a single relation between x, y, z is the eliminant. The surface, which is represented by this equation, is the same whatever values may have been assigned to p and q; thus it is the locus of all the focal points of all the curves, and so it is called the *focal surface* of the congruence.

Any surface of the congruence meets the focal surface at the focal points of any of its curves. At any point on the focal surface, we have

$$\frac{\partial f}{\partial x} dx+\frac{\partial f}{\partial y} dy+\frac{\partial f}{\partial z} dz+\frac{\partial f}{\partial p} dp+\frac{\partial f}{\partial q} dq=0,$$

$$\frac{\partial g}{\partial x} dx+\frac{\partial g}{\partial y} dy+\frac{\partial g}{\partial z} dz+\frac{\partial g}{\partial p} dp+\frac{\partial g}{\partial q} dq=0;$$

and a corresponding equation derived from $J=0$, which (with these two) would give the direction-cosines of the normal to the tangent plane. Corresponding to the focal points on the curve, we have

$$\frac{\partial f}{\partial a}=\kappa \frac{\partial g}{\partial a}, \quad \frac{\partial f}{\partial b}=\kappa \frac{\partial g}{\partial b},$$

so that

$$\frac{\partial f}{\partial x} dx+\frac{\partial f}{\partial y} dy+\frac{\partial f}{\partial z} dz=\kappa\left(\frac{\partial g}{\partial x} dx+\frac{\partial g}{\partial y} dy+\frac{\partial g}{\partial z} dz\right),$$

as the equation for directions in the tangent plane of the focal surface. But at these points on the surface of the congruence derived through our curve, we have $\mu=\kappa$; so that the direction-cosines of the normal to the tangent plane to the latter at these points are proportional to

$$\kappa \frac{\partial g}{\partial x}-\frac{\partial f}{\partial x}, \quad \kappa \frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}, \quad \kappa \frac{\partial g}{\partial z}-\frac{\partial f}{\partial z},$$

and therefore are the same as those of the normal to the tangent plane to the focal surface at the point. Hence any surface of a congruence touches the focal surface at the foci of any of its curves; and any two surfaces, containing a particular curve, touch one another at the foci of the curve.

The last result suggests the consideration of envelopes of the curves in the congruence. We cannot usually have

$$f=0,\quad g=0,\quad \frac{\partial f}{\partial p}=0,\quad \frac{\partial f}{\partial q}=0,\quad \frac{\partial g}{\partial p}=0,\quad \frac{\partial g}{\partial q}=0.$$

So we imagine a family selected according to some law between p and q; and then its envelope is given by

$$f=0,\quad g=0,\quad \frac{\partial f}{\partial p}+\frac{\partial f}{\partial q}\frac{dq}{dp}=0,\quad \frac{\partial g}{\partial p}+\frac{\partial g}{\partial q}\frac{dq}{dp}=0.$$

When we eliminate x, y, z, we have an ordinary equation

$$F\left(\frac{dq}{dp},\, q,\, p\right)=0,$$

of the first order; this will determine the required law. Also, the equations are included in, but are not so extensive as, the set of equations

$$f=0,\quad g=0,\quad J=0;$$

and so the envelope of the selected curves lies upon the focal surface, touching the curves at their focal points.

Surfaces normal to a congruence.

271. Consider the possibility, that the curves in a congruence should be normal to some surface. Along the curve

$$f(x, y, z, p, q)=0,\quad g(x, y, z, p, q)=0,$$

we have

$$\frac{dx}{J\left(\dfrac{f,\, g}{y,\, z}\right)}=\frac{dy}{J\left(\dfrac{f,\, g}{z,\, x}\right)}=\frac{dz}{J\left(\dfrac{f,\, g}{x,\, y}\right)};$$

and so a surface, cutting the curve at right angles, is given by

$$J\left(\frac{f,\, g}{y,\, z}\right)dx+J\left(\frac{f,\, g}{z,\, x}\right)dy+J\left(\frac{f,\, g}{x,\, y}\right)dz=0.$$

If the surface is to cut all the curves of the congruence at right angles, the values of p and q must be imagined as obtained from $f=0$ and $g=0$, and then be substituted in $J\left(\dfrac{f,\, g}{y,\, z}\right)$, $J\left(\dfrac{f,\, g}{z,\, x}\right)$, $J\left(\dfrac{f,\, g}{x,\, y}\right)$, wherever they occur. Let the resulting values of these quantities, which now are functions of the variables alone, be denoted by X, Y, Z; our equation is

$$Xdx+Ydy+Zdz=0.$$

In general and unconditionally, this differential relation is not integrable, in the sense that its integral equivalent consists of only a single equation; and therefore there is no surface orthogonal to all the curves of an arbitrarily assigned congruence.

But the differential relation has an integral equivalent consisting of a single equation, if the condition of integrability is satisfied, viz. we must have

$$I = X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0.$$

As regards this equation there are two possibilities.

First, it may happen not to be an identity. In that event, it is a relation between x, y, z, which is algebraical in form and so can be regarded as providing one or several values of z. If, for any one of these, the equation

$$X dx + Y dy + Z dz = 0$$

is satisfied, then the equation giving the value of z in terms of x and y provides a surface orthogonal to all the curves of the congruence. But it is not usually the case that values of z thus obtained do satisfy the differential relation; even those, which do satisfy it, only provide isolated surfaces orthogonal to the curves; and the number of these isolated surfaces is limited, so that it cannot be greater than the degree of the function I.

Secondly, the condition of integrability may happen to be satisfied identically. In that event, the relation

$$X dx + Y dy + Z dz = 0$$

has an integral equivalent consisting of an equation

$$N(x, y, z) = a,$$

where a is an arbitrary constant; the integral equivalent is obtained in the customary fashion. The integral equation gives a family of surfaces. All the curves of the congruence are cut orthogonally by the family.

Further, we have assumed that the congruence is represented by integral equations $f = 0$ and $g = 0$, which are algebraical. It may be given initially by differential equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z},$$

the primitive of which contains the two necessary parameters and consists of two integral equations. These two integral equations are, however, not necessarily algebraical, even when X, Y, Z are algebraical. The argument is otherwise unaltered; and so we have the result:—

A congruence of curves is usually not capable of orthogonal section by a surface; but there may be isolated surfaces normal to the curves in particular congruences, the number being limited when the curves are algebraical; and it may happen (under the condition indicated) that a congruence is cut normally by a family of surfaces.

Ex. 1. For the congruence

$$\frac{dx}{y(z+1)}=\frac{dy}{z(x+1)}=\frac{dz}{x(y+1)},$$

so that

$$X=y(z+1),\quad Y=z(x+1),\quad Z=x(y+1).$$

The equation $I=0$ is not satisfied identically, being

$$yz+zx+xy+x+y+z=0.$$

The value of z given by this relation does not satisfy the equation

$$Xdx+Ydy+Zdz=0;$$

there is no surface orthogonal to the congruence.

Ex. 2. For the congruence

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y},$$

being a congruence of circles

$$\left.\begin{array}{l} x+y+z=p \\ x^2+y^2+z^2=q \end{array}\right\},$$

the equation $I=0$ is satisfied identically. Hence the circles can be cut orthogonally by a family of surfaces, whose differential equation is

$$(y-z)\,dx+(z-x)\,dy+(x-y)\,dz=0;$$

the integral equation of the family is easily found to be

$$y-z=a(x-z).$$

Ex. 3. Shew that the congruence

$$py-qx=0,\quad x^2+y^2+z^2-2px-2qy+1=0,$$

being a congruence of circles orthogonal to the two particular surfaces

$$z=0,\quad x^2+y^2+z^2=1,$$

has

$$\frac{dx}{x}=\frac{dy}{y}=\frac{2z\,dz}{z^2-x^2-y^2+1}$$

for its differential equations. Prove that the condition of integrability is satisfied; and verify that all the curves of the congruence are cut orthogonally by the family of spheres

$$x^2+y^2+z^2-1=az,$$

where a is the parameter of the family.

272. The general condition of integrability is

$$X\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)+Y\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)+Z\left(\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right)=0.$$

When the congruence is given by the equations

$$f(x, y, z, p, q)=0,\quad g(x, y, z, p, q)=0,$$

of the most general type, then

$$X=\frac{\partial(f, g)}{\partial(y, z)},\quad Y=\frac{\partial(f, g)}{\partial(z, x)},\quad Z=\frac{\partial(f, g)}{\partial(x, y)};$$

and the values of p and q, in terms of x, y, z from $f=0$ and $g=0$, have to be inserted, explicitly or implicitly, in X, Y, Z before the partial derivatives are framed. Let

$$\frac{d}{ds}=X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z},$$

so that d/ds represents derivation along the curve of the congruence; and write

$$A=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2,$$

$$B=\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z},$$

$$C=\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+\left(\frac{\partial g}{\partial z}\right)^2.$$

Then the foregoing general condition can be expressed in the form

$$\begin{vmatrix} \Sigma\frac{\partial g}{\partial x}\frac{d}{ds}\left(\frac{\partial f}{\partial x}\right), & \frac{d}{ds}\left(\frac{\partial f}{\partial p}\right), & \frac{d}{ds}\left(\frac{\partial f}{\partial q}\right) \\ B, & \frac{\partial f}{\partial p}, & \frac{\partial f}{\partial q} \\ C, & \frac{\partial g}{\partial p}, & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} \Sigma\frac{\partial f}{\partial x}\frac{d}{ds}\left(\frac{\partial g}{\partial x}\right), & \frac{d}{ds}\left(\frac{\partial g}{\partial p}\right), & \frac{d}{ds}\left(\frac{\partial g}{\partial q}\right) \\ A, & \frac{\partial f}{\partial p}, & \frac{\partial f}{\partial q} \\ B, & \frac{\partial g}{\partial p}, & \frac{\partial g}{\partial q} \end{vmatrix}.$$

When the equations of the congruence have the simpler form

$$f=\xi+\eta+\zeta-p=0,\quad g=\alpha+\beta+\gamma-q=0,$$

where α and ξ are functions of x alone, β and η are functions of y alone, and γ and ζ are functions of z alone, we have

$$X=\eta'\gamma'-\zeta'\beta',\quad Y=\zeta'\alpha'-\xi'\gamma',\quad Z=\xi'\beta'-\eta'\alpha';$$

and then the condition of integrability becomes

$$\begin{vmatrix} \alpha''\xi'-\alpha'\xi'', & \beta''\eta'-\beta'\eta'', & \gamma''\zeta'-\gamma'\zeta'' \\ \alpha', & \beta', & \gamma' \\ \xi', & \eta', & \zeta' \end{vmatrix}=0.$$

Hence we must have

$$\left.\begin{aligned} \alpha''\xi'-\alpha'\xi''&=\rho\alpha'+\sigma\xi' \\ \beta''\eta'-\beta'\eta''&=\rho\beta'+\sigma\eta' \\ \gamma''\zeta'-\gamma'\zeta''&=\rho\gamma'+\sigma\zeta' \end{aligned}\right\},$$

where ρ and σ are pure constants. And similarly for other forms.

Ex. 1. Shew that the congruence

$$\left.\begin{aligned} \xi+\eta+\zeta&=p \\ a\xi+b\eta+c\zeta&=q \end{aligned}\right\},$$

where a, b, c are constants, can be cut orthogonally by a family of surfaces; and determine the family.

Ex. 2. Find the congruence of curves, lying in parallel planes and upon the surfaces

$$\xi+\eta+\zeta=p,$$

which can be cut orthogonally by a family of surfaces; and determine the orthogonal family.

Rectilinear congruences.

273. We now proceed to consider in some detail the properties of congruences composed of straight lines, commonly called *rectilinear congruences.* Their equation has the form

$$\frac{\xi-x}{X}=\frac{\eta-y}{Y}=\frac{\zeta-z}{Z},$$

where x, y, z, X, Y, Z are functions of two independent parameters p and q. The point x, y, z may be regarded as a point on a director surface; the quantities X, Y, Z are the direction-cosines of the line (often called the *ray*) through x, y, z, and, on the sphere

$$X^2+Y^2+Z^2=1,$$

they give a spherical image of the congruence.

Distinction must be made between a congruence of lines, thus defined, and a complex of lines. The equation of a complex has the above form

$$\frac{\xi-x}{X}=\frac{\eta-y}{Y}=\frac{\zeta-z}{Z},$$

where now x, y, z is any point of space, and the direction-cosines X, Y, Z are any definite functions; so that the complex involves three parameters, while the congruence involves two. We shall deal only with congruences.

Take any distance l from x, y, z along the ray through that point on the director surface; the coordinates of the point so obtained are

$$x+lX, \quad y+lY, \quad z+lZ.$$

The square of an arc-element in space at the point is

$$\Sigma dx^2+2dl\Sigma Xdx+2l\Sigma dxdX+dl^2+l^2\Sigma dX^2.$$

Of the quantities which occur in this expression, Σdx^2 is the square of the arc-element on the surface; its form has been amply studied in earlier chapters, and the significance is of minor importance for the lines in the congruence. The quantity ΣXdx will occur from time to time; its evanescence is the condition that the lines are normals to a surface. The quantity $\Sigma dxdX$ is a new quadratic form; we write

$$a=\Sigma x_1X_1, \quad b=\Sigma x_2X_1, \quad b'=\Sigma x_1X_2, \quad c=\Sigma x_2X_2,$$

where $x_1=\dfrac{\partial x}{\partial p}$, $x_2=\dfrac{\partial x}{\partial q}$, and so for the other quantities; then

$$\Sigma dxdX=a\,dp^2+(b+b')\,dpdq+c\,dq^2.$$

The quantity dl is merely an element of length along the ray; it is, of course, independent of p and q. The quantity ΣdX^2 is the square of the arc-element in the spherical image; we write (as before, § 159)

$$e=\Sigma X_1^2, \quad f=\Sigma X_1 X_2, \quad g=\Sigma X_2^2,$$

and

$$\Sigma dX^2 = d\theta^2 = edp^2 + 2fdp\,dq + gdq^2,$$

so that $d\theta$ is the angle between the rays (p, q) and $(p+dp, q+dq)$.

The parameters p and q are at our choice; the choice can be exercised so as to make

$$f=0, \quad b+b'=0.$$

To prove this, take two new independent variables u and v, which are functions of p and q to be determined. The new quantity f is

$$\frac{\partial X}{\partial u}\frac{\partial X}{\partial v}+\frac{\partial Y}{\partial u}\frac{\partial Y}{\partial v}+\frac{\partial Z}{\partial u}\frac{\partial Z}{\partial v},$$

that is,

$$e\frac{\partial p}{\partial u}\frac{\partial p}{\partial v}+f\left(\frac{\partial p}{\partial u}\frac{\partial q}{\partial v}+\frac{\partial p}{\partial v}\frac{\partial q}{\partial u}\right)+g\frac{\partial q}{\partial u}\frac{\partial q}{\partial v};$$

and so the new quantity f will vanish if p and q, as functions of u and v, satisfy the equation

$$e\frac{\partial p}{\partial u}\frac{\partial p}{\partial v}+f\left(\frac{\partial p}{\partial u}\frac{\partial q}{\partial v}+\frac{\partial p}{\partial v}\frac{\partial q}{\partial u}\right)+g\frac{\partial q}{\partial u}\frac{\partial q}{\partial v}=0.$$

The new quantity $b+b'$ is

$$\Sigma\left(\frac{\partial x}{\partial v}\frac{\partial X}{\partial u}+\frac{\partial x}{\partial u}\frac{\partial X}{\partial v}\right),$$

that is,

$$a\frac{\partial p}{\partial u}\frac{\partial p}{\partial v}+(b+b')\left(\frac{\partial p}{\partial u}\frac{\partial q}{\partial v}+\frac{\partial p}{\partial v}\frac{\partial q}{\partial u}\right)+c\frac{\partial q}{\partial u}\frac{\partial q}{\partial v};$$

and therefore the new quantity $b+b'$ will vanish, if p and q satisfy the equation

$$a\frac{\partial p}{\partial u}\frac{\partial p}{\partial v}+(b+b')\left(\frac{\partial p}{\partial u}\frac{\partial q}{\partial v}+\frac{\partial p}{\partial v}\frac{\partial q}{\partial u}\right)+c\frac{\partial q}{\partial u}\frac{\partial q}{\partial v}=0.$$

Thus two relations have to be satisfied.

Two cases arise. Firstly, let the relations be different from one another; then we have two partial equations of the first order involving two dependent variables; by the general existence-theorem for such equations, they possess integrals which even satisfy assigned conditions. Thus the transformation is possible. Secondly, let the relations be the same; then the single relation can be satisfied by taking q any function of u and v, and using the modified relation to determine p. Thus the transformation is possible in an infinitude of ways, when

$$e:f:g=a:b+b':c;$$

such a congruence is called *isotropic* (§ 279).

In both cases, therefore, we can make $f=0$, $b+b'=0$, without loss of generality.

274. Take two consecutive rays determined by (p, q) and $(p+dp, q+dq)$. Let dn denote the shortest distance between them, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ its direction-cosines; then

$$X\cos\alpha + Y\cos\beta + Z\cos\gamma = 0,$$

$$(X+dX)\cos\alpha + (Y+dY)\cos\beta + (Z+dZ)\cos\gamma = 0,$$

so that

$$\left.\begin{aligned} M\cos\alpha &= YdZ - ZdY = (YZ_1 - ZY_1)\,dp + (YZ_2 - ZY_2)\,dq \\ M\cos\beta &= ZdX - XdZ = (ZX_1 - XZ_1)\,dp + (ZX_2 - XZ_2)\,dq \\ M\cos\gamma &= XdY - YdX = (XY_1 - YX_1)\,dp + (XY_2 - YX_2)\,dq \end{aligned}\right\},$$

where

$$M^2 = dX^2 + dY^2 + dZ^2 = d\theta^2.$$

Then

$$dn = dx\,.\cos\alpha + dy\,.\cos\beta + dz\,.\cos\gamma,$$

so that

$$d\theta dn = \Sigma\left[(x_1dp + x_2dq)\left\{(YZ_1 - ZY_1)\,dp + (YZ_2 - ZY_2)\,dq\right\}\right].$$

On the right-hand side, the coefficient of dp^2 is

$$\Sigma x_1(YZ_1 - ZY_1).$$

Now (§ 162)

$$X_1Y_2 - X_2Y_1 = vZ, \quad Z_1X_2 - Z_2X_1 = vY,$$

where $v = (eg - f^2)^{\frac{1}{2}}$; and so the coefficient of dp^2

$$= \frac{1}{v}\,\Sigma x_1\left\{Z_1(Z_1X_2 - Z_2X_1) - Y_1(X_1Y_2 - X_2Y_1)\right\}$$

$$= \frac{1}{v}\,\Sigma x_1\left\{X_2(X_1^2 + Y_1^2 + Z_1^2) - X_1(X_1X_2 + Y_1Y_2 + Z_1Z_2)\right\}$$

$$= \frac{1}{v}(eb' - fa).$$

Similarly, the coefficient of $dpdq$ is

$$\frac{1}{v}(ec - fb + fb' - ga),$$

and the coefficient of dq^2 is

$$\frac{1}{v}(fc - gb).$$

We thus have

$$d\theta dn = \frac{1}{v}\left\{(eb' - fa)\,dp^2 + (ec - fb + fb' - ga)\,dpdq + (fc - gb)\,dq^2\right\}$$

$$= \frac{1}{v}\begin{vmatrix} edp + fdq, & fdp + gdq \\ adp + bdq, & b'dp + cdq \end{vmatrix}.$$

Further, let t denote the distance along the ray (p, q) between the point x, y, z and the foot of the shortest distance dn between the consecutive rays under consideration; and let $t+dt$ denote the distance along the ray $(p+dp, q+dq)$ between the point $x+dx, y+dy, z+dz$ and the foot of the same shortest distance. Then

$$\begin{aligned} x+tX+dn\,.\cos\alpha &= x+dx+(t+dt)(X+dX) \\ &= x+dx+tX+Xdt+tdX, \end{aligned}$$

when infinitesimal quantities of the second order are neglected; thus

$$dn\cos\alpha = dx+Xdt+tdX.$$

Similarly

$$dn\cos\beta = dy+Ydt+tdY,$$

$$dn\cos\gamma = dz+Zdt+tdZ.$$

Multiplying by X, Y, Z, and adding, we have

$$dt+\Sigma Xdx = 0;$$

multiplying by dX, dY, dZ, and adding, we have

$$\Sigma dxdX + t\Sigma dX^2 = 0,$$

that is,

$$t = -\frac{adp^2+(b+b')\,dpdq+cdq^2}{edp^2+2fdpdq+gdq^2}.$$

These results are general, whatever be the parametric variables (p and q) that may originally have been selected.

The first relation shews that, if the congruence is normal to the director surface, it is normal to any parallel surface; for dt and ΣXdx vanish together.

275. The latter relation makes t a function of the ratio $dp:dq$. There are two values of this ratio for which t has a stationary (maximum or minimum) value; and there is a corresponding quadratic giving the two stationary values of t.

When t is a maximum or minimum, we have (denoting the ratio $dp:dq$ by μ)

$$\left.\begin{aligned} t(e\mu+f)+a\mu+\tfrac{1}{2}(b+b') &= 0 \\ t(f\mu+g)+\tfrac{1}{2}(b+b')\mu+c &= 0 \end{aligned}\right\}.$$

When μ is eliminated, we have the quadratic giving the stationary values of t; it is

$$\begin{vmatrix} et+a\;\;\;, & ft+\tfrac{1}{2}(b+b') \\ ft+\tfrac{1}{2}(b+b'), & gt+c \end{vmatrix} = 0,$$

that is,

$$(eg-f^2)\,t^2+\{ec-f(b+b')+ga\}\,t+ac-\tfrac{1}{4}(b+b')^2 = 0.$$

The directions, determined by the ratio $dp:dq$ on the director surface, for which t has one or other of these two stationary values, are obtained by eliminating t; they are given by the equation

$$\{2fa-e(b+b')\}\,dp^2+2(ga-ec)\,dpdq+\{g(b+b')-2fc\}\,dq^2 = 0.$$

Now $v^2, = eg - f^2$, is not zero; so we may take the directions of the parametric curves to be given by the quantities u and v (of § 273); that is, without loss of generality, we may take

$$f = 0, \quad b + b' = 0;$$

and then the rays that give the maximum or minimum value of t are

$$p = \text{constant}, \quad q = \text{constant}.$$

Let t_1 and t_2 denote the maximum and minimum values of t along the ray; say, let

$$t_1 = -\frac{c}{g}, \qquad t_2 = -\frac{a}{e},$$

so that

$$t = \frac{et_2 dp^2 + gt_1 dq^2}{edp^2 + gdq^2}.$$

As to the form of this result, particular consecutive rays are chosen through the parameters of reference, the ray p = constant giving the value t_1 and the ray q = constant giving the value t_2.

Let $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ denote the direction-cosines of the shortest distance between the former ray and the current ray; and let $\cos\alpha_2$, $\cos\beta_2$, $\cos\gamma_2$ denote those of the shortest distance between the latter ray and the current ray; then, as

$$d\theta_1 \cos\alpha_1 = (YZ_2 - ZY_2)\, dq, \quad d\theta_2 \cos\alpha_2 = (YZ_1 - ZY_1)\, dp,$$

and similarly for the other quantities, we have

$$\begin{aligned} d\theta_1 d\theta_2 \Sigma \cos\alpha_1 \cos\alpha_2 &= \{\Sigma\, (YZ_2 - ZY_2)(YZ_1 - ZY_1)\}\, dpdq \\ &= \{\Sigma X^2 \Sigma X_1 X_2 - \Sigma XX_1 \Sigma XX_2\}\, dpdq \\ &= 0, \end{aligned}$$

with our curves of reference. Thus the two shortest distances are perpendicular to one another.

Further, let ω denote the angle which the shortest distance between the ray $(p + dp, q + dq)$ and the ray (p, q) makes—on the special reference—with the shortest distance between the rays $(p, q + dq)$ and (p, q). Then, as

$$\begin{aligned} d\theta d\theta_1 \Sigma \cos\alpha \cos\alpha_1 &= \Sigma\, \{(YZ_2 - ZY_2)\, dq\}\, \{(YZ_1 - ZY_1)\, dp + (YZ_2 - ZY_2)\, dq\} \\ &= \{\Sigma\, (YZ_2 - ZY_2)^2\}\, dq^2 \end{aligned}$$

on our reference, we have

$$d\theta d\theta_1 \cos\omega = gdq^2.$$

But

$$d\theta_1^2 = gdq^2,$$

and therefore

$$d\theta \cos\omega = g^{\frac{1}{2}}\, dq.$$

Similarly

$$d\theta \sin\omega = e^{\frac{1}{2}}\, dp,$$

the two equations being consistent with

$$d\theta^2 = edp^2 + gdq^2.$$

Hence we have

$$t = t_1 \cos^2 \omega + t_2 \sin^2 \omega,$$

a result due to Hamilton.

The analytical analogy with Euler's theorem as to the curvature of normal sections of a surface is obvious.

The two points determined by t_1 and t_2 are called the *limits* of the ray. The two planes through the ray and the two directions, associated with the limits, are called the *principal planes* of the ray; manifestly they are perpendicular to one another.

276. Next, consider the foci of the ray. We know that the foci of a curve in a congruence

$$f(x, y, z, p, q) = 0, \quad g(x, y, z, p, q) = 0,$$

can be obtained by associating the equations

$$\frac{\partial f}{\partial p} = \kappa \frac{\partial f}{\partial q}, \qquad \frac{\partial g}{\partial p} = \kappa \frac{\partial g}{\partial q},$$

with the equations of the curve. In the case of the ray, its equations are

$$\xi - x = lX, \quad \eta - y = lY, \quad \zeta - z = lZ,$$

where l is independent of p and q; thus the equations to be associated with them are

$$\begin{aligned} x_1 + lX_1 &= \kappa (x_2 + lX_2), \\ y_1 + lY_1 &= \kappa (y_2 + lY_2), \\ z_1 + lZ_1 &= \kappa (z_2 + lZ_2). \end{aligned}$$

Multiplying by X_1, Y_1, Z_1 and adding, we have

$$a + le = \kappa (b + lf);$$

multiplying by X_2, Y_2, Z_2, and adding, we have

$$b' + lf = \kappa (c + lg);$$

and therefore the positions of the foci on the ray are given by the equation

$$\begin{vmatrix} a + le, & b + lf \\ b' + lf, & c + lg \end{vmatrix} = 0,$$

that is,

$$(eg - f^2) l^2 + \{ec - f(b + b') + ga\} l + ac - bb' = 0,$$

the roots of which are their distances along the ray from the director surface. Thus, in general, there are two foci; let their distances from the director surface be l_1 and l_2.

Then

$$l_1 + l_2 = \frac{ec - f(b + b') + ga}{eg - f^2}, \quad l_1 l_2 = \frac{ac - bb'}{eg - f^2};$$

and, from the equation for the limits, we have

$$t_1 + t_2 = \frac{ec - f(b+b') + ga}{eg - f^2}, \quad t_1 t_2 = \frac{ac - \frac{1}{4}(b+b')^2}{eg - f^2};$$

hence

$$t_1 + t_2 = l_1 + l_2,$$

$$(t_1 - t_2)^2 - (l_1 - l_2)^2 = \frac{(b-b')^2}{eg - f^2}.$$

Thus the point midway between the foci is midway between the limits; and when the foci are real, they lie between the limits.

When the ray was referred to its principal planes, we had

$$t = t_1 \cos^2 \omega + t_2 \sin^2 \omega,$$

so that in passing along the ray between the two limits, the magnitude of ω varies from 0 to $\frac{1}{2}\pi$. Let its values for l_1 and l_2 be ω_1 and ω_2; then

$$l_1 = t_1 \cos^2 \omega_1 + t_2 \sin^2 \omega_1, \quad l_2 = t_1 \cos^2 \omega_2 + t_2 \sin^2 \omega_2,$$

and

$$l_1 + l_2 = t_1 + t_2.$$

Hence

$$\cos^2 \omega_1 + \cos^2 \omega_2 = 1;$$

and therefore, as ω_1 and ω_2 are not negative and as neither of them is greater than $\frac{1}{2}\pi$, we have

$$\omega_1 + \omega_2 = \tfrac{1}{2}\pi.$$

The planes, through the ray and the two directions determined by these angles ω_1 and ω_2 associated with the foci, are called the *focal planes*. They are not perpendicular to one another; but, because $\omega_1 + \omega_2 = \frac{1}{2}\pi$, it follows that the plane through the ray bisecting the angle between the focal planes bisects also the angle between the principal planes.

It is natural to consider which rays (if any) meet one another. The shortest distance dn between the two rays (p, q) and $(p + dp, q + dq)$ is given by the equation

$$v\,d\theta\,dn = \begin{vmatrix} e\,dp + f\,dq, & f\,dp + g\,dq \\ a\,dp + b\,dq, & b'\,dp + c\,dq \end{vmatrix};$$

and therefore the two rays will meet if $dp : dq$ satisfies the equation

$$\begin{vmatrix} e\,dp + f\,dq, & f\,dp + g\,dq \\ a\,dp + b\,dq, & b'\,dp + c\,dq \end{vmatrix} = 0.$$

Hence there are two rays, which are consecutive to a given ray and intersect it; and therefore two of the surfaces of a rectilinear congruence are developable surfaces.

Moreover, the intersections of the two rays with a given ray are the foci of the latter. For the intersections are on the edge of regression of the developables, which is given by

$$0 = dx + l dX + X dl,$$

that is,

$$0 = (x_1 + lX_1)\, dp + (x_2 + lX_2)\, dq + X dl;$$

and similarly

$$0 = (y_1 + lY_1)\, dp + (y_2 + lY_2)\, dq + Y dl,$$

$$0 = (z_1 + lZ_1)\, dp + (z_2 + lZ_2)\, dq + Z dl.$$

Multiplying first by X_1, Y_1, Z_1 and adding, and next by X_2, Y_2, Z_2 and adding, we have

$$(a + le)\, dp + (b + lf)\, dq = 0,$$

$$(b' + lf)\, dp + (c + lg)\, dq = 0,$$

and therefore

$$\begin{vmatrix} a + le, & b + lf \\ b' + lf, & c + lg \end{vmatrix} = 0,$$

being the equations for the focal distances.

Clearly the focal planes of a ray are the tangent planes to the developable surfaces of the congruence that contain the ray; and it is easy to shew that, if Ω be the angle between the focal planes through the ray,

$$\sin \Omega = \frac{l_1 - l_2}{t_1 - t_2}.$$

Thus on any ray in the congruence there are five special points, viz., the two limits, the two foci, and the middle point (that is, the point midway between the limits and midway between the foci).

When we take all the rays in the congruence, each of these points generates a surface as its locus; and so we have the limit surfaces, the focal surfaces, and the *middle surface*. The two limit surfaces are two sheets of one and the same surface; it is called the *limit surface*. The two focal surfaces are two sheets of one and the same surface; it is called the *focal surface*. Each ray touches the two sheets of the focal surface at the respective foci; the focal planes of the ray are tangent planes to the focal surface at the foci.

Normal rectilinear congruences.

277. Among rectilinear congruences, which are defined by the equations

$$\xi = x + lX, \quad \eta = y + lY, \quad \zeta = z + lZ,$$

special interest attaches (through various physical theories) to those which are capable of orthogonal intersection by a surface and therefore (excluding exceptional cases such as those indicated in § 272) by a family of surfaces. Such congruences are called *normal*.

If the property is possessed by the rectilinear congruence, then there must be variations of ξ, η, ζ representing directions perpendicular to X, Y, Z; and these, accordingly, are such that

$$X d\xi + Y d\eta + Z d\zeta = 0.$$

When the values of ξ, η, ζ are substituted, this relation becomes

$$X dx + Y dy + Z dz + dl = 0,$$

so that $X dx + Y dy + Z dz$ is a perfect differential as it stands. The analysis is reversible; and therefore *it is necessary and sufficient, that the quantity $X dx + Y dy + Z dz$ should be a perfect differential, in order to secure that the rectilinear congruence should be normal*—a theorem due to Hamilton.

Much analytical and geometrical simplification in the general formulæ arises, when we deal with these special rectilinear congruences. We have

$$-dl = X dx + Y dy + Z dz,$$

while X, Y, Z, x, y, z are functions of p and q; hence, writing

$$P = \Sigma X x_1, \quad Q = \Sigma X x_2,$$

we have

$$-dl = P dp + Q dq.$$

The right-hand side must be a perfect differential: thus

$$\frac{\partial P}{\partial q} = \frac{\partial Q}{\partial p}, \quad = -\frac{\partial^2 l}{\partial p \, \partial q};$$

and so

$$\Sigma X_2 x_1 = \Sigma x_1 X_2.$$

Thus

$$b = b';$$

and therefore (§ 276)

$$t_1 - t_2 = l_1 - l_2.$$

Consequently, as

$$t_1 + t_2 = l_1 + l_2,$$

for all congruences, we have

$$t_1 = l_1, \quad t_2 = l_2$$

for a normal congruence.

Again, all the analysis is reversible. It follows therefore that, for a normal congruence, the focal surface is also the limit surface; and the focal planes, becoming the principal planes, are perpendicular to one another.

Further, as we have

$$-dl = P dp + Q dq,$$

the right-hand side being a perfect differential, the integral determines l save as to an additive constant, which is arbitrary; hence a normal congruence of rays is cut orthogonally by a family of surfaces—a result to be expected (after § 274), since the surface given by a definite value of l does not arise through any singular condition.

The simplest example of a normal congruence occurs when it is composed of the aggregate of normals to a surface. The foci are the centres of principal curvature; and the focal surface is the centro-surface of the original surface.

278. One of the most interesting theorems relating to normal rectilinear congruences is connected with a system of rays, subjected to any number of reflections and refractions, viz.:—*the system, once normal, remains normal throughout**. To establish the theorem, consider the effect of any refracting or reflecting surface on the system. We take the surface as the director surface; x, y, z is any point upon it; we denote by X, Y, Z the direction-cosines of the incident ray, by X', Y', Z' the direction-cosines of the refracted (or reflected) ray, and by X'', Y'', Z'' the direction-cosines of the normal to the surface at x, y, z. Then as the incident ray, the refracted (or reflected) ray, and the normal to the surface, lie in one plane, we have

$$\begin{vmatrix} X'', & Y'', & Z'' \\ X\ , & Y\ , & Z \\ X', & Y', & Z' \end{vmatrix} = 0,$$

and therefore quantities λ and μ exist such that

$$\left.\begin{aligned} X &= \lambda X'' + \mu X' \\ Y &= \lambda Y'' + \mu Y' \\ Z &= \lambda Z'' + \mu Z' \end{aligned}\right\}.$$

Consequently

$$\begin{aligned} YZ'' - ZY'' &= \mu(Y'Z'' - Z'Y''), \\ ZX'' - XZ'' &= \mu(Z'X'' - X'Z''), \\ XY'' - YX'' &= \mu(X'Y'' - Y'X''), \end{aligned}$$

and therefore

$$\{(YZ'' - ZY'')^2 + (ZX'' - XZ'')^2 + (XY'' - YX'')^2\}^{\frac{1}{2}}$$
$$= \mu\,\{(Y'Z'' - Z'Y'')^2 + (Z'X'' - X'Z'')^2 + (X'Y'' - Y'X'')^2\}^{\frac{1}{2}}.$$

The left-hand side is the sine of the angle between the incident ray and the normal to the surface; the radical on the right-hand side is the sine of the angle between the normal to the surface and the emerging ray; hence μ is the constant index when there is refraction, and is -1 when there is reflexion—in either case, μ is a constant.

Now for variations along the director surface, we have

$$X''dx + Y''dy + Z''dz = 0;$$

and therefore

$$X dx + Y dy + Z dz = \mu(X'dx + Y'dy + Z'dz).$$

The quantity $X dx + Y dy + Z dz$ is a perfect differential, because the incident system can be cut orthogonally by a family of surfaces; and μ is a constant.

* The theorem usually is connected with the names of Malus and Dupin; see Darboux, t. ii, pp. 280, 281.

Hence $X'dx + Y'dy + Z'dz$ is a perfect differential; that is, the emerging system can be cut orthogonally by a family of surfaces. This result happens at every refracting or reflecting surface; and so a system of rays, if normal, remains normal after any number of refractions and reflexions.

It is easy to deduce the property that, along any ray in a heterogeneous medium, the value of $\int \mu ds$ between two points of its course is less than the value of the same integral along any other path between the same two points.

279. In connection with rectilinear congruences, it was shewn that a transformation of the variables so as to make $f = 0$ and $b + b' = 0$ is always possible; and such a transformation is possible in an infinite number of ways, if

$$a : b + b' : c = e : 2f : g.$$

For a congruence of this type, (called *isotropic*), we have

$$t_1 = t_2 = t,$$

so that the limits of a ray coincide and its foci are imaginary. The feet of the shortest distances, between the ray and consecutive rays, coincide in the point which is the single limit; and all these shortest distances lie in the plane through the single limit. The two limits coincide with the middle point; and the two limit surfaces (or principal surfaces) coincide. This single surface can be called the middle surface of the isotropic congruence; it is the envelope of the plane through the middle point perpendicular to the normal, as well as the locus of the middle point*.

When we have any rectilinear congruence, we have ruled surfaces in the congruence. For those sets of two which correspond to the variables of the principal planes of the ray, the lines of striction coincide with the loci of the limits. For any ruled surface in an isotropic congruence, the line of striction coincides with the locus of the middle point; and so the middle surface of an isotropic congruence contains all the lines of striction of all the ruled surfaces in the congruence.

Now choose as the parameters of reference the parameters of the nul lines in the spherical representation; we have, as usual,

$$X = \frac{u + v}{1 + uv}, \quad Y = i\,\frac{v - u}{1 + uv}, \quad Z = \frac{uv - 1}{1 + uv},$$

so that

$$e = 0, \quad f = \frac{2}{(1 + uv)^2}, \quad g = 0.$$

Our congruence is to be isotropic; hence

$$a = 0, \quad c = 0,$$

* A middle surface, taken as the envelope of the plane through the middle point of a ray normal to the ray in any rectilinear congruence, also may be considered, in addition to the middle surface in § 276; it is of direct importance in the case of isotropic congruences.

and the position of the middle point of the ray is given by

$$t=\frac{b+b'}{2f}.$$

The director surface is at our disposal; let it be chosen so as to be the unique middle surface of the congruence. Then we always have $t=0$, that is, $b+b'=0$; hence, with the middle surface of the isotropic congruence as its director surface, we have

$$a=0, \quad b+b'=0, \quad c=0.$$

It at once follows that

$$dxdX+dydY+dzdZ=0;$$

and therefore any arc on the middle surface is orthogonal to the corresponding arc in the spherical representation. But the ray is normal to its middle surface; and so the spherical representation of the congruence is a spherical representation of the middle surface. As corresponding arcs on the middle surface and the sphere are always orthogonal to one another, the spherical representation is also conformal; and therefore (§ 169) it is possible, though not certain from this property, that the middle surface is a minimal surface.

280. The property that

$$dxdX+dydY+dzdZ=0,$$

for the middle surface of an isotropic congruence, also suggests an association with Weingarten's method for considering the deformation of surfaces and specially the infinitesimal deformation of surfaces; but the consequences will not be developed here.

The theorem, that the middle surface of an isotropic congruence actually is a minimal surface, is due to Ribaucour*; it can be established as follows. We denote by E, F, G, L, M, N, as usual, the fundamental magnitudes for the middle surface.

Because the congruence is isotropic and because we are dealing with the middle surface, we have

$$a=0, \quad b+b'=0, \quad c=0,$$

and therefore

$$\begin{aligned} x_1X_1+y_1Y_1+z_1Z_1&=0,\\ x_1X_2+y_1Y_2+z_1Z_2&=-\rho,\\ x_2X_1+y_2Y_1+z_2Z_1&=\rho,\\ x_2X_2+y_2Y_2+z_2Z_2&=0.\end{aligned}$$

Also

$$X_1^2+Y_1^2+Z_1^2=0, \qquad X_2^2+Y_2^2+Z_2^2=0,$$

$$X_1X_2+Y_1Y_2+Z_1Z_2=f.$$

* *Mém. Acad. Roy. Belg.*, t. xliv (1882), pp. 1—236.

Combining the first of the above relations with

$$XX_1 + YY_1 + ZZ_1 = 0,$$

we have

$$X_1 : Y_1 : Z_1 = Yz_1 - Zy_1 : Zx_1 - Xz_1 : Xy_1 - Yx_1.$$

Hence

$$(Yz_1 - Zy_1)^2 + (Zx_1 - Xz_1)^2 + (Xy_1 - Yx_1)^2 = 0;$$

and so, as

$$Xx_1 + Yy_1 + Zz_1 = 0$$

because the ray is normal to the middle surface, we have

$$E = x_1^2 + y_1^2 + z_1^2 = 0.$$

Similarly, combining the fourth with

$$XX_2 + YY_2 + ZZ_2 = 0,$$

we have

$$G = x_2^2 + y_2^2 + z_2^2 = 0.$$

Again, resolving the equations

$$\begin{aligned} x_1 X_1 + y_1 Y_1 + z_1 Z_1 &= 0, \\ x_1^2 \quad + y_1^2 \quad + z_1^2 \quad &= 0, \\ x_1 X_2 + y_1 Y_2 + z_1 Z_2 &= -\rho, \end{aligned}$$

we have

$$\frac{x_1}{X_1} = \frac{y_1}{Y_1} = \frac{z_1}{Z_1} = -\frac{\rho}{f} = \lambda.$$

Similarly, from

$$\begin{aligned} x_2 X_2 + y_2 Y_2 + z_2 Z_2 &= 0, \\ x_2^2 \quad + y_2^2 \quad + z_2^2 \quad &= 0, \\ x_2 X_1 + y_2 Y_1 + z_2 Z_1 &= \rho, \end{aligned}$$

we have

$$\frac{x_2}{X_2} = \frac{y_2}{Y_2} = \frac{z_2}{Z_2} = \frac{\rho}{f} = -\lambda.$$

Manifestly

$$-\lambda^2 = \frac{f}{F}.$$

Thus

$$\begin{aligned} x_1 &= \lambda X_1, & y_1 &= \lambda Y_1, & z_1 &= \lambda Z_1, \\ x_2 &= -\lambda X_2, & y_2 &= -\lambda Y_2, & z_2 &= -\lambda Z_2. \end{aligned}$$

Hence

$$x_{12} = \lambda X_{12} + \lambda_2 X_1, \quad x_{12} = -\lambda X_{12} - \lambda_1 X_2;$$

and therefore, by addition,

$$2x_{12} = \lambda_2 X_1 - \lambda_1 X_2.$$

Similarly

$$\begin{aligned} 2y_{12} &= \lambda_2 Y_1 - \lambda_1 Y_2, \\ 2z_{12} &= \lambda_2 Z_1 - \lambda_1 Z_2. \end{aligned}$$

Multiplying by X, Y, Z and adding, we have

$$M = Xx_{12} + Yy_{12} + Zz_{12} = 0.$$

Consequently, as $E=0$, $M=0$, $G=0$, we have

$$EN - 2FM + GL = 0;$$

and so the middle surface of the isotropic congruence is a minimal surface*.

Ex. The extremities of a straight line, the length of which is constant and the direction of which depends upon two parameters, are made to describe two surfaces applicable to one another; shew that the middle point of the line generates an isotropic congruence.

Congruences of circles.

281. Another set of congruences of considerable importance is constituted by those congruences which are composed of circles. When all the circles in a congruence can be cut orthogonally by a family of surfaces, the congruence is said to be *normal*; and it usually is called a *cyclical system*. The elements of the theory of cyclical systems, the initiation of which is due to Ribaucour, can be stated in a form somewhat similar to that adopted for rectilinear congruences.

Any circle in space is given by two equations

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2,$$

$$X(x-a)+Y(y-b)+Z(z-c)=0,$$

where

$$X^2+Y^2+Z^2=1.$$

When a, b, c, r, X, Y, Z are functions of two parameters, we have a congruence of circles, by allowing unlimited variations to the parameters. Any point on the circle is given by the equations

$$x=a+lr, \quad y=b+mr, \quad z=c+nr,$$

where

$$l^2+m^2+n^2=1,$$

$$Xl+Ym+Zn=0;$$

and l, m, n are functions of p and q, as well as of a current variable along the circumference, say the arc $r\theta$ measured from a fixed point. Let the radius through this fixed point have direction-cosines λ, μ, ν, and let a perpendicular radius have direction-cosines λ', μ', ν'; then

$$l\lambda + m\mu + n\nu = \cos\theta,$$

$$l\lambda' + m\mu' + n\nu' = \sin\theta,$$

where

$$\lambda^2+\mu^2+\nu^2=1, \quad \lambda\lambda'+\mu\mu'+\nu\nu'=0, \quad \lambda'^2+\mu'^2+\nu'^2=1.$$

Hence

$$\left.\begin{aligned} l &= \lambda\cos\theta+\lambda'\sin\theta \\ m &= \mu\cos\theta+\mu'\sin\theta \\ n &= \nu\cos\theta+\nu'\sin\theta \end{aligned}\right\},$$

$$X=\mu\nu'-\mu'\nu, \quad Y=\nu\lambda'-\nu'\lambda, \quad Z=\lambda\mu'-\lambda'\mu.$$

* For further developments, see Darboux, t. ii, § 260.

Thus the point on the circle is given by

$$\left.\begin{aligned} x &= a + r(\lambda \cos\theta + \lambda' \sin\theta) \\ y &= b + r(\mu \cos\theta + \mu' \sin\theta) \\ z &= c + r(\nu \cos\theta + \nu' \sin\theta) \end{aligned}\right\},$$

where

$$\Sigma\lambda^2 = 1, \quad \Sigma\lambda\lambda' = 0, \quad \Sigma\lambda'^2 = 1;$$

and the quantities $a, b, c, r, \lambda, \mu, \nu, \lambda', \mu', \nu'$ are functions of the two parameters of the congruence, viz. p and q, while θ is the current variable along the circle.

Both sets of equations will be used, as may be found convenient. For the purposes of the analysis, derivatives of a with regard to p and q will be denoted by a_1 and a_2, and so for other magnitudes. The derivative of l with regard to θ will be denoted by l_3, and so for m and n; these quantities, and their derivatives, alone involve θ, in addition to p and q.

Certain combinations of the derivatives are occasionally useful, particularly the combinations connected with the variations with respect to p and q. We take

$$\left.\begin{array}{cccc} \Sigma\lambda_1^2 = e, & \Sigma\lambda_1\lambda_1' = e'' & , & \Sigma\lambda_1'^2 = e' \\ \Sigma\lambda_1\lambda_2 = f, & \Sigma\lambda_1\lambda_2' = \phi, & \Sigma\lambda_1'\lambda_2 = \psi, & \Sigma\lambda_1'\lambda_2' = f' \\ \Sigma\lambda_2^2 = g, & \Sigma\lambda_2\lambda_2' = g'' & , & \Sigma\lambda_2'^2 = g' \end{array}\right\},$$

where the summation is for the cyclical interchange of λ, μ, ν among one another, and for the simultaneous cyclical interchange of λ', μ', ν' among one another. We take

$$\left.\begin{array}{ll} \lambda'\lambda_1 + \mu'\mu_1 + \nu'\nu_1 = -t, & \lambda\lambda_1' + \mu\mu_1' + \nu\nu_1' = t \\ \lambda'\lambda_2 + \mu'\mu_2 + \nu'\nu_2 = -t', & \lambda\lambda_2' + \mu\mu_2' + \nu\nu_2' = t' \end{array}\right\};$$

and then we easily find

$$\left.\begin{aligned} \lambda_1 &= -t\lambda' + (e - t^2)^{\frac{1}{2}} X \\ \mu_1 &= -t\mu' + (e - t^2)^{\frac{1}{2}} Y \\ \nu_1 &= -t\nu' + (e - t^2)^{\frac{1}{2}} Z \end{aligned}\right\}, \quad \left.\begin{aligned} \lambda_2 &= -t'\lambda' + (g - t'^2)^{\frac{1}{2}} X \\ \mu_2 &= -t'\mu' + (g - t'^2)^{\frac{1}{2}} Y \\ \nu_2 &= -t'\nu' + (g - t'^2)^{\frac{1}{2}} Z \end{aligned}\right\},$$

$$\left.\begin{aligned} \lambda_1' &= t\lambda - (e' - t^2)^{\frac{1}{2}} X \\ \mu_1' &= t\mu - (e' - t^2)^{\frac{1}{2}} Y \\ \nu_1' &= t\nu - (e' - t^2)^{\frac{1}{2}} Z \end{aligned}\right\}, \quad \left.\begin{aligned} \lambda_2' &= t'\lambda - (g' - t'^2)^{\frac{1}{2}} X \\ \mu_2' &= t'\mu - (g' - t'^2)^{\frac{1}{2}} Y \\ \nu_2' &= t'\nu - (g' - t'^2)^{\frac{1}{2}} Z \end{aligned}\right\}.$$

Thus

$$\left.\begin{aligned} e'' &= \lambda_1\lambda_1' + \mu_1\mu_1' + \nu_1\nu_1' = -(e - t^2)^{\frac{1}{2}}(e' - t^2)^{\frac{1}{2}} \\ f &= \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = (e - t^2)^{\frac{1}{2}}(g - t'^2)^{\frac{1}{2}} + tt' \\ \phi &= \lambda_1\lambda_2' + \mu_1\mu_2' + \nu_1\nu_2' = -(e - t^2)^{\frac{1}{2}}(g' - t'^2)^{\frac{1}{2}} \\ \psi &= \lambda_1'\lambda_2 + \mu_1'\mu_2 + \nu_1'\nu_2 = -(e' - t^2)^{\frac{1}{2}}(g - t'^2)^{\frac{1}{2}} \\ f' &= \lambda_1'\lambda_2' + \mu_1'\mu_2' + \nu_1'\nu_2' = (e' - t^2)^{\frac{1}{2}}(g' - t'^2)^{\frac{1}{2}} + tt' \\ g'' &= \lambda_2\lambda_2' + \mu_2\mu_2' + \nu_2\nu_2' = -(g - t'^2)^{\frac{1}{2}}(g' - t'^2)^{\frac{1}{2}} \end{aligned}\right\};$$

and

$$\left.\begin{aligned}\Sigma l_1^2 &= e\cos^2\theta + 2e''\cos\theta\sin\theta + e'\sin^2\theta\\ \Sigma l_1 l_2 &= f\cos^2\theta + (\phi + \psi)\cos\theta\sin\theta + f'\sin^2\theta\\ \Sigma l_2^2 &= g\cos^2\theta + 2g''\sin\theta\cos\theta + g'\sin^2\theta\end{aligned}\right\}.$$

Also

$$\left.\begin{aligned}X_1 &= \lambda'(e'-t^2)^{\frac{1}{2}} - \lambda(e-t^2)^{\frac{1}{2}}\\ Y_1 &= \mu'(e'-t^2)^{\frac{1}{2}} - \mu(e-t^2)^{\frac{1}{2}}\\ Z_1 &= \nu'(e'-t^2)^{\frac{1}{2}} - \nu(e-t^2)^{\frac{1}{2}}\end{aligned}\right\},\quad \left.\begin{aligned}X_2 &= \lambda'(g'-t'^2)^{\frac{1}{2}} - \lambda(g-t'^2)^{\frac{1}{2}}\\ Y_2 &= \mu'(g'-t'^2)^{\frac{1}{2}} - \mu(g-t'^2)^{\frac{1}{2}}\\ Z_2 &= \nu'(g'-t'^2)^{\frac{1}{2}} - \nu(g-t'^2)^{\frac{1}{2}}\end{aligned}\right\}.$$

282. According to the general theory, the foci that lie upon any curve of a congruence

$$f(x,\ y,\ z,\ p,\ q) = 0,\quad g(x,\ y,\ z,\ p,\ q) = 0,$$

are given by combining these equations with the equation

$$\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} = 0.$$

Consequently, the foci of a circle of the congruence

$$f = (x-a)^2 + (y-b)^2 + (z-c)^2 - r^2 = 0,$$
$$g = X(x-a) + Y(y-b) + Z(z-c) = 0,$$

are its intersections with the surface

$$\begin{vmatrix} rr_1 + (x-a)a_1 + (y-b)b_1 + (z-c)c_1, & rr_2 + (x-a)a_2 + (y-b)b_2 + (z-c)c_2\\ \Sigma(x-a)X_1 - \Sigma Xa_1 \quad , & \Sigma(x-a)X_2 - \Sigma Xa_2\end{vmatrix} = 0.$$

As this equation is of the second order, the number of sets of values of x, y, z satisfying the three equations is equal to four (§ 270); but if a, b, c are constant (so that all the circles of the congruence pass through a fixed point), the new equation is only of the first order, and so the number of sets of solutions x, y, z of the three equations is only two.

When the alternative form of the equations of the circle is used, the third equation becomes

$$\begin{vmatrix} r_1 + \Sigma la_1 \quad , & r_2 + \Sigma la_2\\ -\Sigma Xa_1 + r\Sigma lX_1, & -\Sigma Xa_2 + r\Sigma lX_2\end{vmatrix} = 0,$$

in the general case; in the special case when the fixed point is common to all the circles of the congruence, the equations for the foci are

$$x^2 + y^2 + z^2 = r^2,$$
$$Xx + Yy + Zz = 0,$$
$$r_1\Sigma lX_2 - r_2\Sigma lX_1 = 0.$$

We shall deal only with the general case. Let

$$\Sigma\lambda a_1 = \alpha,\quad \Sigma\lambda a_2 = \beta,\quad \Sigma\lambda X_1 = \gamma,\quad \Sigma\lambda X_2 = \delta,$$
$$\Sigma\lambda' a_1 = \alpha',\quad \Sigma\lambda' a_2 = \beta',\quad \Sigma\lambda' X_1 = \gamma',\quad \Sigma\lambda' X_2 = \delta',$$

so that

$$\gamma = -(e-t^2)^{\frac{1}{2}}, \quad \gamma' = -(e'-t^2)^{\frac{1}{2}},$$
$$\delta = -(g-t'^2)^{\frac{1}{2}}, \quad \delta' = -(g'-t'^2)^{\frac{1}{2}};$$

also, let

$$\Sigma X a_1 = \rho, \quad \Sigma X a_2 = \sigma.$$

Then the equation giving the foci of the circle in question is

$$\begin{vmatrix} r_1 + \alpha\cos\theta + \alpha'\sin\theta & , & r_2 + \beta\cos\theta + \beta'\sin\theta \\ -\rho + r(\gamma\cos\theta + \gamma'\sin\theta), & & -\sigma + r(\delta\cos\theta + \delta'\sin\theta) \end{vmatrix} = 0,$$

an equation manifestly giving the values of θ which settle the foci of the circle. The equation leads to four values of θ in general, so that there are four foci upon a circle in a general congruence; when the circles all pass through a fixed point, α, α', β, β' vanish, and then the equation leads to two values of θ determining two foci upon each such circle.

The equations of the focal surface of the congruence of circles are

$$x = a + r(\lambda\cos\theta + \lambda'\sin\theta),$$
$$y = b + r(\mu\cos\theta + \mu'\sin\theta),$$
$$z = c + r(\nu\cos\theta + \nu'\sin\theta),$$

together with the above determinantal equation; its explicit equation results from eliminating p, q, and θ, among the four equations. Manifestly, the focal surface is four-sheeted.

283. We know that, in rectilinear congruences, certain selected consecutive rays intersect one another; the points of intersection of any ray with the (two) different rays, which meet it, are the foci of the ray. It is natural to enquire which circles (if any), consecutive to a given circle, do intersect it; we easily find that there are four such consecutive circles, and that each of the four points of intersection is a focus for the circle.

The proof is simple. Any circle

$$x = a + lr, \quad y = b + mr, \quad z = c + nr,$$

intersects a consecutive circle if

$$0 = da + l\,dr + r\,dl,$$
$$0 = db + m\,dr + r\,dm,$$
$$0 = dc + n\,dr + r\,dn,$$

that is, if the equations

$$\left.\begin{aligned} (a_1 + lr_1 + rl_1)\,dp + (a_2 + lr_2 + rl_2)\,dq + rl_3\,d\theta = 0 \\ (b_1 + mr_1 + rm_1)\,dp + (b_2 + mr_2 + rm_2)\,dq + rm_3\,d\theta = 0 \\ (c_1 + nr_1 + rn_1)\,dp + (c_2 + nr_2 + rn_2)\,dq + rn_3\,d\theta = 0 \end{aligned}\right\}$$

are satisfied at the point. Multiplying the equations by l, m, n, adding, and remembering that

$$l^2 + m^2 + n^2 = 1,$$

we have

$$(r_1 + \Sigma la_1)\,dp + (r_2 + \Sigma la_2)\,dq = 0.$$

Multiplying the equations by $-X$, $-Y$, $-Z$, adding, and remembering that

$$Xl + Ym + Zn = 0,$$

so that

$$\Sigma Xl_1 = -\Sigma lX_1, \quad \Sigma Xl_2 = -\Sigma lX_2, \quad \Sigma Xl_3 = 0,$$

we have

$$(-\Sigma Xa_1 + r\Sigma lX_1)\,dp + (-\Sigma Xa_2 + r\Sigma lX_2)\,dq = 0.$$

Eliminating $dp : dq$, we have

$$\begin{vmatrix} r_1 + \Sigma la_1, & r_2 + \Sigma la_2 \\ -\Sigma Xa_1 + r\Sigma lX_1, & -\Sigma Xa_2 + r\Sigma lX_2 \end{vmatrix} = 0,$$

which is the equation giving the four foci of the circle. Hence the intersections of the circle with consecutive circles are its four foci.

The circles, that are consecutive and intersect, are determined by the quantities $p + dp$, $q + dq$; and the point common with the consecutive circle is given by the value $\theta + d\theta$ on that consecutive circle. Now our two equations are

$$r_1 dp + r_2 dq + \quad (\alpha dp + \beta dq) \cos\theta + \quad (\alpha' dp + \beta' dq) \sin\theta = 0,$$

$$-\rho dp - \sigma dq + r(\gamma dp + \delta dq) \cos\theta + r(\gamma' dp + \delta' dq) \sin\theta = 0;$$

hence

$$r^2 \begin{vmatrix} \alpha dp + \beta dq, & \alpha' dp + \beta' dq \\ \gamma dp + \delta dq, & \gamma' dp + \delta' dq \end{vmatrix}^2$$

$$= \begin{vmatrix} \alpha dp + \beta dq, & rr_1 dp + rr_2 dq \\ \gamma dp + \delta dq, & -\rho dp + \sigma dq \end{vmatrix}^2 + \begin{vmatrix} \alpha' dp + \beta' dq, & rr_1 dp + rr_2 dq \\ \gamma' dp + \delta' dq, & -\rho dp - \sigma dq \end{vmatrix}^2.$$

This is an equation of the fourth order in $dp : dq$; its coefficients are functions of p and q only; and therefore it determines four consecutive values of p and q, which give consecutive values of a, b, c, r, X, Y, Z, and therefore give four consecutive intersecting circles. And then the value of $d\theta$ is given by the expression

$$r\,d\theta = (t + \alpha \sin\theta - \alpha' \cos\theta)\,dp + (t' + \beta \sin\theta - \beta' \cos\theta)\,dq,$$

for each of the ratios of $dp : dq$.

284. These results are general; and they belong to any congruence of circles, without any limitations upon the congruence. Every circle meets four other consecutive circles; it intersects each of the circles in a single point, the four points being the foci of the circle.

Now the greatest number of points in which two circles can intersect is two; and so it is conceivable that a congruence of circles may be such as to allow two of the four foci of a circle to lie on one consecutive circle, and the other two to lie on another consecutive circle. In that event, there will be

two (and not four) values of $dp:dq$, which give consecutive intersecting circles; for each of these two values of $dp:dq$, there will be two values of θ, giving the two points upon the consecutive circle which are foci of the first. In order that this distribution of the foci may be possible, the two equations

$$dr + \Sigma\lambda da \cos\theta + \Sigma\lambda' da \sin\theta = 0,$$

$$-\Sigma X da + r\Sigma\lambda dX \cos\theta + r\Sigma\lambda' dX \sin\theta = 0,$$

cannot be independent; for, if independent, they would determine $\cos\theta$ and $\sin\theta$ uniquely for an assigned value of $dp:dq$, and so there would be only a single focus on the consecutive circle. Accordingly, we must have

$$\frac{-\Sigma X da}{rdr} = \frac{\Sigma\lambda dX}{\Sigma\lambda da} = \frac{\Sigma\lambda' dX}{\Sigma\lambda' da};$$

and these equations are to determine two values of $dp:dq$. Let μ denote either of these values of $dp:dq$; then our equations are

$$\frac{-\rho\mu - \sigma}{rr_1\mu + rr_2} = \frac{\gamma\mu + \delta}{\alpha\mu + \beta} = \frac{\gamma'\mu + \delta'}{\alpha'\mu + \beta'}.$$

The condition, that only two values of μ are thus to be provided, requires that the third fraction shall be unconditionally equal to each of the other two; and so quantities I and J must exist such that

$$\left.\begin{aligned} I\gamma + J\gamma' &= -\rho \\ I\delta + J\delta' &= -\sigma \\ I\alpha + J\alpha' &= rr_1 \\ I\beta + J\beta' &= rr_2 \end{aligned}\right\}.$$

Consequently, the two conditions, represented by the equations

$$\left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \\ rr_1, & rr_2, & -\rho, & -\sigma \end{matrix}\right\| = 0,$$

must be satisfied by the magnitudes that occur in the expression of the congruence. The two values of μ are the roots of the quadratic

$$\begin{vmatrix} \alpha\mu + \beta, & \gamma\mu + \delta \\ \alpha'\mu + \beta', & \gamma'\mu + \delta' \end{vmatrix} = 0;$$

and the two values of θ, that belong to a value of μ, are the roots of the equation

$$rr_1\mu + rr_2 + (\alpha\mu + \beta)\cos\theta + (\alpha'\mu + \beta')\sin\theta = 0.$$

The two conditions may be written in the form

$$rr_1(\gamma\delta' - \gamma'\delta) = \rho(\alpha'\delta - \alpha\delta') + \sigma(\alpha\gamma' - \alpha'\gamma),$$

$$rr_2(\gamma\delta' - \gamma'\delta) = \rho(\beta'\delta - \beta\delta') + \sigma(\beta\gamma' - \beta'\gamma);$$

and therefore the quantities a, b, c, X, Y, Z satisfy the single condition

$$\frac{\partial}{\partial q}\left\{\frac{\rho(\alpha'\delta-\alpha\delta')+\sigma(\alpha\gamma'-\alpha'\gamma)}{\gamma\delta'-\gamma'\delta}\right\}$$
$$=\frac{\partial}{\partial p}\left\{\frac{\rho(\beta'\delta-\beta\delta')+\sigma(\beta\gamma'-\beta'\gamma)}{\gamma\delta'-\gamma'\delta}\right\}.$$

while, when this condition is satisfied, r^2 is given by a quadrature in the form

$$\tfrac{1}{2}r^2=\int\frac{1}{\gamma\delta'-\gamma'\delta}\left[\{\rho(\alpha'\delta-\alpha\delta')+\sigma(\alpha\gamma'-\alpha'\gamma)\}\,dp+\{\rho(\beta'\delta-\beta\delta')+\sigma(\beta\gamma'-\beta'\gamma)\}\,dq\right].$$

The two values of μ are the roots of the quadratic

$$(\alpha\gamma'-\alpha'\gamma)\,\mu^2+(\alpha\delta'-\alpha'\delta+\beta\gamma'-\beta'\gamma)\,\mu+\beta\delta'-\beta'\delta=0;$$

let them be μ_1 and μ_2, and let the primitives of the equations

$$\frac{dq}{dp}=\mu_1,\qquad \frac{dq}{dp}=\mu_2,$$

respectively be

$$\rho=\text{constant},\quad \rho'=\text{constant},$$

so that ρ and ρ' are two independent functions of the two parameters of the congruence.

285. Now let these two quantities ρ and ρ' be taken as the parameters; in other words, we may take p and q to be ρ and ρ'. Then the equations

$$\frac{-\Sigma X\,da}{r\,dr}=\frac{\Sigma\lambda\,dX}{\Sigma\lambda\,da}=\frac{\Sigma\lambda'dX}{\Sigma\lambda'da}$$

are to be satisfied by $dp=0$ and $dq=0$. Let the common value of the fractions be Q when $dp=0$, and be P when $dq=0$; so that Q is a function of q only, and P is a function of p only. Then we have

$$\left.\begin{aligned}\Sigma Xa_1&=-Prr_1\\ \Sigma\lambda X_1&=P\Sigma\lambda a_1\\ \Sigma\lambda'X_1&=P\Sigma\lambda'a_1\end{aligned}\right\},\qquad \left.\begin{aligned}\Sigma Xa_2&=-Qrr_2\\ \Sigma\lambda X_2&=Q\Sigma\lambda a_2\\ \Sigma\lambda'X_2&=Q\Sigma\lambda'a_2\end{aligned}\right\}.$$

From the first set, we at once have

$$\left.\begin{aligned}X_1&=P(a_1-X\Sigma Xa_1)\\ Y_1&=P(b_1-Y\Sigma Xa_1)\\ Z_1&=P(c_1-Z\Sigma Xa_1)\end{aligned}\right\};$$

and from the second set, we have

$$\left.\begin{aligned}X_2&=Q(a_2-X\Sigma Xa_2)\\ Y_2&=Q(b_2-Y\Sigma Xa_2)\\ Z_2&=Q(c_2-Z\Sigma Xa_2)\end{aligned}\right\}.$$

Then

$$\frac{\partial}{\partial p}(X - aP) = -aP_1 - X(P\Sigma Xa_1),$$

$$\frac{\partial}{\partial p}(Y - bP) = -bP_1 - Y(P\Sigma Xa_1),$$

$$\frac{\partial}{\partial p}(Z - cP) = -cP_1 - Z(P\Sigma Xa_1),$$

$$\frac{\partial}{\partial p}\{(\Sigma Xa) - \tfrac{1}{2}(a^2 + b^2 + c^2 - r^2)P\}$$
$$= -\tfrac{1}{2}(a^2 + b^2 + c^2 - r^2)P_1 - (\Sigma Xa)(P\Sigma Xa_1).$$

Now our congruence of circles is given initially by the equations

$$\left.\begin{array}{r} x^2 + y^2 + z^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2 - r^2 = 0 \\ Xx + Yy + Zz - \Sigma Xa = 0 \end{array}\right\};$$

it can therefore be expressed, in an equivalent form, by the equations

$$\left.\begin{array}{l} \tfrac{1}{2}P\,(x^2 + y^2 + z^2) - \{(\Sigma Xa) - \tfrac{1}{2}(a^2 + b^2 + c^2 - r^2)P\} \\ \qquad\qquad + (X - aP)x + (Y - bP)y + (Z - cP)z = 0 \\ \tfrac{1}{2}P_1(x^2 + y^2 + z^2) + (\Sigma Xa)(\Sigma Xa_1) + \tfrac{1}{2}(a^2 + b^2 + c^2 - r^2)P_1 \\ \quad - x\{aP_1 + X(P\Sigma Xa_1)\} - y\{bP_1 + Y(P\Sigma Xa_1)\} - z\{cP_1 + Z(P\Sigma Xa_1)\} = 0 \end{array}\right\}.$$

Comparing these two forms of equation with the preceding relations, that follow from the relations connecting derivatives of X, Y, Z, a, b, c with respect to p, we can express the result in the following form:—

When a congruence of circles is such that each circle, of the set along directions given by a constant value of the parameter q, is intersected in two points by a consecutive circle of the same set, the equations of the congruence can be taken in the form

$$\left.\begin{array}{r} P\,(x^2 + y^2 + z^2) - 2\alpha x\; - 2\beta y\; - 2\gamma z\; + 2\delta\; = 0 \\ P_1(x^2 + y^2 + z^2) - 2\alpha_1 x - 2\beta_1 y - 2\gamma_1 z + 2\delta_1 = 0 \end{array}\right\}.$$

Equations, connecting the derivatives of the quantities P, α, β, γ, δ with respect to q, also exist. Writing

$$\frac{\alpha}{P} = A, \quad \frac{\beta}{P} = B, \quad \frac{\gamma}{P} = C, \quad \frac{\delta}{P} = D,$$

we can take the equations of the congruence in the form

$$\left.\begin{array}{r} x^2 + y^2 + z^2 - 2Ax - 2By - 2Cz + 2D = 0 \\ A_1x + B_1y + C_1z - D_1 = 0 \end{array}\right\};$$

and again there are relations involving the derivatives of A, B, C, D.

A corresponding form could be obtained by using the equations involving X_2, Y_2, Z_2, a_2, b_2, c_2; and its parametric magnitudes would be subject to relations connecting their derivatives with respect to p.

286. Instead of developing the limitations which are imposed by these relations for either of the forms, we shall now assume that the congruence is given by the foregoing pair of equations. Should the congruence be given initially by more general equations, the determination of the appropriate variables p and q can be effected (§ 284) by the integration of two isolated ordinary differential equations, each of the first order; and so no generality is lost by the immediate adoption of the pair of equations, so that we may take

$$a = A, \qquad b = B, \qquad c = C,$$
$$X = \theta A_1, \quad Y = \theta B_1, \quad Z = \theta C_1, \quad \Sigma X a = \theta D_1.$$

Now two of the foci of the circle lie upon another circle given by a consecutive value of p; the equations for these two foci are

$$x^2 + y^2 + z^2 - 2Ax - 2By - 2Cz + 2D = 0,$$
$$A_1 x + B_1 y + C_1 z - D_1 = 0,$$
$$A_{11} x + B_{11} y + C_{11} z - D_{11} = 0,$$

and therefore the equations of the line joining them are

$$\left.\begin{array}{l} A_1 x + B_1 y + C_1 z - D_1 = 0 \\ A_{11} x + B_{11} y + C_{11} z - D_{11} = 0 \end{array}\right\}.$$

As the other two foci are to lie upon a consecutive circle, for which p is unaltered and q has a consecutive value, these other two foci must satisfy the equations

$$x^2 + y^2 + z^2 - 2Ax - 2By - 2Cz + 2D = 0,$$
$$A_1 x + B_1 y + C_1 z - D_1 = 0,$$
$$A_2 x + B_2 y + C_2 z - D_2 = 0,$$
$$A_{12} x + B_{12} y + C_{12} z - D_{12} = 0.$$

The equations are to provide two points; hence the last three equations are not independent, and so two relations are satisfied, viz.

$$\left\| \begin{array}{cccc} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_{12}, & B_{12}, & C_{12}, & D_{12} \end{array} \right\| = 0.$$

These relations can also be expressed in the form

$$A_{12} = \rho A_1 + \sigma A_2,$$
$$B_{12} = \rho B_1 + \sigma B_2,$$
$$C_{12} = \rho C_1 + \sigma C_2,$$
$$D_{12} = \rho D_1 + \sigma D_2,$$

where the quantities ρ and σ can be any functions of p and q. The equations of the line through the two foci are

$$\left.\begin{array}{l} A_1 x + B_1 y + C_1 z - D_1 = 0 \\ A_2 x + B_2 y + C_2 z - D_2 = 0 \end{array}\right\}.$$

Gathering together the various results, we have the theorem*:—

Let four quantities A, B, C, D be taken as solutions of an equation

$$\frac{\partial^2\theta}{\partial p\,\partial q}=\rho\frac{\partial\theta}{\partial p}+\sigma\frac{\partial\theta}{\partial q},$$

where ρ and σ are any functions of p and q; then on each circle of the congruence

$$\left.\begin{aligned}x^2+y^2+z^2-2Ax-2By-2Cz+2D=0\\ x\frac{\partial A}{\partial p}+y\frac{\partial B}{\partial p}+z\frac{\partial C}{\partial p}-\frac{\partial D}{\partial p}=0\end{aligned}\right\},$$

two of the foci lie on one consecutive circle, given by the same value of q and a consecutive value of p, and the other two foci lie on another consecutive circle, given by the same value of p and a consecutive value of q.

The result may also be stated in the form:—

Let any five functions A, B, C, D, E of p and q be chosen from among integrals of an equation

$$M\theta_{11}+N\theta_{12}+P\theta_{22}+M'\theta_1+N'\theta_2+P'=0;$$

also let $m:n$ be chosen so that

$$Mn^2-Nmn+Pm^2=0;$$

then the equations

$$\left.\begin{aligned}A(x^2+y^2+z^2)+Bx+Cy+Dz+E=0\\ \left(m\frac{\partial}{\partial p}+n\frac{\partial}{\partial q}\right)\{A(x^2+y^2+z^2)+Bx+Cy+Dz+E\}=0\end{aligned}\right\}$$

define a congruence of circles of the foregoing type.

287. The surfaces, generated by consecutive circles which intersect, have a relation to the congruence of circles similar to that which is borne to a rectilinear congruence by its developables; and the two-fold locus of the foci of the circles (which, from their equations, manifestly lie upon the envelope of the circles) is a double curve on these surfaces, corresponding to the edge of regression of the developables in the rectilinear congruence. The equations of this two-fold locus for one system of circles are obtained by eliminating p between the equations

$$\left.\begin{aligned}x^2+y^2+z^2-2Ax-2By-2Cz+2D=0\\ A_1x+B_1y+C_1z-D_1=0\\ A_{11}x+B_{11}y+C_{11}z-D_{11}=0\end{aligned}\right\};$$

* Darboux, t. ii, p. 316.

and for the other system of circles through the elimination of q between the equations

$$\left.\begin{aligned} x^2+y^2+z^2-2Ax-2By-2Cz+2D=0 \\ A_1x+B_1y+C_1z-D_1=0 \\ A_{12}x+B_{12}y+C_{12}z-D_{12}=0 \end{aligned}\right\}.$$

Also, the equations of the two surfaces, generated by consecutive circles, are obtained by eliminating p and q respectively between the two equations of the congruence.

288. The equations for obtaining the magnitude, direction, and position of the shortest distance between any two consecutive circles (p, q) and $(p+dp, q+dq)$ of the congruence are as follows. Let its magnitude be $d\tau$ and its direction-cosines L, M, N; then

$$Ll_3 + Mm_3 + Nn_3 = 0,$$

$$Ldl_3 + Mdm_3 + Ndn_3 = 0.$$

Also, let $d\phi$ be the angle between the tangents to the two circles at the feet of the shortest distance, so that

$$d\phi^2 = dl_3^2 + dm_3^2 + dn_3^2;$$

then

$$\left.\begin{aligned} Ld\phi &= m_3dn_3 - n_3dm_3 \\ Md\phi &= n_3dl_3 - l_3dn_3 \\ Nd\phi &= l_3dm_3 - m_3dl_3 \end{aligned}\right\}.$$

Again, by projection on the axes of reference, we have

$$a + lr + Ldt = a + da + (l+dl)(r+dr),$$

that is,

$$Ld\tau = da + rdl + ldr;$$

and similarly

$$Md\tau = db + rdm + mdr,$$

$$Nd\tau = dc + rdn + ndr.$$

We at once have

$$d\phi d\tau = \begin{vmatrix} da + rdl + ldr, & l_3, & dl_3 \\ db + rdm + mdr, & m_3, & dm_3 \\ dc + rdn + ndr, & n_3, & dn_3 \end{vmatrix},$$

which gives an expression for $d\tau$ involving functions of θ and also $d\theta$.

Multiplying the equations for $Ld\tau$, $Md\tau$, $Nd\tau$ by l_3, m_3, n_3, and adding, we have

$$\Sigma l_3da + r\Sigma l_3dl + dr\Sigma ll_3 = 0.$$

Now

$$\Sigma ll_3 = 0, \quad \Sigma l_3dl = d\theta - tdp - t'dq,$$

so that

$$(-\alpha\sin\theta+\alpha'\cos\theta-rt)\,dp+(-\beta\sin\theta+\beta'\cos\theta-rt')\,dq+rd\theta=0,$$

an equation expressing $d\theta$ in terms of dp, dq, and functions of θ.

Multiplying the same three equations by dl_3, dm_3, dn_3, and adding, we have

$$\Sigma da\,dl_3+r\Sigma dl\,dl_3+dr\Sigma l\,dl_3=0.$$

Now

$$\begin{aligned}\Sigma da\,dl_3&=-d\theta\Sigma l\,da-\sin\theta\Sigma da\,d\lambda+\cos\theta\Sigma da\,d\lambda',\\ \Sigma dl\,dl_3&=dp^2\Sigma l_1l_{13}+dp\,dq\,\Sigma(l_1l_{23}+l_2l_{13})+dq^2\Sigma l_2l_{23},\\ \Sigma l\,dl_3&=-d\theta+t\,dp+t'dq.\end{aligned}$$

Inserting these values and substituting from the former equation for $d\theta$, we obtain an equation of the form

$$A\cos 2\theta+B\sin 2\theta+C\cos\theta+D\sin\theta+E=0,$$

where A, B, C, D, E are quadratic functions of the ratio $dp:dq$; they do not involve θ, and they have magnitudes connected with the congruence for their coefficients.

Accordingly, this is the equation for θ. When an appropriate value is found, the earlier equation gives $d\theta$ in terms of dp and dq; and then $d\phi$ is known. We thus have the value of $d\tau$ in terms of dp and dq, and of known magnitudes that do not involve dp or dq.

Cyclical Systems.

289. Just as special importance centres in those rectilinear congruences which can be cut orthogonally by a family of surfaces, so also it is necessary to take particular account of congruences of circles which can be cut orthogonally by a family of surfaces. Such congruences are called *cyclical systems.*

The direction-cosines of the tangent to any circle of a congruence represented by

$$x=a+lr,\quad y=b+mr,\quad z=c+nr,$$

are proportional to l_3, m_3, n_3; hence every direction $dx:dy:dz$ at the point, perpendicular to the tangent to the circle, must satisfy the relation

$$l_3dx+m_3dy+n_3dz=0.$$

If the circles of the congruence can be cut orthogonally, this differential relation must have a single equation as its integral equivalent, the single equation representing of course the family of orthogonal surfaces; and the condition, necessary and sufficient to secure the result, is the customary condition of integrability. When we write

$$\begin{aligned}P&=\Sigma l_3(a_1+rl_1+lr_1)=-\alpha\sin\theta+\alpha'\cos\theta-rt,\\ Q&=\Sigma l_3(a_2+rl_2+lr_2)=-\beta\sin\theta+\beta'\cos\theta-rt',\end{aligned}$$

(with the notation of §§ 281, 282), the foregoing differential relation becomes

$$Pdp + Qdq + rd\theta = 0.$$

The condition of integrability is

$$P\left(\frac{\partial Q}{\partial \theta} - \frac{\partial r}{\partial q}\right) + Q\left(\frac{\partial r}{\partial p} - \frac{\partial P}{\partial \theta}\right) + r\left(\frac{\partial P}{\partial q} - \frac{\partial Q}{\partial p}\right) = 0,$$

which, when the values of P and Q are substituted, becomes

$$T\cos\theta + T'\sin\theta + \Theta = 0,$$

where

$$\left.\begin{aligned} T &= r(t\beta - t'\alpha) + r_1\beta' - r_2\alpha' - r\beta_1' + r\alpha_2' \\ T' &= r(t\beta' - t'\alpha') - r_1\beta + r_2\alpha + r\beta_1 - r\alpha_2 \\ \Theta &= \alpha\beta' - \alpha'\beta \qquad\qquad + r^2t_1' - r^2t_2 \end{aligned}\right\},$$

T, T', Θ manifestly being independent of θ.

The condition of integrability may be satisfied identically, so that

$$T = 0, \quad T' = 0, \quad \Theta = 0.$$

In that case, let

$$\Omega = \text{constant}$$

be the integral equivalent of the differential relation; it is the equation of the family of surfaces cutting the congruence of circles orthogonally.

If the condition is not satisfied identically, it may provide no value of θ, or one value of θ, or two values of θ. In the first case, there is no surface orthogonal to the congruence. In the second case, if (and only if) the value of θ satisfies the differential relation, there is one special surface orthogonal to the congruence. In the third case, if (and only if) one of the values of θ satisfies the differential relation, there is a special surface orthogonal to the congruence; there can, at the utmost, be two special surfaces thus orthogonal to the congruence.

Ex. 1. Consider the congruence of circles, which lie in the tangent planes to a surface and have their centres at the point of contact of the tangent plane with the surface.

Let the surface be referred to the lines of curvature as the parametric curves; then we can take these directions as the axes of reference in the planes of the circles, so that

$$\lambda = a_1 E^{-\frac{1}{2}}, \quad \mu = b_1 E^{-\frac{1}{2}}, \quad \nu = c_1 E^{-\frac{1}{2}},$$

$$\lambda' = a_2 G^{-\frac{1}{2}}, \quad \mu' = b_2 G^{-\frac{1}{2}}, \quad \nu' = c_2 G^{-\frac{1}{2}}.$$

Then

$$\alpha = \Sigma\lambda a_1 = E^{\frac{1}{2}}, \quad \beta = \Sigma\lambda a_2 = 0,$$

$$\alpha' = \Sigma\lambda' a_1 = 0, \qquad \beta' = \Sigma\lambda' a_2 = G^{\frac{1}{2}}.$$

Also

$$\begin{aligned} -t &= \Sigma\lambda'\lambda_1 \\ &= \Sigma a_2 G^{-\frac{1}{2}}(a_{11}E^{-\frac{1}{2}} - \tfrac{1}{2}a_1 E^{-\frac{3}{2}}E_1) \\ &= -\tfrac{1}{2}\frac{E_2}{(EG)^{\frac{1}{2}}}; \end{aligned}$$

and, similarly,

$$t' = -\tfrac{1}{2}\frac{G_1}{(EG)^{\frac{1}{2}}}.$$

Hence

$$\begin{aligned}
T &= r_1 G^{\frac{1}{2}},\\
T' &= r_2 E^{\frac{1}{2}},\\
\Theta &= (EG)^{\frac{1}{2}} - \tfrac{1}{2}\frac{r^2}{(EG)^{\frac{1}{2}}}\left[G_{11} + E_{22} - \frac{1}{2EG}\{G_1(GE_1 + G_1E) + E_2(E_2G + GE_2)\}\right]\\
&= (EG)^{\frac{1}{2}}(1 + r^2K),
\end{aligned}$$

where K is the Gaussian measure of curvature of the surface.

In order that the congruence of circles may be a cyclical system, we are to have

$$T = 0, \quad T' = 0, \quad \Theta = 0.$$

Hence we must have

$$r = \text{constant},$$

$$K = -\frac{1}{r^2};$$

and therefore on a surface, whose Gaussian measure of curvature is constant and equal to $-\dfrac{1}{r^2}$, the congruence of circles of constant radius r, which lie in the tangent planes to the surface and have their centres at the point of contact of the tangent plane with the surface, constitutes a cyclical system.

It is not difficult to prove that the surfaces, orthogonal to these circles, are themselves surfaces of constant negative curvature $-1/r^2$.

Ex. 2. Shew that, when any four of the orthogonal surfaces of a cyclical system are taken, the anharmonic ratio of the four points in which they cut any circle of the system is the same for all the circles.

EXAMPLES.

1. Three surfaces, independent of one another, are given by equations

$$f(x, y, z, p, q) = 0, \quad g(x, y, z, p, q) = 0, \quad h(x, y, z, p, q) = 0;$$

and x_1, x_2 denote the derivatives of x with regard to p, q, and similarly for y and z. Prove that

$$J\left(\frac{f, g, h}{x, y, z}\right)(y_1z_2 - y_2z_1) = J\left(\frac{g, h}{p, q}\right)\frac{\partial f}{\partial x} + J\left(\frac{h, f}{p, q}\right)\frac{\partial g}{\partial x} + J\left(\frac{f, g}{p, q}\right)\frac{\partial h}{\partial x},$$

with two similar relations; and verify that the direction-cosines of the normal to the focal surface of the congruence

$$f = 0, \quad g = 0,$$

at any point are proportional to

$$\kappa\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}, \quad \kappa\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}, \quad \kappa\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}.$$

2. If all the curves of a congruence meet a fixed curve, this fixed curve lies on the focal surface.

3. Shew that, on each sheet of the focal surface of a rectilinear congruence, the curves corresponding to the two sets of developable surfaces of the congruence are a conjugate system. What are these curves?

4. When the two sheets of the focal surface of a rectilinear congruence are the same, the Gaussian measure of curvature of the focal surface at the point where it is touched by the ray is

$$-\frac{1}{(t_1-t_2)^2},$$

with the notation of § 275.

5. In a normal rectilinear congruence, the distance between the foci of a ray is the same for all rays; shew that the two sheets of the focal surface have their Gaussian measure of curvature constant and negative.

6. Shew that, if the rectilinear congruence

$$ay-bx+c'=0, \quad cx-az+b'=0, \quad bz-cy+a'=0$$

is normal, the expression

$$(a^2+b^2+c^2)^{-\frac{3}{2}} \begin{vmatrix} da, & db, & dc \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}$$

must be an exact differential.

7. Prove that the rectilinear tangents to a family of geodesics traced upon a surface constitute a normal congruence.

8. Rays are incident upon a reflecting surface, and the developables of the incident congruence are reflected into the developables of the reflected congruence; shew that they cut the surface in a conjugate system.

9. Shew that, if a congruence of circles normal to two surfaces is such that each of them is met in two points by one of the consecutive circles, the congruence is normal.

10. The envelope of a sphere depending upon a couple of parameters is formed; and the lines of curvature on the two sheets of the envelope correspond. Circles are drawn, each cutting a sphere normally at its two points of contact with its envelope; shew that they form a cyclical system.

11. Shew that with a given rectilinear congruence it is generally possible to associate a unique cyclical system, so that the rays in the former are the axes of the circles in the latter.

12. Shew that, if the planes of all the circles in a cyclical system pass through a fixed point, the circles are cut orthogonally by a sphere having its centre at that point, unless the point is at infinity when the circles are cut orthogonally by a plane.

13. Prove that the family of surfaces, orthogonal to a cyclical system, belongs to a triply orthogonal system.

14. In a normal congruence of plane curves, the envelope of the planes is a surface; shew that, when the surface is deformed so as to carry its tangent planes with it while the curves in the planes are unaltered, the congruence remains normal.

MISCELLANEOUS EXAMPLES.

1. A cubic helix has the plane at infinity for an osculating plane. Shew that it can be represented by the equations

$$x=\lambda\,(a^2t-\tfrac{1}{3}t^3), \quad y=\lambda at^2, \quad z=\mu\,(a^2t+\tfrac{1}{3}t^3).$$

2. In a skew curve of constant torsion, the direction-cosines of the binormal have the form

$$a\cos\theta+b\cos 4\theta, \quad a'\sin\theta+b'\sin 4\theta, \quad c\cos\tfrac{5}{2}\theta.$$

Find constants a, b, a', b', c so that the curve may be algebraic; and obtain its equations.

3. Shew that the binormals of a skew curve generate a scroll on which the curve is the line of striction; that the principal radii of curvature of the scroll at a point on the curve are $\sigma\tan\frac{1}{2}\beta$ and $-\sigma\cot\frac{1}{2}\beta$, where σ is the radius of torsion of the curve and $\tan\beta=2\rho/\sigma$; and that the other inflexional tangent at the point makes an angle β with the generator.

4. Prove that any skew curve is a geodesic on some developable surface; and find the angle between the curve and the generator of the developable at any point.

When a linear relation or a quadratic relation exists between s and ρ/σ, the edge of regression becomes a point or a conic respectively when the surface is developed into a plane.

Shew also that, if the lines of curvature on a developable are spherical curves, the edge of regression is a geodesic on a cone and so belongs to the first class of curves indicated.

5. Denoting by κ and τ the curvature and tortuosity at the origin of the line of curvature of the surface

$$z=\tfrac{1}{2}(ax^2+by^2)+\tfrac{1}{6}(Ax^3+3Bx^2y+3Cxy^2+Dy^3)+\tfrac{1}{24}(ax^4+4\beta x^3y+\ldots),$$

when it is referred to plane $z=0$ as a tangent plane, prove that

$$\kappa^2\tau=\frac{(3a-b)AB-3aBC}{(a-b)^2}-\frac{a\beta}{a-b}.$$

6. The tangent lines to two skew curves are conjugate with respect to a sphere. The distances of their points of contact from the centre of the sphere are r and r'; the perpendiculars upon the tangents from the centre of the sphere are p and p'; and the radii of curvature and of torsion are ρ, σ, and ρ', σ', for the two curves respectively. Prove that

$$\frac{\rho^2p^3}{\sigma r^4}=\frac{\rho'p'^3}{\sigma'r'^4},$$

and that $rr'(\sigma\sigma')^{-\frac{1}{2}}$ is equal to the radius of the sphere.

7. Two skew curves are such that their tangent lines at corresponding points are polar conjugates with regard to the paraboloid $x^2+y^2=2z$. Denoting by n_1 and n_1' the cosines of the angles between those tangents and the axis of z, by ν and ν' the moments of the tangents about that axis, by n_3 and n_3' the cosines of the angles between the binormals and the axis, and the tortuosities by σ and σ', prove that

$$\nu\nu' = -n_1 n_1', \quad \sigma\sigma' = -n_3^2 n_3'^2.$$

8. When a surface is represented in cylindrical coordinates by the equation $z=f(r, \theta)$, its Gaussian measure of curvature is

$$\frac{r^2 \dfrac{\partial^2 z}{\partial r^2}\left(\dfrac{\partial^2 z}{\partial \theta^2} + r\dfrac{\partial z}{\partial r}\right) - \left(r\dfrac{\partial^2 z}{\partial r \partial \theta} - \dfrac{\partial z}{\partial \theta}\right)^2}{\left\{r^2 + r^2\left(\dfrac{\partial z}{\partial r}\right)^2 + \left(\dfrac{\partial z}{\partial \theta}\right)^2\right\}^2}.$$

9. Points $P_1, \ldots, P_n$ move in such a way, that their mean centre is a fixed point and the tangent planes to their loci at corresponding points are parallel. Let R_r and R_r' be the principal radii of curvature of the locus of P_r; and let ω_r be the angle which a line of curvature makes with a line in the tangent plane parallel to a fixed plane; shew that

$$\sum_{r=1}^{n} (R_r + R_r') = 0, \quad \sum_{r=1}^{n} (R_r - R_r') \cos 2\omega_r = 0, \quad \sum_{r=1}^{n} (R_r - R_r') \sin 2\omega_r = 0.$$

10. The surface $z=f(x, y)$ has $ux+vy+wz=1$ for a tangent plane, where w is some function of u and v. Shew that

$$x + z\frac{\partial w}{\partial u} = 0, \quad y + z\frac{\partial w}{\partial v} = 0,$$

$$z^4 \left\{\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2\right\} w^4 \left\{\frac{\partial^2 w}{\partial u^2}\frac{\partial^2 w}{\partial v^2} - \left(\frac{\partial^2 w}{\partial u \partial v}\right)^2\right\} = 1;$$

and apply the last result to compare the measures of curvature of a surface and of its polar reciprocal at corresponding points.

11. When a surface is defined as the envelope of a plane

$$ux + vy + wz = 1,$$

where w is a function of u and v, its principal curvatures are the values of κ given by

$$\kappa (u^2+v^2+w^2)^{\frac{1}{2}} = \lambda (uw_1 + vw_2 - w),$$

the values of λ being roots of the equation

$$\begin{vmatrix} \lambda w_{11} + w_1^2 + 1, & \lambda w_{12} + w_1 w_2, & u + ww_1 \\ \lambda w_{12} + w_1 w_2, & \lambda w_{22} + w_2^2 + 1, & v + ww_2 \\ u + ww_1, & v + ww_2, & u^2+v^2+w^2 \end{vmatrix} = 0.$$

12. Skew surfaces are generated by drawing normals to a given surface at points which lie on different curves traced upon it. Shew that, for surfaces containing an assigned normal, the Gaussian measure of curvature at the centres of principal curvature of the original surface is constant; and that, for surfaces touching along an assigned normal, the principal radii of curvature at the centres of principal curvature of the original surface are the same.

13. The generators of a ruled surface all belong to a linear complex; prove that the asymptotic lines can be determined by a single quadrature. Shew that, for the surface

$$(yz-x)^2 = y^2(x^2+y^2),$$

they are given by

$$\frac{xy}{yz-x} + \frac{2}{y} = \text{constant}.$$

14. On a surface represented by

$$x = U_1 V_1, \quad y = U_2 V_2, \quad z = U_3 V_3,$$

where U_1, U_2, U_3 are functions of u only, and V_1, V_2, V_3 are functions of v only, such that the parametric curves are a conjugate system, shew that the asymptotic lines can be determined by quadratures. Obtain them for the surface

$$x = A(u-a)^m (v-a)^m, \quad y = B(u-b)^m (v-b)^m, \quad z = C(u-c)^m (v-c)^m.$$

15. On the surface

$$\left.\begin{aligned} 2Ax &= (a+u)^3 + (a+v)^3 \\ 2By &= (b+u)^3 + (b+v)^3 \\ 2Cz &= (c+u)^3 + (c+v)^3 \end{aligned}\right\},$$

which is the locus of middle points of the chords of the curve

$$Ax = (a+u)^3, \quad By = (b+u)^3, \quad Cz = (c+u)^3,$$

the asymptotic lines are given by $u \pm v = \text{constant}$.

16. Prove that the curvature of an asymptotic line on any surface is

$$4 \frac{(-\alpha\beta)^{\frac{7}{8}}}{(\alpha-\beta)^{\frac{3}{2}}} \left\{ \frac{\partial}{\partial u} \left(-\frac{\alpha}{\beta^3} \right)^{\frac{1}{8}} \pm \frac{\partial}{\partial v} \left(-\frac{\beta}{\alpha^3} \right)^{\frac{1}{8}} \right\},$$

where α and β are the principal radii of curvature, and where u, v are the parameters of the lines of curvature.

17. The necessary and sufficient condition, that the lines of curvature may divide a surface into infinitesimal squares, is that the quantity

$$\left\{ \frac{\partial(\kappa+\kappa')}{\partial x} dX + \frac{\partial(\kappa+\kappa')}{\partial y} dY + \frac{\partial(\kappa+\kappa')}{\partial z} dZ - d(\kappa\kappa') \right\} (\kappa-\kappa')^{-2}$$

should be a perfect differential, κ and κ' denoting the circular curvatures of the lines of curvature on the surface.

18. At any point P on a pseudosphere (of constant curvature -1) a unit length PQ is taken along a tangent, drawn in such a direction that the tangent plane at Q to the locus of Q passes through PQ and is perpendicular to the tangent plane at P. Shew that the locus of Q is also a pseudosphere.

19. Prove that, on a surface of constant negative curvature $-1/r^2$, the area of the maximum triangle, which can be formed with two of its sides of given lengths a and b, is

$$2r^2 \sin^{-1}\left(\tanh \frac{a}{2r} \tanh \frac{b}{2r} \right).$$

20. Obtain the general equation of geodesics on the surface

$$\begin{aligned} 2x &= \int (1-u^2) u^{n-1} du + \int (1-v^2) v^{n-1} dv, \\ 2yi &= -\int (1+u^2) u^{n-1} du + \int (1+v^2) v^{n-1} dv, \\ z &= \frac{1}{n+1} (u^{n+1} + v^{n+1}), \end{aligned}$$

where n is a constant, in the form

$$\log \frac{cu}{v} = a \int \{a^2 + \theta^n (1+\theta^2)\}^{-\frac{1}{2}} \frac{d\theta}{\theta},$$

θ denoting uv, and a and c being arbitrary constants.

21. Shew that one system of the lines of curvature on the surface

$$x=f(\lambda)\cos\lambda-f'(\lambda)\sin\lambda+F(\mu)\cos\lambda,$$
$$y=f(\lambda)\sin\lambda+f'(\lambda)\cos\lambda+F(\mu)\sin\lambda,$$
$$z=\mu,$$

is composed of curves in parallel planes.

22. A point in space is determined by the parameters λ, μ, ν of the quadrics, which pass through it and are confocal with $x^2/a+y^2/b+z^2/c=1$. Obtain the equations of a straight line in the form

$$\Sigma\frac{d\lambda}{\{(\lambda-a)(\lambda-\beta)f(\lambda)\}^{\frac{1}{2}}}=0,\quad \Sigma\frac{\lambda d\lambda}{\{(\lambda-a)(\lambda-\beta)f(\lambda)\}^{\frac{1}{2}}}=0,$$

where

$$f(\lambda)=(a+\lambda)(b+\lambda)(c+\lambda),$$

and a, β are constants. Hence shew that the tangent lines, common to the quadrics of parameters a and β, are normals to a family of parallel surfaces given by

$$\Sigma\left\{\frac{(\lambda-a)(\lambda-\beta)}{f(\lambda)}\right\}^{\frac{1}{2}}d\lambda=0.$$

23. The coordinates of a point on a quadric $x^2/a+y^2/b+z^2/c=1$ are given in the form

$$\frac{x^2}{a}=\frac{a-b}{a-c}\operatorname{sn}^2 u\operatorname{sn}^2 v,\quad \frac{y^2}{b}=\frac{b-a}{b-c}\operatorname{cn}^2 u\operatorname{cn}^2 v,\quad \frac{z^2}{c}=\frac{c-a}{c-b}\operatorname{dn}^2 u\operatorname{dn}^2 v,$$

where the modulus of the elliptic functions is $(a-b)^{\frac{1}{2}}(a-c)^{-\frac{1}{2}}$. Shew that the equations of the lines of curvature are

$$u=\text{constant},\quad v=\text{constant};$$

and that those of the generators are

$$u+v=\text{constant},\quad u-v=\text{constant}.$$

24. A point on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is represented by

$$x=a\operatorname{sn} u\operatorname{dn} v,\quad y=b\operatorname{cn} u\operatorname{cn} v,\quad z=c\operatorname{dn} u\operatorname{sn} v,$$

where the modulus is $2^{-\frac{1}{2}}$, and the ellipsoid is such that $2b^2=a^2+c^2$. Shew that u and v are the parameters of its lines of curvature; discuss the surface of centres; and prove that the curve $u+v=\gamma$ is the intersection of the ellipsoid with the quadric

$$ac(y^2-b^2)\operatorname{cn}\gamma-2b^2xz\operatorname{dn}^2\gamma+abcy\operatorname{sn}^2\gamma=0.$$

25. Find the geodesics on the surface $ds^2=(u+v)^{-2}du\,dv$; and use the integral equation to shew that, on a surface generated by rotating a tractrix about its asymptote, the geodesics lie upon the cylinders

$$r^2(\phi^2+A\phi+B)+a^2=0,$$

and cut the cuspidal edge at an angle a, where

$$\frac{c^2}{r^2}=\sec^2 a-\phi^2.$$

26. A pseudosphere is represented by the equations

$$x=ak\cos\omega\cos\phi,\quad y=ak\cos\omega\sin\phi,\quad z=a\int_0^\omega(1-k^2\sin^2\theta)^{\frac{1}{2}}d\theta;$$

obtain the integral equation of its geodesics in the form

$$\tan\omega=A\sin k(\phi+\beta).$$

Prove also that the equation of the geodesic parallel, having its centre at $\omega=a$, $\phi=0$, and having $\frac{1}{2}a\pi$ for its radius, is

$$\cos k\phi+\tan\omega\tan a=0.$$

27. A closed circuit of given perimeter is drawn on a surface, so as to cut off a maximum area of the surface. Prove that the geodesic curvature of the circuit is constant; and shew that, in general, only a limited number of such circuits of the same perimeter can be drawn through any assigned point of the surface.

28. The geodesic distances of any point on a surface from two fixed points on the surface, geodesically distant c from one another, are r and r'. The surface is such that the curves

$$(1+a)\,r^2+(1-a)\,r'^2=a$$

are parallel curves (a being the parameter of the family); shew that the angle between the curves

$$r=\text{constant},\quad r'=\text{constant},$$

at a point of intersection is

$$\cos^{-1}\frac{r^2+r'^2-c^2}{2rr'}.$$

29. Obtain the equation of geodesics on Enneper's minimal surface in the form

$$\frac{du}{u}-\frac{dv}{v}+\{1+auv\,(1+uv)^2\}^{-\frac{1}{2}}\left(\frac{du}{u}+\frac{dv}{v}\right)=0,$$

where a is an arbitrary constant.

30. The line of striction of a scroll is one of its asymptotic curves; prove that the angle at which it cuts any generator is equal to its angle of contingence t, and that the asymptotic curves cut the generators at a distance r from the curve measured along the generators, where

$$\frac{2}{r\tau^{\frac{1}{2}}}=a+\int\tau^{\frac{1}{2}}\cos t\,ds,$$

τ being the tortuosity of the curve at the point.

31. A skew curve has assigned terminal points and assigned directions for its tangents at the terminal points; and it is to have the property that $\int\kappa^2 ds$ along the curve has a stationary value, where κ is the circular curvature at any point. Denoting the torsion at the point by τ, prove that

$$\kappa^2\tau=a,$$

$$\kappa^2\left(\frac{d\kappa}{ds}\right)^2+a^2+b\kappa^2+\tfrac{1}{4}\kappa^6=0,$$

where a and b are constants.

32. The arc-element on a surface is given by

$$ds^2=\frac{a^2}{w^2}(du^2+dv^2+dw^2),$$

where $u^2+v^2+w^2=1$.

Shew that the geodesics are given by linear equations between u and v.

Denoting the geodesic distance between two points u, v, w and u', v', w' by ρ, prove that

$$4\eta\eta'\sinh^2(\rho/2a)=(\xi-\xi')^2+(\eta-\eta')^2,$$

where

$$\xi=\frac{au}{1-v},\quad \eta=\frac{aw}{1-v},$$

and so for ξ' and η'. Obtain the arc-element in terms of ξ and η; and determine the curves in the plane of ξ, η which correspond to geodesics on the surface.

Prove that the Gaussian measure of curvature for the surface is constant.

33. All surfaces, whose lines of curvature can be spherically represented by two systems of orthogonal circles, can be generated as envelopes of the plane

$$lx+my+nz+lf\left(\frac{n-\cos a}{l}\right)+mF\left(\frac{1-n\cos a}{m}\right)=0,$$

where $l^2+m^2+n^2=1$, a is a constant depending upon the particular set of orthogonal circles, and the axes of x and z are parallel to the lines through which the planes of the two circles pass.

34. When a surface is geodesically represented on a plane, curves of finite constant geodesic curvature on the surface do not in general become circles on the plane.

35. Prove that the surface, represented (in the ordinary notation of elliptic functions) by the equations

$$\left.\begin{aligned}2\kappa^2x&=E(u)+E(v)-(1-\kappa^2)(u+v)\\ 2\kappa^2y&=i\{E(u)-E(v)-(1+\kappa^2)(u-v)\}\\ e^{\kappa z}&=(\operatorname{dn}u-\kappa\operatorname{cn}u)(\operatorname{dn}v-\kappa\operatorname{cn}v)\end{aligned}\right\},$$

is minimal, and that its principal curvatures are

$$\pm 2(\operatorname{cn}u\operatorname{dn}u\operatorname{cn}v\operatorname{dn}v)^{\frac{1}{2}}(1+\operatorname{sn}u\operatorname{sn}v)^{-2}.$$

36. Prove that the surface

$$\begin{aligned}x&=\lambda\cos a+\sin\lambda\cosh\mu,\\ y&=\mu+\cos a\cos\lambda\sinh\mu,\\ z&=\sin a\cos\lambda\cosh\mu,\end{aligned}$$

is a minimal surface; that the parametric curves are plane lines of curvature; and that the Gaussian measure of curvature is

$$-(\cosh\mu+\cos a\cos\lambda)^{-4}\sin^2 a.$$

37. A rigid boundary consists of two finite perpendicular straight lines OP and OQ, and two infinite straight lines through P and Q perpendicular to the plane POQ drawn in the same sense. Shew that the minimal surface with this boundary is obtained by taking

$$F(u)=-ik(1+2u^2\cos a+u^4)^{-1}$$

in the Weierstrass equations, where k and a are real, and OP, OQ are the axes of x and y. Obtain relations between k, a, and the lengths of OP and OQ.

38. Taking the conics subsidiary to the construction of a Dupin cyclide as the focal conics of a system of confocal quadrics, prove that the equation of the cyclide can be expressed in the form

$$a_1+a_2+a_3=\text{constant},$$

where a_1, a_2, a_3 are the primary semi-axes of the confocals through any point.

39. When the fundamental magnitudes E, F, G are (i) functions of one parameter only, or are (ii) homogeneous functions of two parameters of degree -2, the surface is applicable on a surface of revolution.

40. A portion of a sphere is deformed, and θ is the angle which the normal to the deformed surface makes with the axis of z; prove that $H\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)\sec\theta$ cannot be negative for the deformed surface, H denoting the mean measure of curvature.

41. A surface is generated by the revolution of the curve

$$x = k(a + b\,\mathrm{cn}\,u), \quad y = bE(u),$$

round the axis of y, where k is the modulus and $a > b$. Prove that the zone between the planes $u = 2K$ and $u = -2K$ is applicable to a portion of an anchor ring.

Shew that the real branches of the asymptotic lines upon the surface are closed curves, if

$$\int_0^{2K} \left(\frac{bk'\,\mathrm{sn}\,v}{a\,\mathrm{dn}\,v - bk'\,\mathrm{sn}\,v}\right)^{\frac{1}{2}} dv = \frac{m}{n}\pi,$$

where m and n are integers.

42. Prove that helicoids of a special type exist, which are applicable to scrolls of a special type so that the helices of the former coincide with the orthogonal trajectories of the generators of the latter.

Prove also that the surfaces of revolution to which the helicoids in question can be applied are generated, by rotation round the axis of y, of one or other of the curves

$$x\,\mathrm{dn}\,u = 1, \quad y = u - E(u) + k^2 \frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u};$$

$$x\,\mathrm{cn}\,u = 1, \quad ky = u - E(u) + \frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u}.$$

43. A geodesic circle is defined as the locus of points on a surface at a constant geodesic distance from a centre. It might also be defined as a curve of constant geodesic curvature. Prove that, if the definitions agree for one centre, the surface is applicable to a surface of revolution; if they agree for all centres, the Gaussian measure of curvature is constant.

44. Prove that an infinite number of scrolls can usually be found applicable to a given scroll, so that their generators correspond; and that scrolls, with their generators parallel to those of a given cone, can be found similarly applicable to the given scroll.

Let the given scroll be the cylindroid

$$x = u\cos v, \quad y = u\sin v, \quad z = p\sin 2v,$$

and the given cone be $x^2 + y^2 = z^2 \cot^2 a$; shew that the equations of the line, on the scroll applicable as above to the cylindroid, which corresponds to the axis of z on the cylindroid, are

$$x = \pm p\sin a\left[\frac{\cos\{(\sec a + 2)v + \beta\}}{\sec a + 2} + \frac{\cos\{(\sec a - 2)v + \beta\}}{\sec a - 2}\right],$$

$$y = \mp p\sin a\left[\frac{\sin\{(\sec a + 2)v + \beta\}}{\sec a + 2} + \frac{\sin\{(\sec a - 2)v + \beta\}}{\sec a - 2}\right],$$

$$z = \pm p\cos a\sin 2v,$$

where β is an arbitrary constant. Shew also that this line is the line of striction on its scroll.

45. Prove that the ruled surface, which is applicable to the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

and has its generators parallel to those of the hyperboloid in the same sense, is given by the equations

$$\frac{x}{a}-\frac{u}{\Delta}\cos v=\quad\sin v-\frac{2bc}{(a^2+c^2)^{\frac{1}{2}}(a^2-b^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{c(a^2-b^2)^{\frac{1}{2}}}{b(a^2+c^2)^{\frac{1}{2}}}\sin v\right\},$$

$$\frac{y}{b}-\frac{u}{\Delta}\sin v=-\cos v+\frac{2ac}{(b^2+c^2)^{\frac{1}{2}}(a^2-b^2)^{\frac{1}{2}}}\tanh^{-1}\left\{\frac{c(a^2-b^2)^{\frac{1}{2}}}{a(b^2+c^2)^{\frac{1}{2}}}\cos v\right\},$$

$$\frac{z}{c}-\frac{u}{\Delta}\quad=\quad\frac{2ab}{(a^2+c^2)^{\frac{1}{2}}(b^2+c^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{a(b^2+c^2)^{\frac{1}{2}}}{b(a^2+c^2)^{\frac{1}{2}}}\tan v\right\},$$

where

$$\Delta^2=a^2\cos^2 v+b^2\sin^2 v+c^2.$$

46. The quadric $z=xy$ is deformed into a surface

$$Z=f(x,\ y),\quad X=g(x,\ y),\quad Y=h(x,\ y);$$

prove that a solution of the partial equation of the second order for Z is

$$Z=(x^2y^2+axy)^{\frac{1}{2}}+a\log\{(xy)^{\frac{1}{2}}+(a+xy)^{\frac{1}{2}}\},$$

where a is an arbitrary constant; and obtain the values of X and Y to be associated with this value of Z.

47. Three quantities α, β, γ connected with the parameters λ, μ, ν of three quadrics, passing through a point in space and confocal with $x^2/a+y^2/b+z^2/c$, are defined by the relations

$$\Sigma(a+\lambda)^{-\frac{1}{2}}d\lambda=d\alpha,\quad \Sigma(b+\lambda)^{-1}(a+\lambda)^{-\frac{1}{2}}d\lambda=d\beta,\quad \Sigma(c+\lambda)^{-1}(a+\lambda)^{-\frac{1}{2}}d\lambda=d\gamma.$$

Prove that the surfaces defined by α, β, γ as parameters are a triply orthogonal system; and obtain the arc-element in space as given by

$$4ds^2=d\alpha^2+\frac{f(b)}{c-b}d\beta^2+\frac{f(c)}{b-c}d\gamma^2,$$

where

$$f(\theta)=(\theta+\lambda)(\theta+\mu)(\theta+\nu).$$

48. Obtain the equation of confocal cyclides in the form

$$\frac{a\xi^2}{a+\lambda}+\frac{b\eta^2}{b+\lambda}+\frac{c\zeta^2}{c+\lambda}=1,$$

where λ is the parametric variable, and ξ, η, ζ are the parameters (to be replaced by the functions of the variables) of a system of triply orthogonal spheres.

Shew that the cyclides constitute a triply orthogonal system.

49. Obtain a triply orthogonal system of surfaces such that

$$H_1=1,\quad H_2=1,\quad H_3=Au+Bv+C,$$

where A, B, C are functions of w alone.

50. Shew that the surfaces

$$xy = uz^2, \quad x^2+y^2+z^2 = v, \quad x^2+y^2+z^2 = w(x^2-y^2),$$

are a triply orthogonal system.

51. The coordinates of a point in space are given by the equations

$$x = 2a\frac{\sin u}{1+(w-v)^2\sin^2 u}\{\cos v - (w-v)\sin v\},$$

$$y = 2a\frac{\sin u}{1+(w-v)^2\sin^2 u}\{\sin v + (w-v)\sin v\},$$

$$z = a\left\{\log(\tan\tfrac{1}{2}u) + \frac{2\cos u}{1+(w-v)^2\sin^2 u}\right\}.$$

Shew that the u-surface is a sphere having its centre on the axis of z, that the w-surface is a pseudosphere having its measure of curvature equal to $-1/a^2$, and that the v-surface is a surface of revolution round the axis of z. Verify that the parametric surfaces are a triply orthogonal system, by obtaining the arc-element in space in the form

$$ds^2 = \frac{a^2\operatorname{cosec}^2 u}{1+(w-v)^2\sin^2 u}[\{1-(w-v)^2\sin^2 u\}\,du^2 + 4(w-v)^2\sin^4 u\,dv^2 + 4\sin^2 u\,dw^2].$$

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